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A 2d Field Solver for Periodic Structures with Special Corner Elements*

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Abstract

A 2d finite element field solver has been written which allows quasi-periodic boundary conditions, making it ideal for calculating travelling waves in periodic structures. Special elements are used at corners for improved accuracy. Comparisons with URMEL[1], URMEL-T[2], SUPERFISH[3] and analytic solutions are made, showing that this code yields better eigenvalues than the URMELs despite the use of a coarser mesh.

I. Introduction

YAP (an acronym for Yet Another Program) is a 2d finite element field solver capable of finding TE and TM modes in planar structures and monopole (TM_0 and TE_0) modes in axisymmetric structures. The structures can include symmetry and periodic boundaries. YAP's algebraic eigenvalue solver uses the inverse power method with an eigenvalue shift, which yields the mode with eigenvalue closest to a specified target eigenvalue. A typical problem uses up to 1000 nodes, consuming about 1 minute of IBM 3091 CPU time.

Readers not interested in the details of the finite element method (section II) or the special corner elements (section IV A) can skip these sections and still understand the rest of the paper.

II. Finite Element Formulation

The finite element formulation follows closely the formulation used by the PRUD-W[4] code. One case is presented here: the TE mode of a planar periodic structure.

Let the field B_z be complex and have an assumed time dependence $e^{-i\omega t}$, and let the interior of the structure be Ω (in the (x,y) plane) with three boundaries: Γ_{metal}, Γ_{right} and Γ_{left}. The latter two boundaries are the right and left boundaries of one cell of the periodic structure, and are connected by the rigid motion $R : \Gamma_{\text{left}} \rightarrow \Gamma_{\text{right}}$. This rigid motion may include rotation as well as translation. The fields in the periodic structure are decomposed into modes with phase advance ϕ, in accordance with Floquet's theorem. The phase advance can be any real number.

A. Strong Formulation

The usual statement of Maxwell's equations is the strong formulation: given the phase advance ϕ, find the eigenvalues ω^2/c^2 and eigenmodes B_z such that

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) B_z = 0 \quad \text{in } \Omega, \tag{1a}$$

$$\hat{n} \cdot \nabla B_z = 0 \quad \text{on } \Gamma_{\text{metal}}, \tag{1b}$$

$$B_z(R\vec{x}) = B_z(\vec{x})e^{i\phi} \quad \forall \vec{x} \in \Gamma_{\text{left}}, \tag{1c}$$

$$\hat{n} \cdot \nabla B_z(R\vec{x}) = -\hat{n} \cdot \nabla B_z(\vec{x})e^{i\phi} \quad \forall \vec{x} \in \Gamma_{\text{left}}. \tag{1d}$$

Equations (1c) and (1d) are the quasi-periodic boundary condition.

B. Weak Formulation

An equivalent statement of the problem is the weak formulation: given the phase advance ϕ, find the eigenvalues ω^2/c^2 and eigenmodes $B_z \in \mathcal{V}$ such that $\forall v \in \mathcal{V}$

$$\int_\Omega \nabla v^* \cdot \nabla B_z - \frac{\omega^2}{c^2} v^* B_z \, d\Omega = 0 \tag{2a}$$

where

$$\mathcal{V} = \left\{ v \in H^1(\Omega) : v(R\vec{x}) = v(\vec{x})e^{i\phi} \quad \forall \vec{x} \in \Gamma_{\text{left}} \right\} \tag{2b}$$

and $H^1(\Omega)$ is a complex Hilbert space. The weak formulation is obtained from the strong formulation by multiplying equation (1a) by v^* and integrating by parts.

C. Galerkin and Finite Element Formulations

The Galerkin formulation restricts v and B_z to a finite dimensional subspace $\mathcal{V}^h \subset \mathcal{V}$. This formulation reduces to an algebraic eigenvalue problem which is solved to yield approximate eigenvalues and eigenmodes.

