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Dressed Skeleton Expansion and the Coupling Scale Ambiguity Problem*

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Abstract

We present the Dressed Skeleton Expansion (DSE) as a method of perturbative calculation in quantum field theories, without the scale ambiguity problem. We illustrate the application of the DSE method to the two-particle elastic scattering amplitude in ϕ_6^3 theory, and compare this method with the usual perturbative expansion, combined with scale setting prescriptions.

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Perturbative calculations in quantum field theories are usually expressed as a power series in a fixed coupling constant. At high transferred squared momentum, the fixed coupling constant must be replaced by a running coupling constant. This procedure is usually referred to as the renormalization-group-improved perturbation, which leads to the absorption of the large logarithmic terms into the running coupling constant. In simple words, given a truncated series of a physical quantity expanded in powers of a coupling constant in a given scheme:

$$R_n = \alpha^s(\mu)\ [r_0 + r_1(\mu)\ \alpha(\mu) + \ldots r_n(\mu)\ \alpha^n(\mu)]\ , \tag{1}$$

the coupling scale μ must be chosen appropriately for the perturbative series to be useful. The unknown dependence of the truncated series on μ is commonly referred as the coupling scale ambiguity problem. There is also another source of ambiguity in the perturbative expansion arising from the freedom in the choice of the renormalization scheme [1,2]. In our opinion, the freedom to select various renormalization schemes is no more than the freedom to adopt 'meter' or 'foot' as the basic unit of length. As long as a scheme is well defined, we can always agree on expressing the result in a particular scheme. (See, however, the next discussion about renormalization scheme invariant method.) Notice that in the process of translating results from one scheme to another—namely, replacing one coupling constant by another—unavoidably we re-encounter the problem of scale setting. More precisely, two coupling constants $\alpha_1(\mu)$ and $\alpha_2(\mu)$ of different schemes are related by an equation:

$$\alpha_1(\mu_1) = \alpha_2(\mu_2) + C_1(\mu_1/\mu_2)\ \alpha_2^2(\mu_2) + C_2(\mu_1/\mu_2)\ \alpha_2^3(\mu_2) + \ldots\ . \tag{2}$$

There is clearly a scale ambiguity problem: an appropriate value of μ_2 must be chosen for each value of μ_1. In a sense, the scale ambiguity is a more fundamental problem than the corresponding scheme ambiguity problem: once one has solved the scale ambiguity problem, there is no ambiguity in how to implement different schemes.

Several methods have been proposed to solve the coupling scale ambiguity. Among them we shall mention:

1. *Fastest Apparent Convergence (FAC) [1,3]:*

The idea behind FAC is that one should choose the coupling scale that makes the series look like most convergent. Operationally we will define this method as setting the contribution of the second order term (i.e., next to tree level) to be zero.

2. *Principle of Minimal Sensitivity (PMS)[1]:*

We define this method here as the choice of the coupling scale at the stationary point of the truncated series:

$$\left.\frac{dR}{d\mu}\right|_{\mu} = 0 \; . \tag{3}$$

The PMS method also aims toward the choice of a renormalization scheme. Beyond two-loop order, this method requires the variation of scheme parameters besides the coupling scale.

3. *Automatic Scale Fixing (BLM) [4]:*

This method is inspired by QED. The philosophy is to absorb all fermionic vacuum polarization effects into the running coupling constant. In 1-loop order massless QCD, it is operationally equivalent to the condition of a vanishing coefficient of the n_f (number of light fermions) term. Therefore BLM results are formally invariant under the change of number of light flavors:

$$\frac{\partial R}{\partial n_f}\,[\alpha(\mu), n_f] = 0\ . \tag{4}$$

4. Renormalization Scheme Invariant Calculation (RSI) [3,5]:

This is yet another point of view on the subject. Given a physical quantity, we can define an effective coupling (or effective charge) associated with it (which we shall call the R-scheme coupling constant):

$$\begin{aligned} R &= \alpha^s(\mu)\,[r_0 + r_1(\mu)\ \alpha(\mu) + \dots\,] \\ &\equiv r_0\ \alpha_R^s \end{aligned} \tag{5}$$

