SLAC–PUB–5579
May 1991
(A/E)

DIFFERENTIAL LUMINOSITY UNDER BEAMSTRAHLUNG*

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Abstract

For the next generation of e^+e^- linear colliders in the TeV range, the energy loss due to *beamstrahlung* during the collision of the e^+e^- beams is expected to be substantial. One consequence is that the center-of-mass energy between the colliding particles can be largely degraded from the designed value. The knowledge on the differential luminosity as a function of the center-of-mass energy is essential for particle physics analysis on the interesting events. In this paper we derive an analytic formula for such a differential luminosity, which agrees very well with computer simulations. A major characteristic of this formula is discussed.

I. INTRODUCTION

It is known that *beamstrahlung*, i.e., the synchrotron radiation from the colliding e^+e^- beams, will carry away a sbustantial fraction of beam particle energy in future linear colliders. This, for one thing, will result in a degradation of the center-of-mass energy of the colliding beams. From high energy physics point of view, it is important to know the luminosity as a function of the spreaded center-of-mass, so as to analyze the data attained from the collider.

When the average number of beamstrahlung photons radiated per beam particle is much less than unity, the energy spectrum for the final e^+ or e^- beams is simply the well-known Sokolov-Ternov spectrum [1] for the radiated photons with the fractional photon energy, $y(\equiv E_\gamma/E_0)$, replaced by the corresponding final electron (or positron) energy, $x = 1 - y$. When the condition is such that the average number of photons radiated is not much less than unity, the effect of succesive radiations becomes important. Previously, the multi-photon beamstrahlung process has been studied by Blankenbecler and Drell [2], and independently by Yokoya and Chen [3]. In this paper, we shall adopt the formulation developed in Ref. 3 as the basis for our derivation of the differential luminosity. In section 2, we will review the electron spectrum under multi-photon beamstrahlung. Section 3 will be devoted to the derivation of the differential luminosity. The characteristic feature of our formula is discussed and comparison to computer simulation is presented in the last section.

II. ELECTRON ENEERGY SPECTRUM

Let $\psi(x,t)$ be the energy spectral function of the electron for energy $x \equiv E/E_0$ at time t normalized as $\int \psi(x,t)dx = 1$. We assume that the emmision of the photon takes place in an infinitesimally short time interval. Then the evolution of the spectral function can be described by the rate equation

$$\frac{\partial\psi}{\partial t} = -\nu(x)\psi(x,t) + \int_x^\infty F(x,x')\psi(x',t)dx' \quad , \qquad (1)$$

where the first term corresponds to the *sink*, and the second term the *source*, for the evolution of $\psi(x,t)$. Here $\nu(x)$ is the average number of photons radiated per unit time and F is the spectral function of radiation, i.e., $F(x,x')dx'$ is the transition probability of an electron from energy x' to the energy interval $(x, x+dx)$ per unit time. Obviously, $F(x,x') = 0$ if $x \geq x'$. Notice, however, that F does not include the probability for electrons to remain at the same energy without photon emmision.

The spectral function of radiation can be characterized by the beamstrahlung parameter Υ, defined as

$$\Upsilon = \gamma\frac{B}{B_c} \quad , \qquad (2)$$

where B is the effective field strength of the beam, and $B_c = m^2c^3/e\hbar \sim 4.4\times10^{13}$ Gauss is the Schwinger critical field. For historical reasons, this parameter is related to the parameter ξ introduced by Sokolov and Ternov,[1] by a simple factor

$$\xi = \frac{\omega_c}{E} = \frac{3}{2}\frac{r_e\gamma^2}{\alpha\rho} = \frac{3}{2}\Upsilon \equiv \frac{x}{\kappa} \quad , \qquad (3)$$

where r_e is the classical electron radius, α the fine structure constant, ω_c the critical frequency of radiation, and ρ the instantaneous radius of curvature, which is proportional to γ. Thus the introduced parameter K is independent of energy. Since the two parameters are trivially related, one may employ either of them depending on the convenience of the situation.

The transition probability F derived by Sokolov and Ternov is

$$F(x,x') = \frac{\nu_{cl}\kappa}{xx'}f(\xi,\eta) \quad , \qquad (4)$$

$$f(\xi,\eta) = \frac{3}{5\pi}\frac{1}{1+\xi\eta}\Big[\int_\eta^\infty K_{5/3}(u)du + \frac{\xi^2\eta^2}{1+\xi\eta}K_{2/3}(\eta)\Big] \quad ,$$

where $\eta = \kappa[(1/x') - (1/x)]$, K_ν's are the modified Bessel functions and ν_{cl} is the number of photons per unit time calculated by the classical theory of radiation,

$$\nu_{cl} = \nu_{\eta=0} = \frac{5}{2\sqrt{3}}\frac{\alpha\gamma}{\rho} \quad . \qquad (5)$$

Note that for a given field strength ν_{cl} is independent of the particle energy. In general, however,

$$\nu(x) = \int_0^x F(x,x')dx' \equiv \nu_{cl}U_0(\xi) \quad . \qquad (6)$$

The function $U_0(\xi)$ is normalizedsuch that $U_0(0) = 1$, and can be represented by the following approximate expression:

$$U_0(\xi) = \frac{1 - 0.598\xi + 1.061\xi^{5/3}}{1 + 0.922\xi^2} \quad , \qquad (7)$$

where the relative error is within 0.7% for any ξ.

