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A New Valley Method for Instanton Deformation^{*}

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ABSTRACT

Instanton and anti-instanton configurations used in evaluating the baryon and lepton number violation process in the standard model are investigated. Although not solutions of the field equations, such configurations play a crucial role in evaluating the relevant amplitudes. The streamline method, which has been employed for this purpose, is examined. A new method that has better features than the streamline method is proposed. The trajectory in configuration space that leads to the instanton and anti-instanton configuration is traced from vacuum using the new method.

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1. Introduction

Baryon and lepton number violation is a nonperturbative process in the standard model. Attempts to calculate its cross sections were made in various different formalism; ideal-gas instanton calculus,^[1-4] coherent state ansatz for Sphaleron,^[5] Gervais-Sakita formalism,^[6] interacting instantons,^[7-10] and other methods.^[11,12]

Among them, the first quantitative prediction at high energy was given by Ringwald and Espinosa, who predicted large but unitarity-violating cross sections. This unitarity problem was solved by analysing of the multi-instanton sectors.^[7,8,9] It was found that at TeV scale a configuration with many instantons (I) and anti-instantons (\bar{I}), which interact among themselves, contributes an amount comparable to that of the single I . It was further proven that the whole set of these configurations give unitary amplitudes. This, however, was a formal solution and only qualitative estimates were possible. There were two problems associated with it; (a) the I - \bar{I} interaction, derived from the asymptotic region, is singular at short distance and causes fictitious divergences in higher order calculations, (b) contribution of fluctuations becomes comparable to that of the background at TeV scale and is hard to estimate. These problems were expected to be solved by incorporating deformation of the original configuration.

In the context of baryon number violation, the deformation of $I\bar{I}$ was treated by Arnold and Mattis,^[13] and more recently by Khoze and Ringwald,^[14] The latter authors constructed a candidate for separated, but deformed $I\bar{I}$ configuration by applying a conformal transformation on co-central $I\bar{I}$ (with different radius). One essential component in their analysis is on “streamline method”, developed for QCD analysis.^[15-17]

Streamline method uses a collective coordinate in functional space: A configuration with I and \bar{I} separated by an infinite distance is a solution of the field equations. The action is twice that of I . When they approach each other, the action decreases and eventually they annihilate each other, resulting in a vacuum. This series of configurations defines a collective coordinate, and is called a “val-

ley”, since it is expected that the action rises if the configuration is altered in any direction other than the distance between I and \bar{I} . Although no configuration in this valley region is a solution, it is important to integrate over the valley to obtain relevant (baryon-number violating) amplitudes. In order to carry out the integral, one needs to choose the trajectory that properly follows the valley bottom and do Gaussian integration in the directions other than the trajectory.

The streamline method defines a trajectory for the valley bottom. It however contains some undesirable features. In this paper we point out those and propose a new definition, which avoids them. In the next chapter, we shall review the Gaussian integration method around a trajectory and the streamline method. The difficulties of the streamline method are pointed out. The new proposal for the trajectory is given in the chapter 3. We shall also show difference between the streamline and our trajectory for a nontrivial two-dimensional model. In chapter 4 we trace the growth of the $I\bar{I}$ configuration starting from the vacuum for a double well quantum mechanics. Discussion and summary are given in the last chapter.

2. The Streamline

The streamline method was originally proposed to circumvent the danger of Gaussian integral over a quasi zero-mode around a given classical solution. It is not an exact zero-mode but has a small eigenvalue compared with the other nonzero modes. Indeed, the path-integral around the largely separated I and \bar{I} suffers from the danger due to the interaction between them. That is, when ideal gas approximation is used, all the displacements of I and \bar{I} are zero-modes, and thus all the coordinates of centers of I and \bar{I} are collective coordinates. Ideal gas approximation, however, has limited validity. (In quantum mechanics, it can deal with only the shift of the ground state. In high energy physics, it leads to unitarity violating amplitudes.) Therefore the interaction between I and \bar{I} has to be taken into account. This interaction makes the relative distance between them a quasi zero-mode. One would still want to integrate over this collective coordinate, since

Gaussian approximation is in no sense valid for variations of this coordinate. In order to do this, one needs to adopt a trajectory in the functional space that defines the collective coordinate and do Gaussian integrals in other directions.

Let us first discuss the Gaussian integration, since it can be defined for a general collective coordinates, independent of its definition. We take a partition function,

$$\mathcal{Z} \equiv \mathcal{N} \int \prod_{\mathbf{i}} \frac{d\phi_{\mathbf{i}}}{\sqrt{2\pi}} e^{-S[\phi]}, \quad (2.1)$$

where \mathcal{N} is a suitably chosen normalization factor. Here we have introduced a cut off (either in the configuration space or momentum space) to discretise the field variable ϕ , for brevity. We further assume that $(\phi_{\mathbf{i}})$ has trivial metric, $g_{ij} = \text{diag}(1, 1, 1, \dots)$. (Otherwise, we need to insert the metric factor in the appropriate places in the following equations, which can be done without ambiguity.) In order to evaluate \mathcal{Z} using a collective coordinate α , we apply the Fadeev-Popov method by inserting the following;

$$1 = \int d\alpha \delta((\phi_{\mathbf{i}} - \phi_{\mathbf{i}}(\alpha))R_{\mathbf{i}}(\alpha)) \Delta(\phi(\alpha)), \quad (2.2)$$

where $\phi_{\mathbf{i}}(\alpha)$ is a valley trajectory and $R_{\mathbf{i}}$ is the normalized gradient vector,

$$R_{\mathbf{i}} \equiv \frac{1}{\sqrt{A}} \frac{\partial S}{\partial \phi_{\mathbf{i}}}, \quad A = \sum_{\mathbf{i}} \left(\frac{\partial S}{\partial \phi_{\mathbf{i}}} \right)^2. \quad (2.3)$$

The Fadeev-Popov constraint (2.2) restricts the fluctuation space (the space of integration) to be orthogonal to the gradient. This enables us to do the perturbative calculation systematically, even though, in general, a point on the trajectory is not a classical solution. The Fadeev-Popov determinant $\Delta(\phi(\alpha))$ is given by

$$\Delta(\phi(\alpha)) = \left| \frac{d\phi_{\mathbf{i}}(\alpha)}{d\alpha} \left(R_{\mathbf{i}}(\alpha) - (\phi_{\mathbf{j}} - \phi_{\mathbf{j}}(\alpha)) \frac{\partial R_{\mathbf{i}}}{\partial \phi_{\mathbf{j}}}(\alpha) \right) \right|. \quad (2.4)$$