If R depends on a single external momentum p^2, then the evolution of $R(p^2)$—or equivalently $\alpha_R(p^2)$—on p^2 can be studied very nicely without the necessity of additional inputs, such as $\Lambda_{\rm QCD}$. This is usually claimed to be renormalization-scheme independent calculation, but one should bear in mind that implicitly one has prefered a particular scheme: the R-scheme. The R-scheme is, in a sense, a natural scheme for the study of the evolution properties of a given field theory, because the coupling constant itself in this case is experimentally measured, and hence there is no need for other exogenous coupling constants. But this is not the end of the story; we know that in massless QCD the bare coupling constant is the only parameter in the theory, so ideally we should be able to make one single measurement and predict all other results. For example, the total hadron decay width of heavy quarkonia possesses no lab controllable momentum (and thus no evolution to talk about); nevertheless, QCD should be able to predict this value. Another problem with the

RSI method is a proliferation of coupling constants: effectively, one coupling constant is introduced for each physical process. Further, the problem of scale ambiguity resurges when we want to relate one effective coupling to another.

The usual impression is that as long as the coupling scale μ^2 is chosen near the typical scale Q^2 of a given process, the perturbation series would give a reasonable result. We should notice, however, that due to dimensional transmutation (i.e., the presence of Λ_{QCD}) the correct scale might in some cases not be proportional to Q^2, but rather to some other power of Q^2, or in an even more complicated form. So the naive form of assigning coupling scale to typical physical scales runs the danger of being too simplistic. Also, for processes involving many scales, in general it is not clear how a "typical scale" can be defined.

For multi-scale processes, the usual way of assigning a uniform coupling throughout all the vertices becomes questionable. Consider for instance the exclusive process $e^+e^- \rightarrow \mu^+\mu^-\gamma$ (fig. 1). In QED the vertices **a** and **b** should have a coupling strength $\sim \alpha^{1/2}(Q^2)$, whereas the vertex involving the raidated photon should have a strength $\sim \alpha^{1/2}(0) = 1/\sqrt{137}$.

This observation and controversy on the various scale setting procedures induce us to use the Dressed Skeleton Expansion (DSE) [6], instead of the conventional power series expansion. To illustrate this calculation procedure, we shall consider ϕ^3 theory in six dimensions, which is infrared safe and asymptotically free. To avoid the extra complication coming from mass renormalization, let us assume that the physical mass is negligibly small (the meaning of this will become more precise later). The basic idea of skeleton type calculation is rather simple:

(1) The basic vertex functions are calculated by using renormalization group equation.

(2) Any other Green's function is expanded in skeleton graphs of the basic vertices.

One property of this calculational procedure is that it is automatically scale ambiguity free, because there is no exogenous coupling constant. This resembles BLM's observation of the automatic scale setting procedure for QED. Another observation is that results in DSE calculations are not a simple power series in a coupling constant. In general the results in DSE calculations are expressed directly in terms of functions that involve a scale analogous to Λ_{QCD}. This should not come as a surprise. In fact, the concept of coupling constant is also lost in conventional perturbation theory with scale fixing procedure. In QCD, after scale fixing, the results are directly expressed in term of Λ_{QCD}. In this sense, the coupling constant merely serves as an intermediate device and is discarded after scale fixing (analyze for example, eq. (20)).

As a simple example, let us apply this idea to two-particle elastic scattering amplitude in ϕ_6^3 theory. We shall perform our calculation within dimensional regularization [7]. In the following we use $d = 6 + 2\epsilon$ and λ_μ is the $\overline{\mathrm{MS}}$ scheme [8] dimensionless coupling constant:

$$\begin{aligned} \lambda_0 &= \mu^{-\epsilon} \left[\lambda_\mu + \frac{3}{8\hat{\epsilon}} \frac{\lambda_\mu^2}{(4\pi)^3} + \dots \right] . \\ \frac{1}{\hat{\epsilon}} &= \frac{1}{\epsilon} - \log(4\pi) + \gamma_E \end{aligned} \tag{6}$$