The finite element formulation is the Galerkin formulation with a particular choice of $\mathcal{V}^h$. The domain Ω is partitioned into elements Ω_e, and $\mathcal{V}^h$ is chosen to be a space of functions which are piecewise simple (e.g. linear or quadratic on each element) and continuous. Figure 1(a) is an example of a partition. The choice of $\mathcal{V}^h$ is where YAP and PRUD-W diverge. YAP uses 6-node quadratic lagrange-type triangular elements and special corner elements while PRUD-W uses 8-node curvilinear quadrilateral elements.

In YAP, the fields are quadratic on the master element $\widehat{\Omega}$ shown in figure 1(b), where each node represents a separate basis function. These fields are mapped to the actual element Ω_e using a quadratic transformation $T_e : \widehat{\Omega} \rightarrow \Omega_e$. This allows the sides of an element to be curved in order to closely follow the boundary of Ω, as seen in elements Ω_1, Ω_3 and Ω_7 of figure 1(a).

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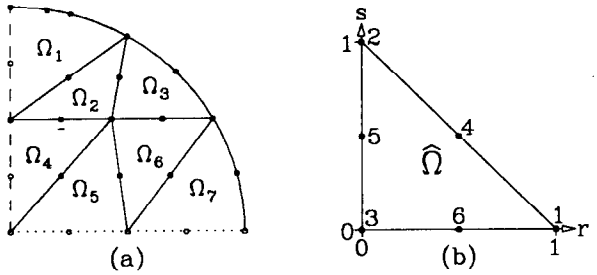

Figure 1. (a) A sphere partitioned into elements. The dotted line is the axis and the dashed line is a symmetry plane. (b) The master element $\hat{\Omega}$ and its 6 nodes.

III. Analytic Tests

YAP was tested on some analytically solvable structures. Results for two test structures are shown in figure 2. The cutoff TM_{01} mode of a 1 cm length of circular waveguide with 1 cm radius was calculated with YAP and URMEL. The analytic eigenvalue is $\omega^2/c^2 = 5.783185964/\rho^2$, where ρ is the radius of the waveguide. Both programs converged smoothly as the mesh was refined. This allows extrapolation to an infinitely refined mesh, which yields an eigenvalue with significantly reduced error.

The lowest TM_{010} mode of a sphere with 1 cm radius was calculated with YAP, URMEL, URMEL-T and SUPERFISH. The analytic eigenvalue is $\omega^2/c^2 = 7.527929583/r^2$, where r is the radius of the sphere. URMEL and URMEL-T do not converge smoothly as the mesh is refined, making extrapolation difficult. SUPERFISH and YAP converge smoothly.

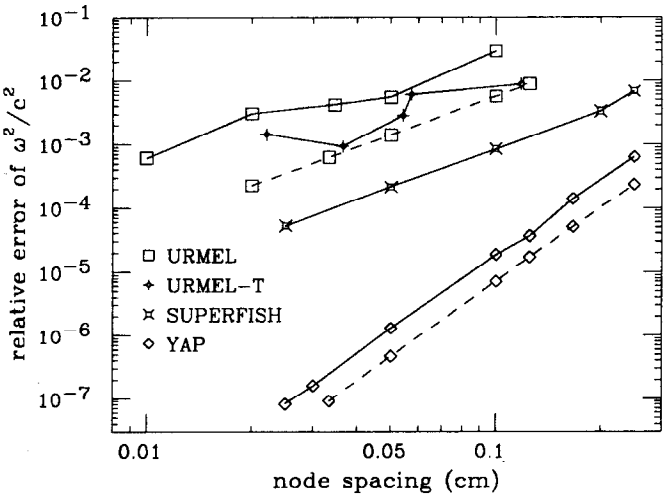

Figure 2. Relative error versus mesh size. The solid lines are tests on a sphere ($r = 1.0$ cm) and the dashed lines are tests on a circular waveguide ($\rho = 1.0$ cm).

