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Non-harmonic Gauge Coupling Constants in Supersymmetry and Superstring Theory*

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ABSTRACT

Recent developments in understanding non-harmonic gauge coupling constants in supersymmetry and superstring theory are summarized.

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INTRODUCTION

The gauge coupling constant in $N=1$ supersymmetric field theories arises as a chiral integral over the supersymmetric field strength $W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$ (V being the vector superfield) in the following way:[1]

$$\tfrac{1}{4}\sum_a \int d^2\theta f_a(\phi)(W^\alpha W_\alpha)_a + \text{h.c.} = -\tfrac{1}{4}\sum_a (\text{Re}f_a (F_{\mu\nu}F_{\mu\nu})_a - \text{Im}f_a (F\tilde{F})_a + \ldots)\ . \tag{1}$$

The index a labels different factors in the gauge group $G = \prod_a G_a$. f_a is an arbitrary holomorphic function of the chiral superfields in the theory.† Eq. (1) identifies $\text{Re}f_a$ as the field-dependent gauge couplings and $\text{Im}f_a$ as the θ-angle:

$$f_a(\phi) = \frac{1}{g_a^2(\phi)} - \frac{i\theta_a(\phi)}{8\pi^2}\ . \tag{2}$$

One loop corrections of the gauge coupling constant are of the generic form

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2}{g^2_{\text{GUT}}} + b_a \ \ln \frac{M^2_{\text{GUT}}}{\mu^2} + \Delta_a \tag{3}$$

where b_a is related to the one loop β-function via $\beta_a = b_a\, g_a^3/16\pi^2$. Δ_a are the infrared finite one loop threshold corrections which generally arise from integrating out the massive modes of the theory.‡ A field-dependent Δ_a can be induced for example by field-dependent mass terms. From eqs. (1) and (2) one might expect Δ_a to be the real part of a holomorphic function (harmonic) in a supersymmetric theory. However, non-harmonic threshold corrections were found in orbifold vacua of the heterotic string and in a particular class of renormalizable $N=1$ supersymmetric field theories.[2] In this talk we summarize the results of ref. 2 and report

† Here we restrict our attention to gauge neutral functions f_a. The most general f could transform in the adjoint representation of the gauge group.[1]

‡ As we will see later this statement is not completely correct. Massless fields can also contribute to Δ_a.

on some of our more recent work.[3,4] We show that non-harmonic gauge couplings can appear if massless particles are in the spectrum and induce non-local terms in the effective action.

Recently, non-harmonic gauge couplings were also discussed by Shifman and Vainshtein who show that they arise generically at the two loop level due to wave function renormalization.[5] Derendinger, Ferrara, Kounnas and Zwirner reanalyzed eq. (1) and (2) for the case that a linear multiplet is present in the superfield spectrum and found that non-harmonic gauge couplings appear naturally in this setting.[6] (The results of this paper are summarized in J. P. Derendingers talk at this conference.) It was further noted that sigma model anomalies are intimately related to the phenomenon.[7,6]

Related topics were discussed at this conference by I. Antoniadis, M. Cvetič, M. K. Gaillard and T. Taylor.

NON-HARMONIC GAUGE COUPLINGS IN $N=1$ SUPERSYMMETRIC FIELD THEORIES

Let us first discuss non-harmonic gauge couplings in $N=1$ supersymmetric field theories. In these theories the Lagrangian is specified by the Kähler potential K, the superpotential W and the gauge function f which is defined in eq.(1).[1] We focus on the following generic model (in flat space):

$$\begin{aligned} K &= \sum_i \phi^i e^V \bar{\phi}^{\bar{i}} + \sum_I T^I \bar{T}^{\bar{I}} \\ W &= \tfrac{1}{2} M_{ij}(T) \phi^i \phi^j \\ f &= 1 \end{aligned} \tag{4}$$

where ϕ^i denote the charged matter fields (we suppress the gauge index of ϕ) and T^I are gauge neutral scalar fields (moduli). Supersymmetry requires M to be a holomorphic function of T. Let us first assume that all matter fields ϕ^i are massive and calculate the T dependence of f induced by one loop corrections. To first order

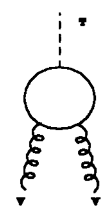

Figure 1.

in T one needs to calculate the quantity $h_I(\langle T^J \rangle) \int d^2\theta T^I W^\alpha W_\alpha$ which arises via the following supergraph

The coupling at the top vertex is proportional to $\partial M / \partial \langle T^I \rangle$ which results in

$$h_I \sim Tr\left(c_R \frac{\partial M}{\partial \langle T^I \rangle} M^\dagger \frac{1}{MM^\dagger + O(p^2)}\right) \tag{5}$$

where p^2 is some off-shell momentum but the precise form of the $O(p^2)$ is irrelevant. (We have also normalized the gauge group generators T^a according to $Tr_R T^a T^b = c_R \delta^{ab}$ for the representation R.) Since all ϕ^i's are massive M has no zero eigenvalue and $MM^\dagger$ is invertible. Therefore we can take the limit $p \to 0$ and obtain

$$\begin{aligned} h_I &\sim Tr\left(c_R \frac{\partial M}{\partial \langle T^I \rangle} M^{-1}\right) = \frac{\partial f(\langle T \rangle)}{\partial \langle T^I \rangle} \\ f &= \sum_R c_R \ \ln \det M_R(\langle T \rangle) \end{aligned} \tag{6}$$

in agreement with eqs. (1) and (2). (M_R is the mass matrix of the superfields in representation R.)