1. The propagator to 1-loop order (fig. 2a) in the massless limit is given by:

$$\begin{aligned} \Delta(p^2) &= Z(p^2)\,\frac{i}{p^2} \\ Z(p^2) &= 1 + \frac{\lambda_\mu^2}{12(4\pi)^3}\left[\frac{1}{\hat{\epsilon}} + \log\left(-\frac{p^2}{\mu^2} - i\epsilon\right) - \frac{8}{3}\right] \end{aligned} \quad . \tag{7}$$

On mass-shell, we can no longer neglect the mass of the particles, and the on-shell wave function renormalization can be shown to be:

$$Z_{\rm OS} = Z(m^2) = 1 + \frac{\lambda_\mu^2}{12(4\pi)^3}\left[\frac{1}{\hat{\epsilon}} + \log\left(\frac{m_{ph}^2}{\mu^2}\right) - \frac{5}{6}\right] . \tag{8}$$

This will be the only place where m_{ph} cannot be taken to be zero.

Notice that we have placed all renormalization effects into $Z(p^2)$ and $Z_{\rm OS}$. That is, the propagator retains its bare form. The effects of $Z(p^2)$ and $Z_{\rm OS}$ enter in the renormalization group equation for the 3-point function. In other words, there is no renormalization group equation for the two-point function [9]. In the massive case, the same idea applies, that is, the renormalized propagator is to be kept in the bare form, with the bare mass replaced by the physical mass, and all renormalization effects of the self-energy are to be absorbed into the wave function renormalization constant.

2. The unrenormalized three-point function with one off-shell leg (fig. 2a) in the massless limit to 1-loop order is given by:

$$\Gamma = -i\lambda_0\left\{1 - \frac{\lambda_\mu^2}{2(4\pi)^3}\left[\frac{1}{\hat{\epsilon}} + \log\left(-\frac{p^2}{\mu^2} - i\epsilon\right) - 3\right]\right\} . \tag{9}$$

The corresponding renormalized vertex function is:

$$\begin{aligned} \Gamma_R &= Z_{\rm OS} Z^{1/2}(p^2)\,\Gamma \equiv -i\lambda(p^2) \\ &= -i\lambda_0\left\{1 + \frac{\lambda_\mu^2}{24(4\pi)^3}\left[-\frac{9}{\hat{\epsilon}} + \frac{95}{3} + 2\log\left(\frac{m_{ph}^2}{\mu^2}\right) - 11\log\left(-\frac{p^2}{\mu^2} - i\epsilon\right)\right]\right\} . \end{aligned} \tag{10}$$

The renormalization group equation to this order is an algebraic equation that simply expresses the fact that λ_0 is unique:

$$\frac{1}{\lambda_0^2} = \frac{1}{\lambda^2(p^2)} + \frac{1}{12(4\pi)^3}\left[-\frac{9}{\hat{\epsilon}} + \frac{95}{3} + 2\ \log\left(\frac{m_{ph}^2}{\mu^2}\right) - 11\ \log\left(-\frac{p^2}{\mu^2} - i\epsilon\right)\right] .$$

Its solution is given by:

$$\lambda^2(p^2) = \frac{12(4\pi)^3}{11\ \log(-p^2/\Lambda_{\rm DS}^2 - i\epsilon)} , \tag{11}$$

where $\Lambda_{\rm DS}$ (DS=Dressed Skeleton) mimics the role of Λ_{QCD}, and is a quantity to be fixed by experiment. This concludes our renormalization program of the fundamental vertices.

The 2-particle scattering amplitude to tree skeleton level (fig. 2b), and to 1-loop renormalization in the fundamental vertices is given by:

$$\begin{aligned} iM_{\rm tree} &= [-i\lambda(s)]^2\ \frac{i}{s} + [-i\lambda(t)]^2\ \frac{i}{t} + [-i\lambda(u)]^2\ \frac{i}{u} \\ &= i\ \frac{12}{11}\ (4\pi)^3 \left[\frac{i}{s\ (\log|s/\Lambda_{\rm DS}^2| - i\pi)} + \frac{i}{t\ \log|t/\Lambda_{\rm DS}^2|} + \frac{i}{u\ \log|u/\Lambda_{\rm DS}^2|}\right] . \end{aligned} \tag{12}$$

The squared scattering amplitude in the DSE approach to the first skeleton loop is given by:

$$|M|^2 = |M_{\rm tree}|^2 . \tag{13}$$

Notice that in the DSE method, no scale setting procedure has to be employed.