The eigenvalues returned by YAP are good, even for coarse meshes. The coarsest sphere test used the mesh of figure 1(a) consisting of only 7 elements and gives an eigenvalue with less than 0.1% error. This is much better than SUPERFISH or the URMELs using a coarse mesh. Also, YAP converges faster as the mesh is refined. For these tests the error dependence on element size, h, is $\mathcal{O}(h^4)$ for YAP, while for SUPERFISH and the URMELs it is $\mathcal{O}(h^2)$. This is the expected behavior, but unfortunately it deteriorates when the structure has sharp corners.

IV. Corners

Consider a sharp corner as shown in figure 3(a). The field w (for example, B_z in a planar TE mode) near the corner has the form

$$w(\rho,\phi) = a_0 + a_1\cos(\pi\phi/\beta)\rho^{\pi/\beta} + a_2\cos(2\pi\phi/\beta)\rho^{2\pi/\beta} + \cdots \tag{3}$$

where ρ is the distance from the corner, β is the corner angle and ϕ is the angle, $0 \le \phi \le \beta$. Some fields may use sin instead of cos and have $a_0 = 0$ in order to satisfy boundary conditions at the metal wall. In either case, the field is not approximated well by the quadratic elements, so a special 7-node corner element[5] is used instead.

A. Details of the Special Corner Element

Figure 3 contains a diagram of the special corner element. Node 3 is the corner node. The basis functions on the master corner element $\hat{\Omega}$ are:

$$\begin{aligned}
N_1 &= u^\alpha(2^\alpha u^\alpha - 1)(1-v)(1-2v)/(2^\alpha - 1)\\
N_2 &= u^\alpha(2^\alpha u^\alpha - 1)(2v-1)v/(2^\alpha - 1)\\
N_3 &= (1 - 2^\alpha u^\alpha)(1-u^\alpha)\\
N_4 &= u^\alpha(2^\alpha u^\alpha - 1)4v(1-v)/(2^\alpha - 1)\\
N_5 &= 4^\alpha u^\alpha(1-u^\alpha)v(2v-1)/(2^\alpha - 1)\\
N_6 &= 4^\alpha u^\alpha(1-u^\alpha)(1-v)(1-2v)/(2^\alpha - 1)\\
N_7 &= 4^\alpha u^\alpha(1-u^\alpha)4v(1-v)/(2^\alpha - 1)
\end{aligned} \tag{4}$$

where $\alpha = \pi/\beta$ and

$$\begin{aligned}
u &= r + s & r &= u(1-v)\\
v &= s/(r+s) & s &= uv.
\end{aligned} \tag{5}$$

The transformation $T_e : \hat{\Omega} \rightarrow \Omega_e$ is the standard quadratic one. The global basis functions are continuous provided the elements are properly assembled.

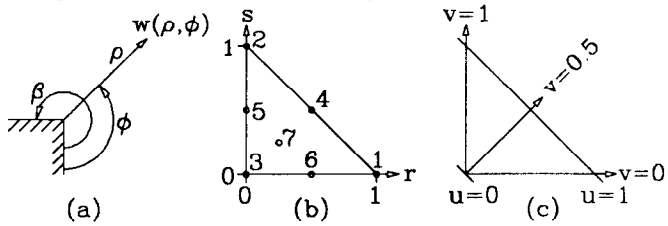

Figure 3. A corner (a), and the nodes (b) and (u,v) coordinates (c) of the master corner element $\hat{\Omega}$.

B. Tests on a Ridged Waveguide

Figure 4 shows test results of URMEL and YAP on the lowest TE mode of a ridged waveguide. The waveguide dimensions are 1 cm by 0.5 cm with a 0.5 cm by 0.25 cm ridge. Extrapolation to an infinitely refined mesh yields $\omega^2/c^2 = 5.06016\ \mathrm{cm}^{-2}$.