Hereafter, we concentrate on the leading order contribution (*i.e.*, up to the one-loop term). The second term in the above is irrelevant at this order and can be

dropped. The determinant $\Delta(\phi(\alpha))$ is the Jacobian containing cosine of the angle between the trajectory and (R_i) . By choosing the α to be in the direction for which action S increases, one obtains up to the one-loop order,

$$\mathcal{Z} = \int d\alpha \frac{d\phi_i(\alpha)}{d\alpha} R_i(\alpha) e^{-S[\phi(\alpha)]} \mathcal{Z}_1, \quad (2.5)$$

where \mathcal{Z}_1 is the one-loop integral,

$$\mathcal{Z}_1 = \mathcal{N} \int \prod_i \frac{d\tilde{\phi}_i}{\sqrt{2\pi}} \delta(\tilde{\phi}_i R_i(\alpha)) \exp\left[-\frac{1}{2} D_{ij}(\alpha) \tilde{\phi}_i \tilde{\phi}_j\right]. \quad (2.6)$$

(We used the variable $\tilde{\phi}_i \equiv \phi_i - \phi_i(\alpha)$). This can be formally evaluated by exponentiating the δ -function and then doing the integrals as in the following;

$$\begin{aligned} \mathcal{Z}_1 &= \mathcal{N} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int \prod_i \frac{d\tilde{\phi}_i}{\sqrt{2\pi}} \exp(ik\tilde{\phi}_i R_i(\alpha)) \exp[\dots] \\ &= \frac{\mathcal{N}}{\sqrt{\det D}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-\frac{1}{2} R D^{-1} R k^2\right) \\ &= \frac{\mathcal{N}}{\sqrt{2\pi (R D^{-1} R) \det D}}. \end{aligned} \quad (2.7)$$

The factor $(R D^{-1} R) \det D$ is actually a determinant of D_{ij} restricted to the space orthogonal to (R_i) . Thus the final result is valid as long as D_{ij} is positive definite in that subspace. In choosing the trajectory, one should keep in mind this constraint.

Although the complete partition function \mathcal{Z} is independent of the choice of the trajectory $\phi(\alpha)$, it is a functional $\mathcal{Z}[\phi(\alpha)]$ at any given order. Therefore it is important to choose the trajectory to obtain a decent approximation.

The streamline method gives a simple and intuitive prescription for choosing

the trajectory $\phi(\alpha)$. Their main equation is

$$\frac{d\phi_i}{d\alpha} = f(\alpha) \frac{\partial S}{\partial \phi_i}, \quad (2.8)$$

where $f(\alpha)$ is an arbitrary function corresponding to reparametrization invariance of α .

There are several disadvantages with this approach. One of them is the problem with its boundary condition: The equation (2.8) only defines a flow (a gradient line) and does not define where the streamline is by itself. Balitsky and Yung^[15] proposed to start from a classical solution and head toward the direction of the minimum curvature. (The initial direction has to be specified, since (2.8) is ill-defined at the solution.) There are actually two ways to do this:

- (a) Starting from the higher end of the valley. One needs to choose the starting point to be $I\bar{I}$ configuration in order to insure that one is following the valley of physical significance. This $I\bar{I}$, however, should be with infinite separation. This calculation thus becomes rather involved.
- (b) Starting from the bottom of the valley, the vacuum. This however does not work due to an instability in the streamline method: This can be easily seen in a toy model defined by the following action with two degrees of freedom;

$$S = \frac{1}{2} (\epsilon \phi_1^2 + \phi_2^2), \quad (0 < \epsilon < 1). \quad (2.9)$$

In this example, the valley trajectory is trivially $\phi_2 = 0$. However, solving the flow equation (2.8) and eliminating α , one finds the gradient lines,

$$\phi_2 = C \phi_1^{1/\epsilon}, \quad (2.10)$$

where C is a an arbitrary parameter. This set of the gradient lines are illustrated in Fig.1. All these lines are tangent to the real valley $\phi_2 = 0$ at the origin. Therefore, the direction does not define a particular line. In actual numerical calculation, even if one makes a finite step to the right direction, any numerical error would drive one up the valley wall.

For high energy problems at hand, the $I\bar{I}$ configuration with short distance is important. One such example is the analysis by Yung,^[18] where the overlapping of the nonperturbative and perturbative piece at high energy is evaluated. (By subtracting this overlap, he obtains a milder behavior for the baryon number violating amplitude at high energy.) The case (b) thus has physical significance.

3. New Valley Method

The new definition we propose is the following: At every point on the trajectory $\phi(\alpha)$, the gradient vector is the eigenvector of the curvature with the smallest eigenvalue. Namely,

$$\frac{\partial^2 S}{\partial \phi_i \partial \phi_j} \frac{\partial S}{\partial \phi_j} = \lambda_{\min} \frac{\partial S}{\partial \phi_i}, \quad (3.1)$$

where λ_{\min} is the minimum eigenvalue of the matrix $\partial^2 S / \partial \phi_i \partial \phi_j$ ($\equiv D_{ij}$). This defines a trajectory, since it has one parameter λ_{\min} , which can play the role of α except for few degenerate cases. In other words, according to (3.1), the gradient vector is orthogonal to all the eigenvectors of D except for the one with smallest eigenvalue. If the total number of the variables ϕ_i is N , this gives $N - 1$ conditions. Therefore we have one degree of freedom left and thus have a trajectory.

One property of the trajectory defined by (3.1) is that the norm of the gradient A is extremized under a constant S . This can be shown by applying the standard Lagrange multiplier method. (For that matter, A can be a gradient of any function of S .) It however should be noted that the variational method is not equivalent to (3.1), since it in general would have solutions with larger eigenvalues. Our valley trajectory in fact follows the points where the gradient is *minimized* along the lines with constant S .

The streamline and our new trajectory generally differ. This is because if we substitute (2.8) into (3.1), we obtain

$$l.h.s. = \frac{1}{f} D_{ij} \frac{d\phi_j}{d\alpha} = \frac{1}{f} \frac{d}{d\alpha} \left(\frac{\partial S}{\partial \phi_i} \right). \quad (3.2)$$

This leads to

$$\frac{d}{d\alpha} \left(\frac{\partial S}{\partial \phi_i} \right) = f \lambda_{\min} \frac{\partial S}{\partial \phi_i}. \quad (3.3)$$

Integrating the above, we find

$$\frac{d\phi_i}{d\alpha} = f \frac{\partial S}{\partial \phi_i} = f \exp \left(\int f \lambda_{\min} d\alpha \right) C_i, \quad (3.4)$$

where C_i is a constant. Therefore our trajectory and the streamline can coincide only when they are straight. Thus they are generally incompatible.