We shall only comment about higher order skeleton diagrams. Diagrams like the box diagram in fig. 2c are to be calculated by inserting the renormalized vertex functions in the momentum integrals. Notice that the propagators in the graph keep their bare form. For the box diagram shown in fig. 2c, we have the expression:

$$iM_{box}^{(u)} = \int \frac{d^6k}{(2\pi)^6} \frac{\lambda(k_1^2,k_2^2)\ \lambda(k_2^2,k_3^2)\ \lambda(k_3^2,k_4^2)\ \lambda(k_4^2,k_1^2)}{(k_1^2+i\epsilon)\ (k_2^2+i\epsilon)\ (k_3^2+i\epsilon)\ (k_4^2+i\epsilon)}\ , \tag{14}$$

where $\lambda(p^2,q^2)$ is the 3-point vertex function with two off-shell legs, obtained in a similar fashion as $\lambda(p^2)$. In the massless limit ($|p^2|, |q^2| \gg m_{ph}^2$):

$$\lambda^2(p^2,q^2) = 12(4\pi)^3 \Bigg\{ 12\ \frac{p^2 \log\left(-p^2/\Lambda_{\rm DS}^2 - i\epsilon\right) - q^2 \log\left(-q^2/\Lambda_{\rm DS}^2 - i\epsilon\right)}{p^2-q^2} - \log\left(-\frac{p^2}{\Lambda_{\rm DS}^2} - i\epsilon\right) - \log\left(-\frac{q^2}{\Lambda_{\rm DS}^2} - i\epsilon\right)\Bigg\}^{-1} . \tag{15}$$

Notice that when $|p^2| \gg |q^2|$, we recover the one off-shell leg vertex; that is,

$$\lim_{|p^2| \gg |q^2|} \lambda(p^2,q^2) = \lambda(p^2)\ . \tag{16}$$

Observe that the expression of the box diagram eq. (14) contains no undetermined momentum scales. That is, higher order skeleton diagrams in general are also scale ambiguity free.

Let us now return to FAC and PMS methods of scale fixing in $\overline{\rm MS}$. The running coupling constant to 1-loop order is

$$\alpha_{\overline{\rm MS}}(\mu) = \frac{\lambda_\mu^2}{(4\pi)^3} = \frac{4}{3}\ \frac{1}{\log|\mu^2/\Lambda_{\overline{\rm MS}}|}\ . \tag{17}$$

The squared renormalized scattering amplitudes to 1-loop order (fig. 3) results in the expression [10]

$$|M|^2 = (4\pi)^6 \alpha_{\overline{\rm MS}}^2(\mu) \Big(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\Big)^2 \Big\{1+\alpha_{\overline{\rm MS}}(\mu)\Big[\frac{3}{2}\ \log\Big|\frac{\mu^2}{\Lambda_{\overline{\rm MS}}^2}\Big| - \frac{11}{6}\ {\rm H}(s,t,u,\tilde{\Lambda}^2)\Big]\Big\}\ , \tag{18}$$

where

$$\tilde{\Lambda} = \Lambda \ \exp\left\{ \frac{95}{66} + \frac{6}{11}\ \zeta(2) + \frac{1}{11}\ \log\left|\frac{m_{ph}^2}{\Lambda_{\overline{\mathrm{MS}}}^2}\right| \right\} ,$$