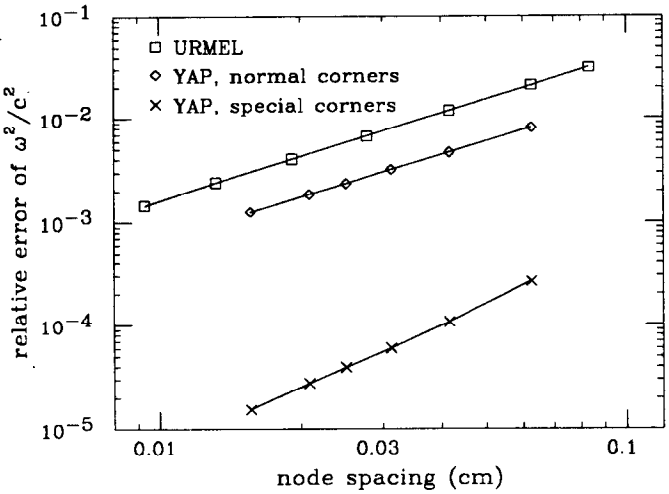

Figure 4. Relative error versus mesh size for a ridged waveguide.

For this structure URMEL and YAP without special corners are similar. The error dependence on the mesh size is $\mathcal{O}(h^{4/3})$ for both programs. This agrees with the theoretical eigenvalue error, $\mathcal{O}(h^{2\pi/\beta})$, where $\beta > \pi$ is the sharpest corner in the structure. Without special corner elements, YAP's error is only a factor of two better than URMEL for a given mesh size.

The special corner elements give YAP significantly improved accuracy and improve the error dependence to $\mathcal{O}(h^2)$, but this is still short of the behavior seen in smooth structures.

V. Quasi-Periodic Boundary Conditions

The quasi-periodic boundary conditions are helpful and sometimes necessary when modelling periodic structures. Only a single period of the structure is modelled, yielding the travelling wave solution for any desired phase advance. If only metal and symmetry boundaries are available, many periods of the structure must be modelled in order to obtain an equivalent standing wave solution for only a few phase advances. Furthermore, only symmetric periodic structures can be modelled this way. No such restriction applies when quasi-periodic boundary conditions are used.

Figure 5 is an example of the use of quasi-periodic boundaries to generate a dispersion curve. Generating the same data using metal and symmetry boundary conditions would require modelling at least six periods and consume more computer time or yield less accurate results. Identifying the phase advance of each mode would also be required.

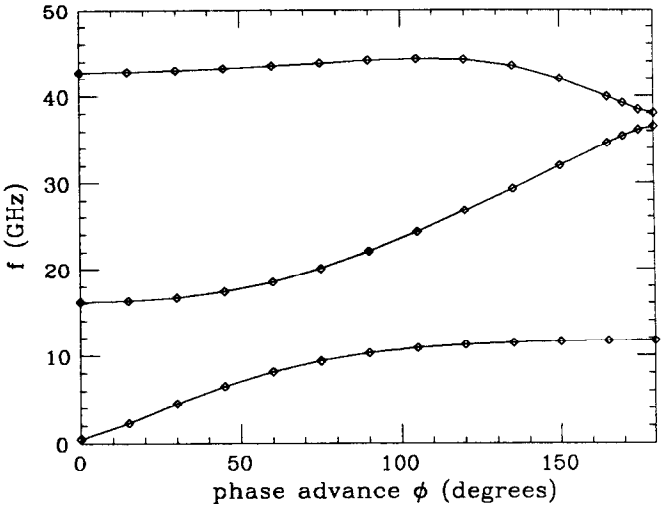

Figure 5. Dispersion curves obtained with quasi-periodic boundaries on a single period of a cross-field amplifier structure.

VI. Conclusion

The finite element method employed by YAP is more accurate than URMEL, URMEL-T and SUPERFISH.

The special corner elements help significantly for structures with sharp corners, but their performance does not match the performance seen with smooth structures. This indicates that the computational effort spent on corner regions is still inadequate compared to other regions where the field is smooth. This can be remedied by either an improved special corner element or a technique like adaptive mesh refinement.

The quasi-periodic boundary condition is an important part of a field solver because it allows asymmetric periodic structures to be modelled and saves time and effort for both the computer and the user.

I would like to thank Juwen Wang for running SUPERFISH tests on a sphere.

VII. References

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