There are several advantages of the new definition (3.1). The equation (3.1) is not a flow equation and gives a *local* definition; given a point in ϕ space, one can determine if it belongs to the new valley or not. (For ordinary field theory, it yields a fourth order differential equation.) Therefore it does not need any boundary conditions at the end points of the trajectory. In fact, when applied to the toy model in the previous chapter (2.9), the new definition (3.1) gives,

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \epsilon \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (3.5)$$

since $\lambda_{\min} = \epsilon$. This has a unique solution $\phi_2 = 0$, unlike the streamline method. This toy model analysis actually applies to neighborhood of a classical vacuum or any other classical solution, since in that region the action is quadratic in field variable. Therefore we find that in general our valley trajectory approaches the classical solution from the direction that is exactly the direction of the eigenvector of D with the smallest eigenvalue at the solution.

Note that in our prescription the problem of Gaussian approximation for small (or even negative) eigenvalue does not exist at any point (including the end points) on the trajectory. By definition (3.1) as well as the Fadeev-Popov constraint (2.2), the direction with the smallest eigenvalue is completely removed from the fluctuation space. This desirable property is not necessarily achieved in the streamline

method, in which the fluctuation space is orthogonal to R_i and therefore its basis vectors are generally different from eigenvectors of D_{ij} . Thus the Gaussian integral for the streamline method may still suffer from a small eigenvalue. This may result in large error at a given order of perturbation in the streamline prescription.

Let us next discuss a case when there is actually a zero-mode. In such a case, the trajectory has to be chosen along the related collective coordinate. The streamline method is rather singular in this case, for all the flow given by (2.8) is *orthogonal* to the flat valley bottom. Even if one starts from a point on the right trajectory, the right hand side of (2.8) is zero and thus one cannot trace it. (One has to deal with this case as a limit of a case with a small but nonzero eigenvalue.) In contrast, our new prescription (3.1) gives the correct trajectory of the collective coordinate, since (3.1) is trivially satisfied on it.

It is interesting to note that our choice of the trajectory (3.1) extremizes the tree part of (2.5) that also includes the Jacobian, which is defined by the following,

$$\mathcal{Z}_0[\phi(\alpha)] \equiv \int d\alpha \frac{d\phi_i(\alpha)}{d\alpha} R_i(\alpha) e^{-S[\phi(\alpha)]}. \quad (3.6)$$

In order to show this, let us first rewrite $\mathcal{Z}_0[\phi(\alpha)]$ as the following,

$$\begin{aligned} \mathcal{Z}_0[\phi(\alpha)] &= \int d\alpha \left(\frac{d}{d\alpha} F(\alpha) \right) G(\alpha), \\ F(\alpha) &\equiv -e^{-S[\phi(\alpha)]}, \quad G(\alpha) \equiv \frac{1}{\sqrt{A}}. \end{aligned} \quad (3.7)$$

After a partial integration, we find that the variation of $\mathcal{Z}_0[\phi(\alpha)]$ is to the first order,

$$\delta \mathcal{Z}_0[\phi(\alpha)] = \int d\alpha \delta\phi_j \frac{d\phi_i}{d\alpha} \frac{\partial F}{\partial \phi_i} \frac{\partial G}{\partial \phi_j}. \quad (3.8)$$

Using our definition of the trajectory (3.1), we find that

$$\begin{aligned}\frac{\partial F}{\partial \phi_{[i}} \frac{\partial G}{\partial \phi_{j]}} &= -\frac{1}{\sqrt{A}} R_i D_{jl} R_l + [i \leftrightarrow j] \\ &= -\frac{\lambda_{\min}}{\sqrt{A}} R_i R_j + [i \leftrightarrow j] = 0.\end{aligned}\tag{3.9}$$

Thus our prescription can also be obtained from a variational principle for $\mathcal{Z}_0[\phi(\alpha)]$.

In actual calculations, it is more practical to use a flow equation obtained from the definition. For this purpose, we take the derivative of (3.1) with respect to α and find that

$$H_{ij} \frac{d\phi_j}{d\alpha} = \frac{d\lambda_{\min}}{d\alpha} \frac{\partial S}{\partial \phi_i},\tag{3.10}$$

where

$$H = (D - \lambda_{\min} \cdot 1)D + \delta H, \quad \delta H_{ij} = \frac{\partial^3 S}{\partial \phi_i \partial \phi_j \partial \phi_l} \frac{\partial S}{\partial \phi_l}.\tag{3.11}$$

For the toy model (2.9), $\delta H = 0$ and $\lambda_{\min} = \epsilon$, a constant. Thus we find that (3.10) leads to

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d\phi_1}{d\alpha} \\ \frac{d\phi_2}{d\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\tag{3.12}$$

Therefore, we find that $(d\phi_i/d\alpha) \parallel (1, 0)$. This implies that in general near a classical solution the flow equation (3.10) does not suffer from the instability, which plagued the streamline method.

Except for rare cases such as the toy model (2.9), $\det H \neq 0$ and λ_{\min} is not a constant. Therefore one can invert H to obtain the vector $d\phi_i/d\alpha$ explicitly. In practice we fix the reparametrization invariance in α which (3.10) possesses. We have found it most convenient to choose it so that the vector $d\phi/d\alpha$ is normalized.

Namely,

$$d\alpha^2 = \sum_i d\phi_i^2. \quad (3.13)$$

In such a case, we have

$$\frac{d\phi_i}{d\alpha} = \frac{(H^{-1}R)_i}{|H^{-1}R|}. \quad (3.14)$$

We shall use this flow equation to trace the trajectory in the following.