$$\begin{aligned} \mathrm{H}(s,t,u,\tilde{\Lambda}^2) = &\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^{-1} \\ &\left\{\frac{1}{s}\left(\log\left|\frac{s}{\tilde{\Lambda}^2}\right| - \frac{12}{11}\ \log^2\left|\frac{t}{u}\right| + \frac{12}{11}\ \log\left|\frac{t}{s}\right|\log\left|\frac{u}{s}\right|\right)\right. \\ &+\frac{1}{t}\left(\log\left|\frac{t}{\tilde{\Lambda}^2}\right| - \frac{12}{11}\ \log^2\left|\frac{u}{s}\right| + \frac{12}{11}\ \log\left|\frac{u}{t}\right|\log\left|\frac{s}{t}\right|\right) \\ &\left.+\frac{1}{u}\left(\log\left|\frac{u}{\tilde{\Lambda}^2}\right| - \frac{12}{11}\ \log^2\left|\frac{s}{t}\right| + \frac{12}{11}\ \log\left|\frac{s}{u}\right|\log\left|\frac{t}{u}\right|\right)\right\} \end{aligned} \qquad (19)$$

The $\alpha_{\overline{\mathrm{MS}}}^3$ term in eq. (18) comes from the interference of the 1-loop diagrams with the tree diagrams.

We now apply the usual scale setting prescriptions. To this order in $\overline{\mathrm{MS}}$, the two methods—FAC and PMS—predict the same result:

$$|M|^2_{PMS-FAC} = (4\pi)^6\ \left(\frac{12}{11}\right)^2 \left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^2 \frac{1}{\mathrm{H}^2(s,t,u,\tilde{\Lambda}^2)} \ . \qquad (20)$$

Moreover, it can be easily shown that RSI calculation (the effective charge method) leads also to the same result, provided that $\tilde{\Lambda}^2$ is measured at a physical point by through the formula (20). .

To compare the result of PMS-FAC with DSE, we need to know the relationship between $\tilde{\Lambda}$ and Λ_{DS}. Let us take Λ_{DS} as our unit of momentum: $\Lambda_{\mathrm{DS}} = 1$, and express all other momentums in unit of Λ_{DS}. We take the physical point $s = 2|t| = 2|u| = 10^6$ as the matching point. This leads to

$$\tilde{\Lambda} = 0.7167 \ . \qquad (21)$$

In fig. 4 we show the s dependence of $|M|^2_{DSE}$ and $|M|^2_{PMS-FAC}$ for the "symmetrical point" $s = 2|t| = 2|u|$, assuming that $\Lambda_{\mathrm{DS}} = 1$. In fig. 5 we show the

t dependence for a fixed value of s. We see from these figures that the DSE has no qualitative discrepancy with results obtained by usual scale setting procedures. The difference in the lower momentum region is within the expectation of higher order contributions [11].

To summarize, we have presented the Dressed Skeleton Expansion (DSE) as an alternative perturbative calculation method, which has the advantage of being scale ambiguity free and has the property of assigning different coupling strength at different vertices. We have illustrated the usage of the DSE with the two-particle scattering amplitude in massless ϕ_6^3 theory, and compared it with standard perturbative calculation with scale setting prescriptions. Evidently, one drawback of the DSE method is that calculation beyond the tree skeleton level becomes very complicated. However, for many-scale processes, the DSE method in tree skeleton level provides simple but concrete way of obtaining results without scale ambiguity problem. Applications of the DSE method to field theory models in $1+1$ dimension and to QCD are discussed in forthcoming papers.

We thank Prof. Stanley Brodsky for motivation of the scale setting problem and helpful discussions.

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[9] This should be contrasted with QED. In QED, due to the Ward identity $Z_1 = Z_2$, it is not necessary to use the full skeleton expansion. The only vertex function that needs to be renormalized is the photon propagator multiplied by the squared bare charge (see ref. 4).

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[11] A numerical analysis reveals that the discrepancy in fig. 5 away from the symmetrical point is mainly due to the omission of the box skeleton diagrams. However, the good agreement of the overall scale dependence (fig. 4) and the observable difference in the relative scale dependence (fig. 5) hints that multi-scale processes, rather than one-scale processes, are the true test ground for the various scale setting methods. For a recent experimental comparison of the performance of the different scale setting methods in a multi-scale process, see e.g., G. Kramer and B. Lampe, CERN–TH–5810/90 or Nucl. Phys. (Proc. Suppl.) **B16** (1990) 254.