* Work supported by Department of Energy contract DE–AC03–76SF00515.

Presented at the IEEE Particle Accelerator Conference, San Francisco, CA, May 6–9, 1991

To look for a compact analytic solution for ψ in Eq.(1), the exact Sokolov-Ternov spectral function in Eq.(4) is somewhat cumbersome. In the classical regime of radiation, i.e., $\xi \ll 1$, one can instead invoke an approximate expression to replace $f(\xi, \eta)$ in Eq.(4):

$$g(\eta) = \frac{1}{\Gamma(1/3)} \eta^{-2/3} e^{-\eta} \quad . \tag{8}$$

With this approximation, Eq.(1) can be solved by proper Laplace transformations. The details can be found from Ref. 3. The solution is

$$\psi(x,t) = e^{-N_{cl}} \left[\delta(1-x) + \frac{e^{-\eta}}{1-x} h(\eta^{1/3} N_{cl}) \right] \quad , \tag{9}$$

where $N_{cl} = \nu_{cl} t$ is the average number of photons radiated up to time t, and

$$h(u) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \exp(u p^{-1/3} + p) dp = \sum_{n=1}^{\infty} \frac{u^n}{n! \Gamma(n/3)} \quad , \tag{10}$$

with $\lambda > 0$ and $0 \le u \le \infty$. The first term in Eq.(9) represents the electron population that suffers no radiation. The n^{th} term in the Taylor expansion of the second term corresponds to the process of n-photon emissions.

For finite values of ξ, the rate equation cannot be solved exactly since $\nu(x)$ is not constant in time any more. However, in the intermediate regime where $\xi \lesssim \mathcal{O}(10)$, $\nu(x)$ should not deviate from ν_{cl} too significantly. This suggests a solution based upon minor perturbation from the above classical result. It is found[3)] that

$$\psi(x,t) = e^{-N_\gamma} \left[\delta(1-x) + \frac{e^{-\eta}}{1-x} h(\eta^{1/3} \bar{N}(\eta)) \right] \tag{11}$$

for the intermediate regime, where

$$N_\gamma = U_0(\xi) N_{cl} \quad , \tag{12}$$

$$\bar{N} = \frac{1}{1+\xi\eta} N_{cl} + \frac{\xi\eta}{1+\xi\eta} N_\gamma \quad . \tag{13}$$

III. CENTER-OF-MASS LUMINOSITY

To find the differential luminosity $\mathcal{L}(s)$ as a function of the center-of-mass energy squared,s, one needs to convolute the energy spectrum of one beam, $\psi(x_1, t)$, with the other, $\psi(x_2, t)$. Let $t = 0$ when the e^+e^- bunches first meet. Then the first z-slice in beam #1 will always encounter a "fresh" beam #2:

$$\frac{d\mathcal{L}(0)}{dz} \propto \frac{2}{l} \int_0^{l/2} dt \psi(x_1, t) \psi(x_2, 0) \quad , \tag{14}$$

where l is the total length of each bunch. A slice at z in beam #1, however, will see a beam #2 which has evolved for a time $t = z/2$:

$$\frac{d\mathcal{L}(z)}{dz} \propto \frac{2}{l} \int_0^{l/2} dt \psi(x_1, t) \psi(x_2, z/2) \quad . \tag{15}$$

Adding all z-slices in beam #1 together, we have

$$\begin{aligned} \mathcal{L} &\propto \frac{4}{l^2} \int_0^{l/2} dt \psi(x_1, t) \int_0^{l} dz \psi(x_2, z/2) \\ &= \frac{4}{l^2} \int_0^{l/2} dt \psi(x_1, t) \int_0^{l/2} dz \psi(x_2, z) \quad . \end{aligned} \tag{16}$$

Note that the above two integrals are functionally identical. Inserting the spectral function in Eq.(9), we find, for $\xi \ll 1$,

$$\begin{aligned} \phi(x) &\equiv \frac{2}{l} \int_0^{l/2} dt \psi(x, t) \\ &= \frac{1}{N_{cl}} \left[(1 - e^{-N_{cl}}) \delta(1-x) + \frac{e^{-\eta(x)}}{1-x} g(\eta) \right] \quad . \end{aligned} \tag{17}$$

The function $g(\eta)$ in the second term is

$$g(\eta) = \sum_{n=1}^{\infty} \frac{\eta^{n/3}}{n! \Gamma(n/3)} \gamma(n+1, N_{cl}) \quad , \tag{18}$$

where $\gamma(n+1, N_{cl})$ is the incomplete gamma function.