Where one already knows an approximate solution for a valley trajectory, one could use a perturbation theory to obtain corrections. One such a situation is for $I\bar{I}$ separated by a large but finite distance. In Ref.7~9 and 14, the authors took sum of I and \bar{I} solutions for such a case. This sum is not a solution of the trajectory equation (3.1), but is expected to be close to it. In such a case, one can do a perturbation theory to obtain the right trajectory. Let us carry out a first order calculation in a general framework. Let us denote the approximate solution by $\phi_i^{(0)}(\alpha)$, so that

$$\delta V_i \equiv (D^{(0)} - \lambda_{\min}^{(0)}) \frac{\partial S}{\partial \phi_i^{(0)}}, \quad (3.15)$$

where $D^{(0)} \equiv D(\phi^{(0)})$, etc, is a small quantity. (In the above example, $\delta V = O(\exp(-I\bar{I} \text{ separation}))$.) We further denote the exact solution by $\phi_i = \phi_i^{(0)} + \delta\phi_i$. Expanding (3.1) around $\delta\phi_i = 0$, we find

$$\delta V_i + H^{(0)}\delta\phi_i = \delta\lambda_{\min} \frac{\partial S}{\partial \phi_i^{(0)}}. \quad (3.16)$$

On the right hand side, $\delta\lambda_{\min}$ is to the first order,

$$\delta\lambda_{\min} = R^{(0)} \delta D R^{(0)} = R^{(0)} \frac{\partial D^{(0)}}{\partial \phi_i^{(0)}} R^{(0)} \delta\phi_i. \quad (3.17)$$

This can be easily checked in the representation where $D^{(0)}$ is diagonal. Thus we

find that

$$\delta\phi_i = -G^{(0)-1}(D^{(0)} - \lambda_{\min}^{(0)})\frac{\partial S}{\partial\phi_i^{(0)}}, \quad (3.18)$$

where

$$G_{ij}^{(0)} \equiv H_{ij}^{(0)} - \frac{1}{A} \frac{\partial S}{\partial\phi_i} \frac{\partial S}{\partial\phi_k} \frac{\partial S}{\partial\phi_l} \frac{\partial^3 S}{\partial\phi_k\partial\phi_l\partial\phi_j} \Big|_{\phi=\phi^{(0)}} \quad (3.19)$$

For weak coupling, cubic derivative terms in $G^{(0)}$ can be ignored and we find that

$$\delta\phi_i = -D^{(0)-1}P_0\frac{\partial S}{\partial\phi_i^{(0)}}, \quad (3.20)$$

where $P_0 \equiv (D^{(0)} - \lambda_{\min}^{(0)})^{-1}(D^{(0)} - \lambda_{\min}^{(0)})$ is the projection operator to the space orthogonal to the eigenvector of lowest eigenvalue. This method can also be applied to a case with $\lambda_{\min}^{(0)} = 0$, since the projection operator takes this direction out of $\partial S/\partial\phi_i^{(0)}$ and makes the operation of $D^{(0)-1}$ well-defined.

So far we have treated only a trivial example, (2.9). Let us now deal with a nontrivial but simple example and demonstrate the formalism we presented so far. The action we choose is

$$S = \frac{1}{2}(\phi_2 - f(\phi_1))^2 + \epsilon g(\phi_1). \quad (3.21)$$

In this action, for $\epsilon = 0$ there exists exact zero-mode, or collective coordinate

$$\phi_2 = f(\phi_1). \quad (3.22)$$

We will choose f and g to satisfy

$$f(0) = 0, \quad g(\phi_1) = \frac{1}{2}\phi_1^2 + O(\phi_1^3) \quad (\text{for } \phi \ll 1), \quad (3.23)$$

so that the our valley trajectory always passes the origin. (Near the origin this action reduces that of the toy model (2.9).) The ϵ -term is introduced to give nonzero gradient on the valley bottom.

Our trajectory can be obtained as follows. Instead of solving (3.1) itself, we can require that *l.h.s.* is parallel to the gradient vector $\partial S/\partial\phi_i$. For this purpose, we calculate

$$K(\phi) = \frac{\partial S}{\partial\phi} \cdot O \cdot D \cdot \frac{\partial S}{\partial\phi}, \quad O \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.24)$$

and look for solution(s) of $K(\phi) = 0$ that has the smaller eigenvalue of D . For $\epsilon = 0$, we find that

$$K(\phi) = (\phi_2 - f)^3 f'' f', \quad \left(f' \equiv \frac{df}{d\phi_1}, \text{ etc.} \right) \quad (3.25)$$

There are three solution to this equation: For the solution $f'' = 0$, the eigenvalues of D are 0 and $f'^2 + 1$. The gradient vector $\partial S/\partial\phi$ is an eigenvector of the larger eigenvalue. Therefore this is not a solution of (3.1). For the second solution $f' = 0$, one can also see that $\partial S/\partial\phi$ is the eigenvector with larger eigenvalue near the origin. Therefore even if it gives a solution of (3.1), the trajectory does not pass the origin and is irrelevant. Thus we are left with the right solution (3.22).

For $\epsilon > 0$, we can either solve for $K = 0$ numerically or do perturbation with respect to ϵ around (3.22) (if $\epsilon \ll 1$). We find that

$$\phi_2 = f + \epsilon \frac{f' g'}{f'^2 + 1} + \epsilon^2 \frac{f' g'}{(f'^2 + 1)^3} \left(g'' - \frac{f'' f' g'}{f'^2 + 1} \right) + O(\epsilon^3). \quad (3.26)$$

The $O(\epsilon)$ term agrees with the general result (3.20).

One can also start from the origin and use the flow equation (3.14). Since this is a nontrivial model, $\det H \neq 0$ and $\lambda_{\min} \neq \text{constant}$. Thus we can use the explicit form of the normalized tangent vector (3.14) instead of (3.10).

For actual calculation, we have chosen

$$f = \frac{1 - \cos \phi_1}{2}, \quad g = 4 \left(1 - \cos \frac{\phi_1}{2} \right). \quad (3.27)$$

This way, the trajectory is not straight and we should be able to see difference between the streamline and our trajectory. The function g is chosen so that we

have a saddle point P at the point $(2\pi, 0)$. Since P is a classical solution, both the streamline and our trajectory pass the point P . This enables to start the streamline from P . (The point P is thus an analog of the higher end of the valley, $I\bar{I}$ at infinite separation.)

The streamline and the new trajectory are plotted in Fig.2, together with the contour lines for $\epsilon = 0.5$. In this figure one can see the predicted features of the trajectories. They differ when they are curved. The new trajectory is not perpendicular to the contour. They approach the classical solutions (origin and the saddle point) from the direction of smallest eigenvalue.

4. Instanton and Anti-instanton

In this chapter, we shall take a one-dimensional quantum mechanics with a double-well potential and examine the trajectory that results from our new prescription (3.1). The question is whether our local definition (3.1) does really give the trajectory that approaches the configuration with $I\bar{I}$ for large separation. If this were not the case, our valley method would be of no use for the instanton calculus.