Figure captions

Fig. 1. A typical QED process, where the coupling strength at vertices **a** and **b** is expected to be stronger than the coupling strength at **c**.

Fig. 2. Diagrams involved in the skeleton calculation of two particle scattering amplitude.

Fig. 3. Diagrams involved in the usual perturbative calculation.

Fig. 4. The s dependence of the probability amplitude along the "symmetric" line $s = -2t = -2u$.

Fig. 5. The t dependence of the probability amplitude of a fixed value of $s = 10^6$.

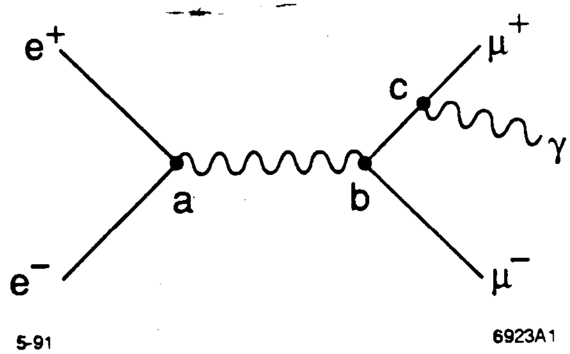

Fig. 1

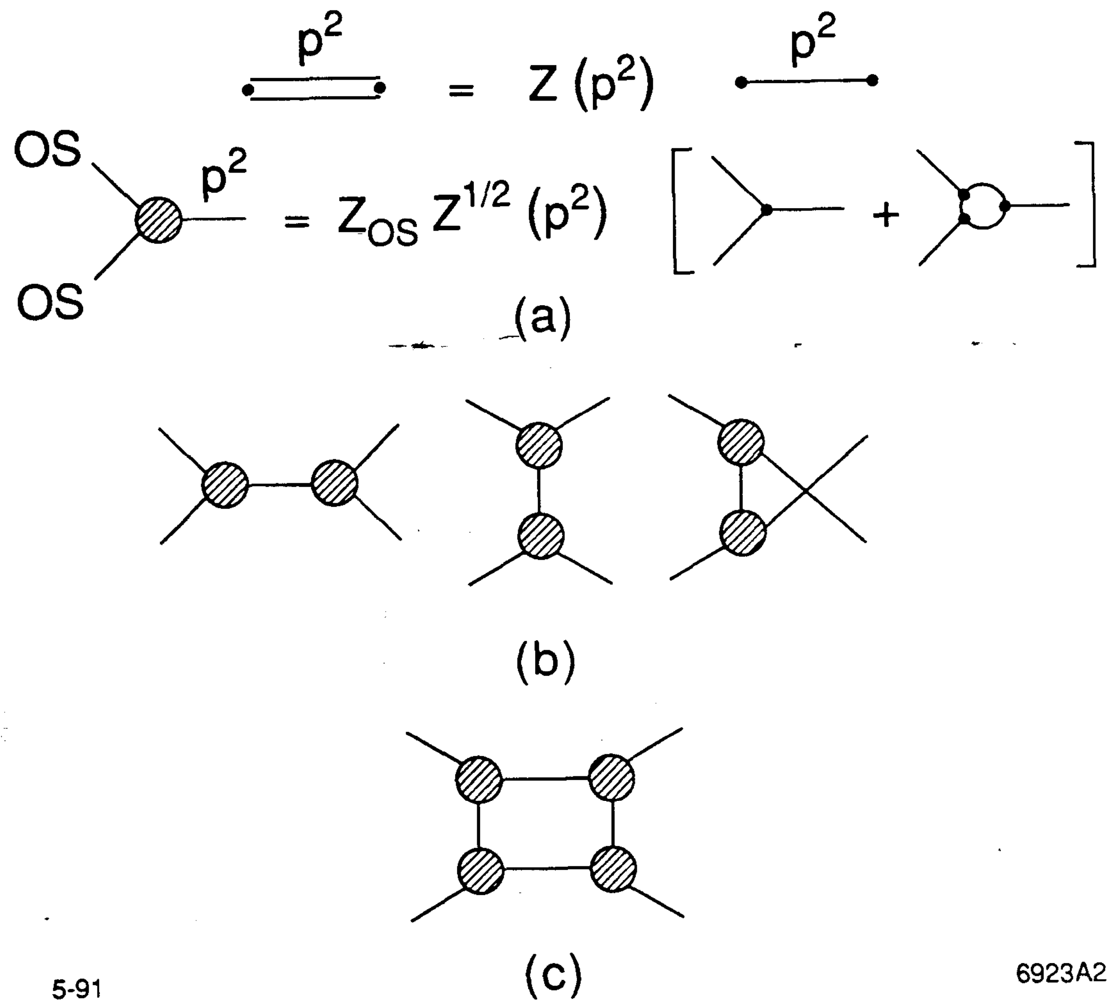

Fig. 2

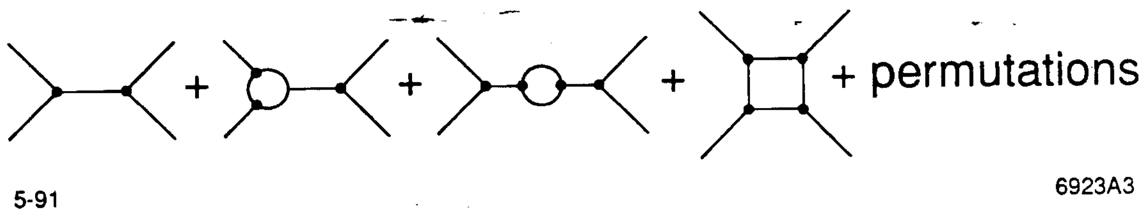

Fig. 3

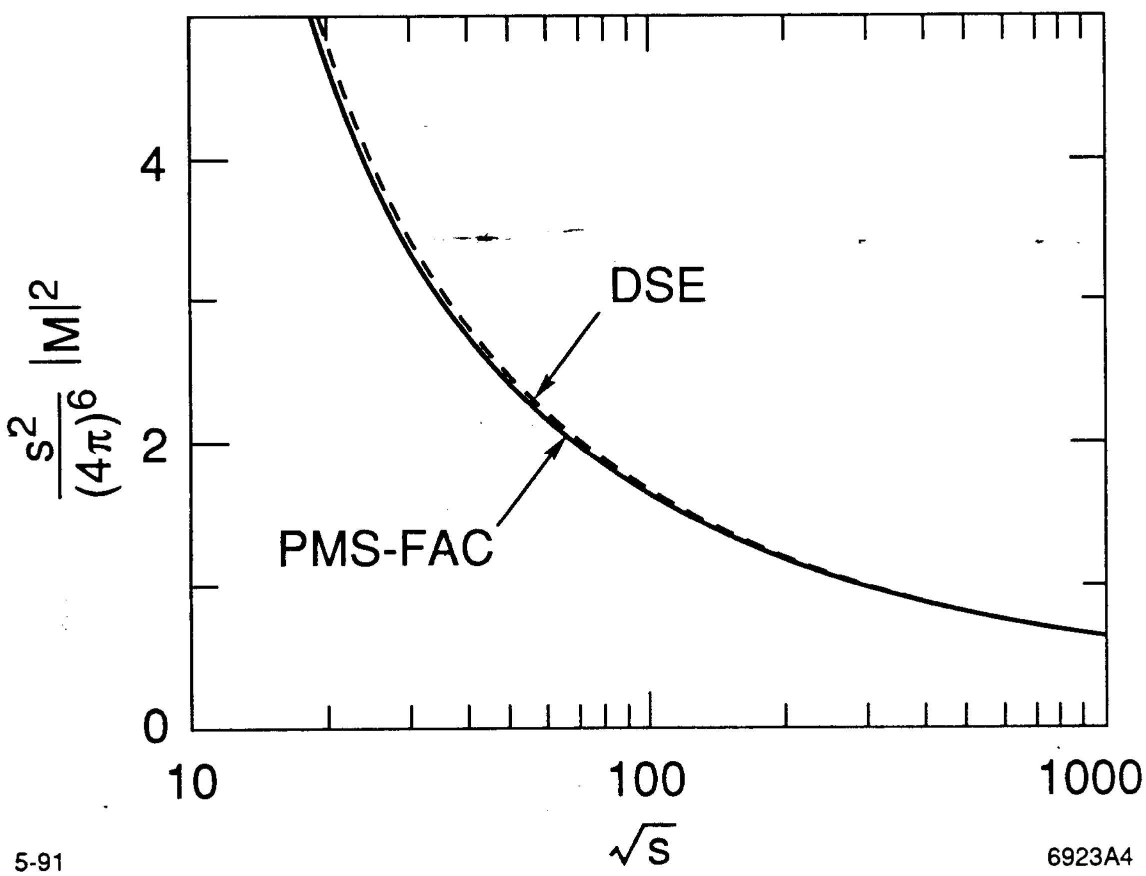

Fig. 4

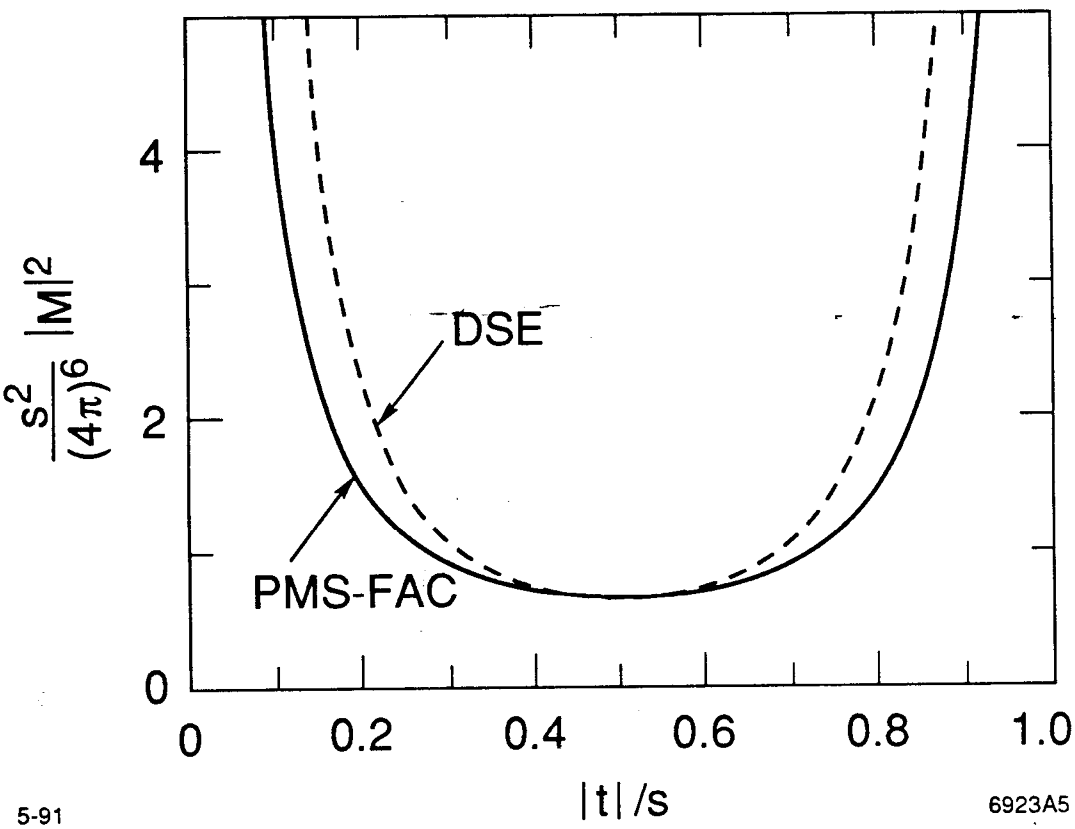

Fig. 5