The center-of-mass energy squraed is $s \equiv x_1 x_2$. The differential luminosity as a function of s is therefore

$$\mathcal{L}(s) = \mathcal{L} \int_s^1 \int_0^1 dx_1 dx_2 \delta(s - x_1 x_2) \phi(x_1) \phi(x_2) \quad . \tag{19}$$

It is straight forward to show that

$$\begin{aligned} \mathcal{L}(s) = \frac{\mathcal{L}}{N_{cl}^2} \Big\{ &\left[1 - e^{-N_{cl}} \right]^2 \delta(1-s) + 2 \left[1 - e^{-N_{cl}} \right] \frac{e^{-\eta}}{1-s} g(\eta) \\ &+ \int_s^1 dx \frac{e^{\eta(x) - \eta(s/x)}}{(1-x)(1-s/x)} g(\eta(x)) g(\eta(s/x)) \Big\} \quad . \end{aligned} \tag{20}$$

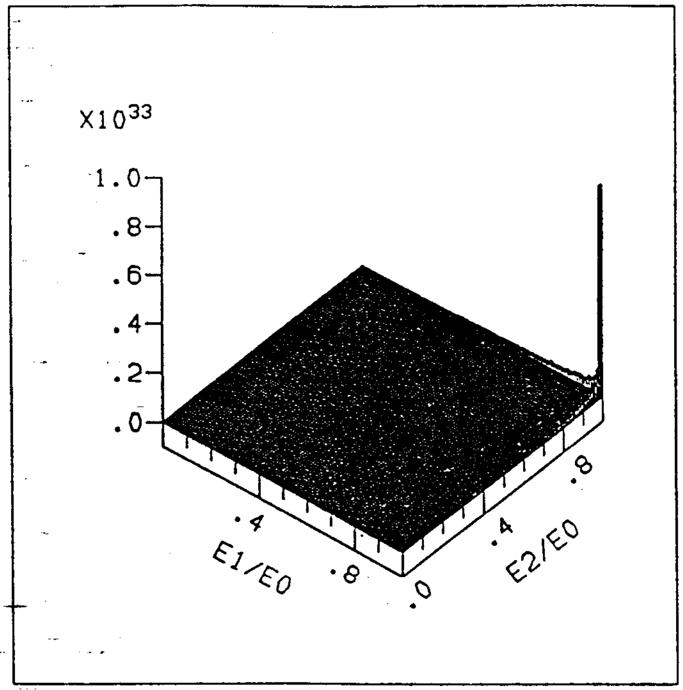

Figure 1. Two-dimensional histogram of the luminosity as a function of x_1 and x_2.

It can be shown that the last term is much smaller than unity, and is negligible. Thus

$$\mathcal{L}(s) = \frac{\mathcal{L}}{N_{cl}^2}\left\{\left[1-e^{-N_{cl}}\right]^2\delta(1-s)+2\left[1-e^{-N_{cl}}\right]\frac{e^{-\eta}}{1-s}g(\eta)\right\}. \tag{21}$$

For the intermediate regime, the spectral function of Eq.(9) should be replaced by Eq.(11). The derivation is essentially the same, and we find

$$\mathcal{L}(s) = \frac{\mathcal{L}}{N_\gamma^2}\left\{\left[1-e^{-N_\gamma}\right]^2\delta(1-s)+2\left[1-e^{-N_\gamma}\right]\frac{e^{-\eta}}{1-s}\bar{g}(\eta)\right\}, \tag{22}$$

where

$$\bar{g}(\eta) = \sum_{n=1}^{\infty}\left(\frac{\bar{N}}{N_\gamma}\right)^n \frac{\eta^{n/3}}{n!\Gamma(n/3)}\gamma(n+1, N_\gamma) \quad . \tag{23}$$

IV. DISCUSSIONS

To confirm our theoretical formulas, we perform computer simulations using the code ABEL [4]. The parameters of a linear collider with a center-of-mass energy 1/2 TeV designed by Palmer [5] (the Machine G in Table 1) was used. The parameter $\xi = 0.45$ in this example, and the bunch length is $l = \sqrt{2\pi}\sigma_z = 0.28$ mm. A two dimansional plot of $\mathcal{L}$ as a function of x_1 and x_2 is shown in Fig. 1. We see that the most striking character of the luminosity spectrum is that, aside from the sharp delta function at the nominal machine energy, other contribution to the luminosity comes essentially from the matching between a full energy particle and a beamstrahlung degraded particle. This is evidenced by the "walls" on the edges of the 2-D plot, which corresponds to the second term in Eq.(22).

ACKNOWLEDGEMENT

The author appreciates helpful discussions with T. Barklow of SLAC.

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