The Euclidean action we take is

$$S = \int d\tau \left(\frac{1}{2} \left(\frac{d\phi}{d\tau} \right)^2 + \frac{1}{2} \phi^2 (1 - \phi)^2 \right). \quad (4.1)$$

This has an instanton solution,

$$\phi(\tau) = \frac{1}{2} \left(1 + \tanh \frac{\tau}{2} \right), \quad (4.2)$$

with action $S(I) = 1/6$.

In order to follow the growth of the $I\bar{I}$ configuration starting from a classical vacuum $\phi = 0$, we restrict the time to be from 0 to L and impose a boundary

condition $\phi(0) = \phi(L) = 0$. We then expand the $\phi(\tau)$ field by the normalized eigenmodes around $\phi = 0$ as in the following,

$$\phi(\tau) = \sum_{n=1}^N \phi_n \sqrt{\frac{2}{L}} \sin(\lambda_n \tau), \quad \lambda_n = \frac{(2n-1)\pi}{L}. \quad (4.3)$$

Since ϕ_n are coefficients of the normalized fluctuations, their metric in the functional space is trivial and enables us to apply the analysis in the chapter 3.

In (4.3), we have restricted ourselves only to the sector that is symmetric in $\tau \rightarrow L - \tau$. The anti-symmetric sector decouples from our calculation and is irrelevant. This is because of the following: Starting from vacuum, the trajectory first heads to the ϕ_1 direction, since it has the smallest eigenvalue. Any anti-symmetric mode $\phi^{(A)}$ couples to the symmetric modes $\phi^{(S)}$ as $\phi^{(A)}\phi^{(A)}\phi^{(S)}$ or $\phi^{(A)}\phi^{(A)}\phi^{(S)}\phi^{(S)}$. Therefore substituting this into H_{ij} defined in (3.11), we find that when the index i is in symmetric sector and j is in anti-symmetric sector all the terms in H_{ij} contain at least one power of $\phi^{(A)}$. Thus if the flow has no component in the anti-symmetric sector initially, it never does.

In carrying out the actual calculation, it is important that (a) L is large enough to contain well-separated I and \bar{I} , and (b) N is large enough so that the instanton shape is well described. For (a), in order to obtain instanton tail to the accuracy of $O(\epsilon)$, we need to have a time span for each I to $\Delta\tau \sim 2|\ln \epsilon|$ and the total length $L \sim 2\Delta\tau$. By choosing $\epsilon = 0.01$, we find that we need $\Delta\tau \sim 9.2$. In actual calculation we have chosen $L = 20$. As for (b), the Fourier analysis of the instanton (4.2) shows that the Fourier coefficient is $e^{-\pi k}$. Thus in order to reproduce the coefficients to $O(\epsilon')$, we need to satisfy

$$N > \frac{L}{2\pi^2} |\ln \epsilon'|. \quad (4.4)$$

Running Mathematica on a NEWS workstation, this constraint is easy to satisfy for $L = 20$. We have calculated the action in terms of ϕ_i for $N = 8$ analytically (it

contains 221 terms). Since our definition (3.1) is fourth order algebraic equation of 8 variables, it does not allow an analytical solution. Thus we followed the trajectory numerically, using the flow equation (3.14) starting from the vacuum. This calculation corresponds to accuracy of $\epsilon' = 3.2 \times 10^{-4}$.

The calculation is done by starting from the origin and continuing the trajectory in the direction (3.14) by finite steps $d\alpha = 0.01$. The shape of $\phi(\tau)$ at various points on the trajectory is shown in Fig.3. One can see that the lowest eigenvalue mode first grows to an $I\bar{I}$ -like configuration with half-width ~ 5 . It is worth mentioning that this width is independent of the value of L . (We have observed this for $L = 10$ and 40 .) This development of a definite width is one characteristic feature of $I\bar{I}$ deformation in our prescription. The action density of the same $\phi(\tau)$'s are shown in the Fig.4. From these, one sees that in fact the well-separated $I\bar{I}$ configuration has been obtained toward the end of the trajectory. The action along the trajectory is given in Fig.5. From this we confirm that I and \bar{I} are well separated at the end. For $\alpha = 3.1$, I and \bar{I} are almost at the far opposite ends of the τ -space. Thus if one continues along the trajectory further, one sees another nontrivial deformation of $I\bar{I}$. The action then begins to rise again.

If one traces the trajectory starting from the vacuum and in the $-\phi_1$ direction, one merely sees monotonically rising action. One has then two choices for integrating over the $\phi_1 < 0$ half-space: (a) apply the trajectory method for the $\phi_1 > 0$ space and do the ordinary perturbation in the $\phi_1 < 0$ space. (b) apply the trajectory method even for $\phi_1 < 0$ space. Since no nonperturbative configurations are expected for $\phi_1 < 0$, these two methods do not yield any differences.

For $L \rightarrow \infty$, there is an exact zero mode, that corresponds to motion of the center of I and \bar{I} . This mode however is anti-symmetric and did not appear in our calculation. This implies necessity for extension of our valley trajectory to higher dimension, which we shall discuss in the next chapter.

5. Discussion

In this paper we have proposed a new prescription (3.1) to choose the valley trajectory. It gives results different from the streamline method. It satisfies two variational principles, one for the gradient and the other for the tree-level partition function. We summarize the desirable features of the new trajectory below.

- It is a local definition. It needs no boundary conditions. Thus one does not have to follow the valley from far away region to obtain points on the trajectory.
- It removes the most dangerous eigenvalue from the Gaussian integral completely, making the one-loop result reliable.
- A case with an exact zero eigenvalue can be treated without any special precautions, while in order to apply the streamline method one has to define it as a limit of a model with a quasi zero-mode.
- The flow equation obtained from it has no instability near the vacuum. This allows us to trace the trajectory from the lower end, a classical vacuum.

We have demonstrated these features for simple models. For a double-well quantum mechanics we have successfully traced the trajectory starting from a classical vacuum and reaching an isolated $I\bar{I}$ configuration, as is seen in Fig.3~5.

In order to deal with the cases with multiple collective coordinates, we need to extend the valley method to multi-dimensional valley “membrane”. This is implied by the problem with c.m.s. coordinate as was mentioned in the previous chapter, as well as multi-pair configuration, $(I\bar{I})^n$ for $n > 1$. We can define a d -dimensional membrane so that at every point on it $\partial S/\partial\phi_i$ lies in the subspace spanned by the lowest d eigenvalues of D . In addition, The Fadeev-Popov constraint should also be modified to the following,

$$\prod_{p=1}^d \delta((\phi_i - \phi_i(\{\alpha\}))V_i^{(p)}(\{\alpha\})), \quad (5.1)$$

where $\{\alpha\} = \{\alpha_1, \dots, \alpha_d\}$, are the collective coordinates and $V_i^{(p)}$ is the p -th lowest eigenvector of D . This prescription divides the whole degrees of freedom in the path-integral into two categories; one is the fluctuation at a given point on the valley membrane and the other is the collective coordinates. Note that the direction of gradient is removed from the fluctuation space and thus Gaussian approximation can be done straightforwardly.

It should be noted that this definition leads to membrane without any boundary. At first sight, this seems strange, since the membrane should have natural boundary made from configurations with fewer collective coordinates. (By pair-annihilating I and \bar{I} , one can obtain $(I\bar{I})^{(n-1)}$.) This however is of no problem, since although the membrane would extend beyond the natural boundary the extra region is simply made of perturbative piece. This is exactly what happens if one chooses to include the trajectory in the $\phi_1 < 0$ region in the previous chapter. In the double-well model the $(I\bar{I})^2$ configuration would have the four-dimensional membrane, which converges toward the vacuum form the subspace spanned by ϕ_1 , ϕ_2 and the two directions of smallest eigenvalues in the asymmetric sector. Detailed treatment of this issue will be presented elsewhere.

There are other areas where our method could readily be applied. One of them is an instanton itself in the electroweak theory. In this theory, there is no finite-size isolated instanton solution. Any finite-size finite-action configuration is unstable against shrinkage to a point. In order to deal with this, Affleck introduced an idea of constrained instanton.^[19] For obtaining a finite-size object, he first introduces a constraint to keep it from shrinking and then integrates over the parameter in the constraint (*i.e.*, radius) to recover the whole functional space. Our new valley method should also be useful for defining the finite-size instanton, following the trajectory that winds down toward the point-like instanton.

ACKNOWLEDGEMENTS

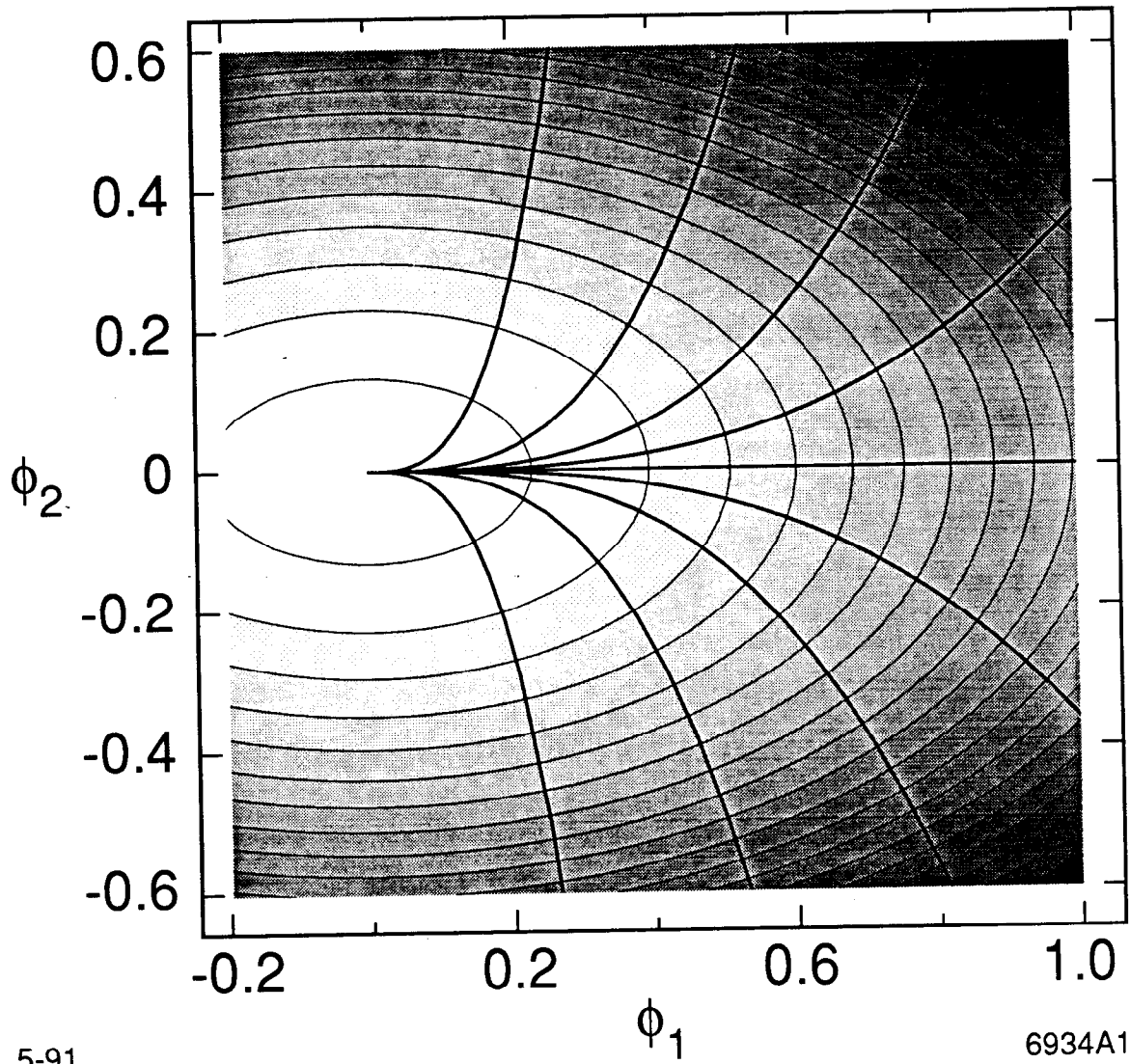
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FIGURE CAPTIONS

- 1) The gradient lines (2.10) near the origin for the free action (2.9). The contour lines correspond to constant values of the action; the darker the shading, the larger the action.
- 2) The streamline (dashed line) and our trajectory (solid line) for the action (3.21) with (3.27).
- 3) The shapes of $\phi(\tau)$ given by the new prescription (3.1). These are taken at points separated by distance $\delta\alpha = 0.31$ along the trajectory up to $\alpha = 3.1$.
- 4) The action density $S(\tau)$ of the $\phi(\tau)$ given in Fig.3.
- 5) The behavior of the action along the trajectory, divided by $2S^{(I)}$. The horizontal axis is the distance α along the trajectory defined by (3.13).

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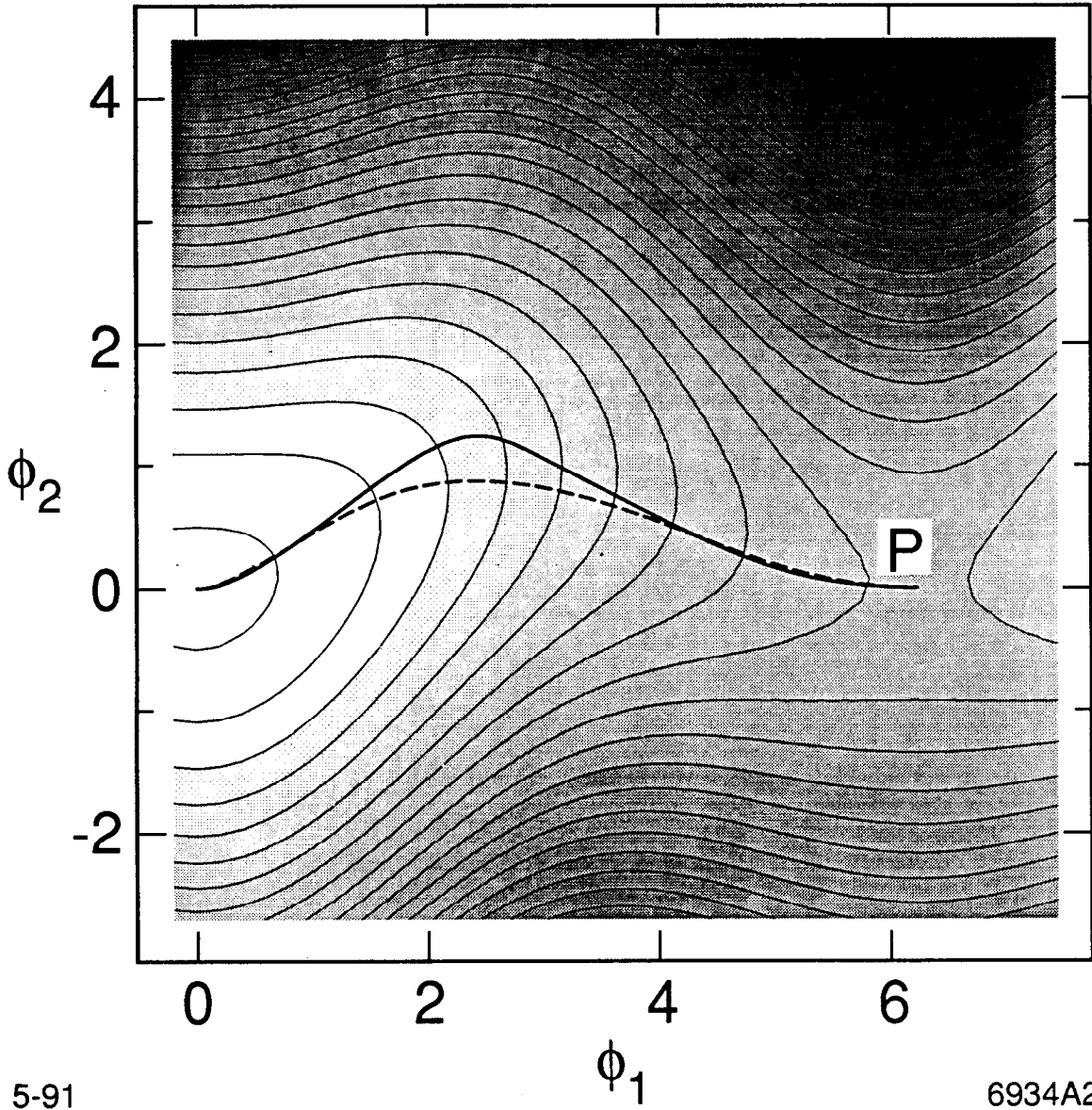
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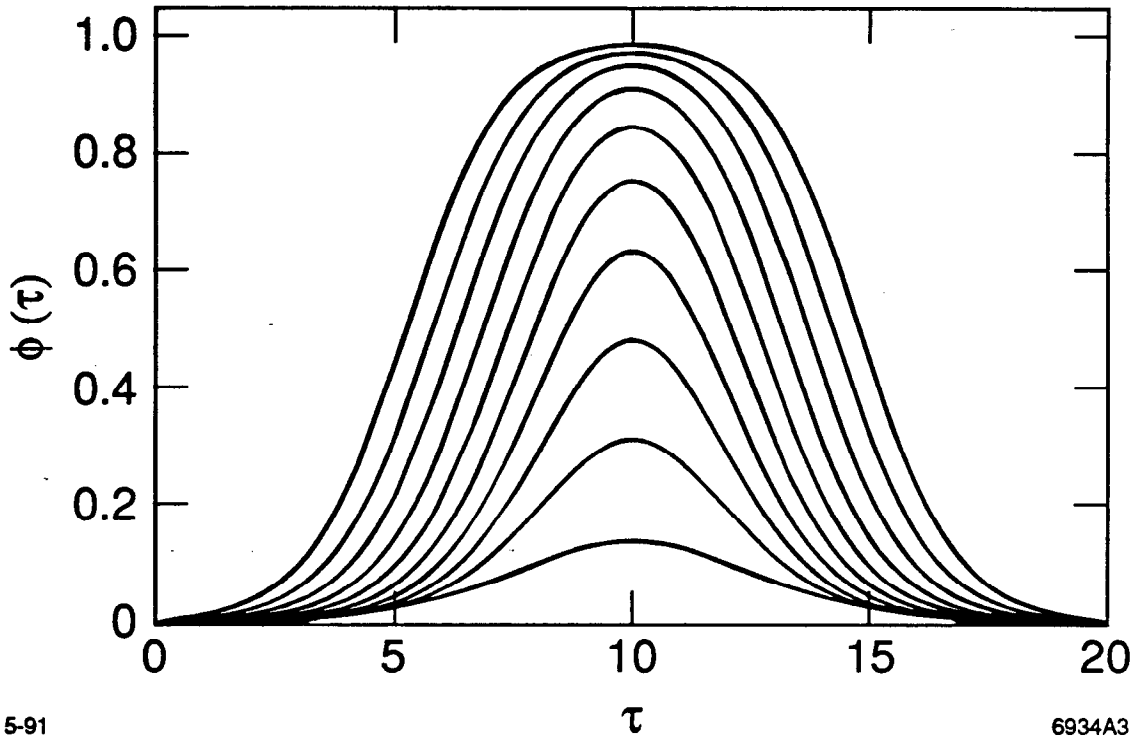
Fig. 1



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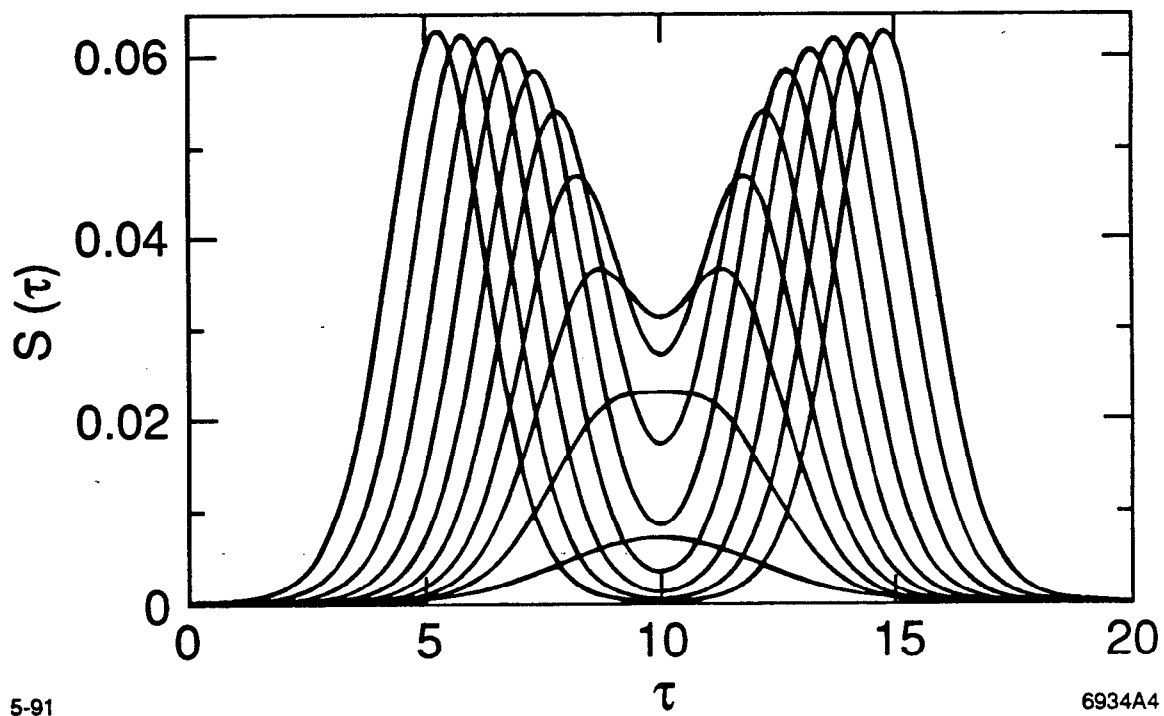
Fig. 2



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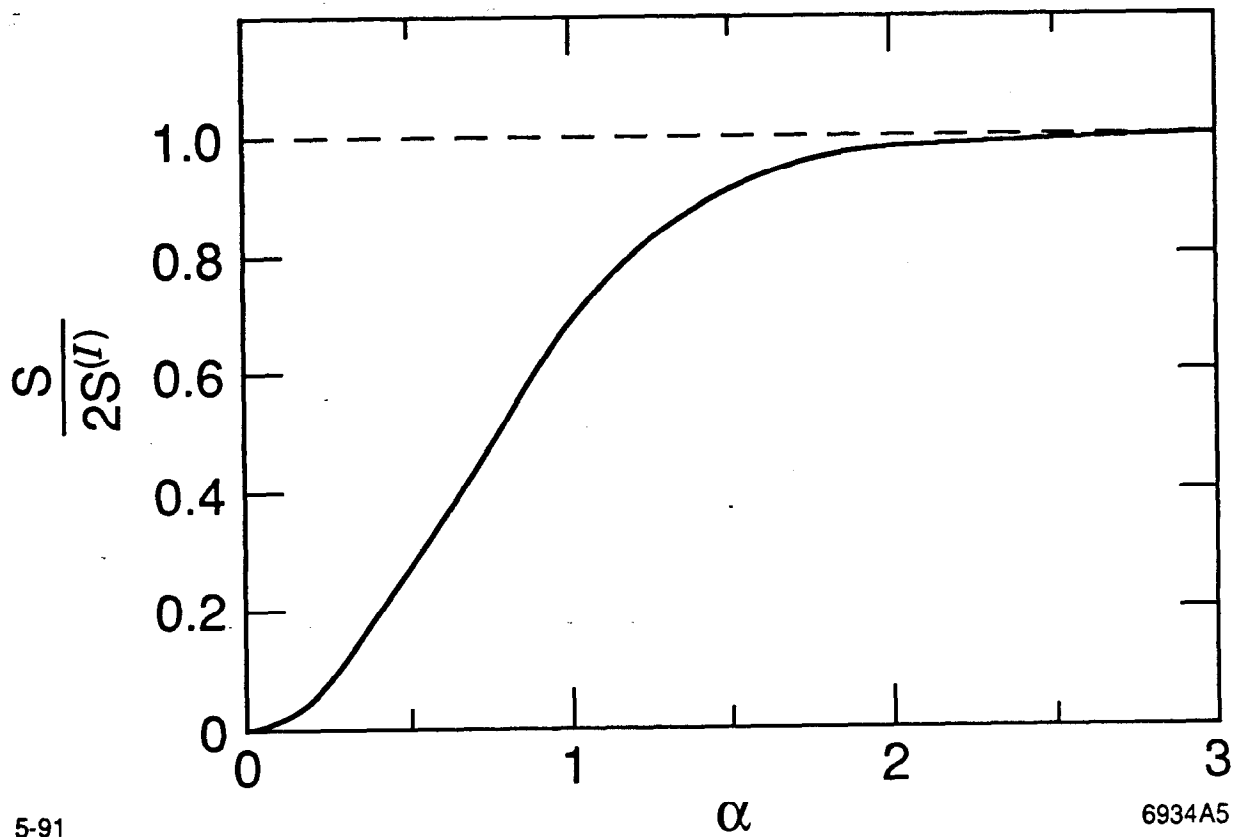
Fig. 3



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Fig. 4



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Fig. 5