If instead some of the charged matter fields are massless M has a subspace of zero eigenvalues and consequently $MM^\dagger$ is not invertible. In this case eq. (6) has no reason to hold and non-harmonic gauge couplings can appear. As an example of such a situation consider $\phi^{1,2}$ to be $\mathbf{27}$'s of E_6 and ϕ^3 a $\overline{\mathbf{27}}$ with a mass matrix of the form*

$$M_{ij} = \begin{pmatrix} 0 & 0 & T^1 \\ 0 & 0 & T^2 \\ T^1 & T^2 & 0 \end{pmatrix}. \tag{7}$$

The zero eigenvalue of M corresponds to one of the **27**'s remaining massless but which linear combination of $\phi^{1,2}$ is the massless mode depends on T^I. Evaluating (5) for the mass matrix (7) leads to

$$h_I \sim \frac{\langle \bar{T}^I \rangle}{|\langle T^1 \rangle|^2 + |\langle T^2 \rangle|^2} \tag{8}$$

which is not the derivative of any holomorphic function. This example clearly shows that h_I is nonholomorphic due to the massless mode in the spectrum.† Similarly one can calculate $\bar{h}_{\bar{I}}$ and infer the gauge coupling constant and the θ-angle in the component expansion. One finds that $1/g^2$ is perfectly well defined but non-harmonic whereas θ is ill-defined. (See ref. 2 for details.)

One should also note that the calculation of h_I was performed in superfields. Thus its nonholomorphicity does not indicate an anomaly in supersymmetry; at every step in the calculation supersymmetry was manifest.

As always when massless particles are present one needs to make a distinction between two different kinds of effective actions. First, there is the generator of 1 light particle irreducible Feynman diagrams often denoted by S_Γ which can be nonlocal if massless particles are in the theory. Its gauge coupling constant g_Γ is the physical coupling constant. Second, one can define the Wilson action $S_W(\mu)$ which only incorporates the massive modes and the high momentum modes of the massless states and therefore includes a cutoff μ. S_W by definition excludes

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⋆ E_6 is chosen as the gauge group to ensure gauge anomaly cancellation.

† The reverse of this statement is not true. Massless particles do not lead necessarily to nonholomorphic h_I. For example eq. (8) becomes holomorphic for $T^2 = 0$ (say). The important point is that if the separation between massless and massive modes is field-dependent a nonholomorphic h_I appears.

infrared divergences and thus is local. The difference between the two actions are graphs of momentum $p < \mu$. (In the context of supersymmetry the exact definition of S_Γ and S_W and their properties are discussed in ref. 8.)

The above calculation of h_I evaluates g_Γ and θ_Γ. Its nonholomorphicity arises precisely in the case where massless modes are present and contribute in the loop of graph 1. This suggests that the h_I of the Wilson action does not suffer from this problem and is holomorphic. We will show that this is indeed the case by recalculating eq. (8) in such a way that the separation between massless and massive modes is manifest.[3] This is achieved by a holomorphic but T dependent field redefinition of the charged fields. Let $\phi^i = S^i_j(T)\,\rho^j$ such that

$$S^T M S = \begin{pmatrix} m(T) & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

where $m(T)$ is the mass matrix of the massive subspace. This field redefinition introduces a nontrivial metric for the charged fields

$$K = g_{i\bar{j}}(T,\bar{T})\rho^i e^V \bar{\rho}^{\bar{j}} + \dots \,. \tag{10}$$

In these coordinates only massive modes run in the loop of graph 1 and in the coupling at the top vertex $M(T)$ is replaced by $m(T)$. From eq. (6) we learn that the contribution to h_I from this graph is now holomorphic. Instead, the non-holomorphicity of h_I reappears through the following supergraph

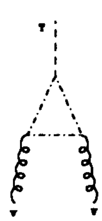

Figure 2.

Only massless modes run in the loop (indicated by the dotted line) and the coupling at the top of the triangle is proportional to the Christoffel connection Γ of the metric g defined in eq. (10). ($\Gamma^k_{Ij} = g^{k\bar{l}} g_{\bar{l}j,I}$). We find the contribution of graph 2 to h_I to be

$$h_I \sim tr\Gamma_I \tag{11}$$

where the trace runs over the massless subsector of the connection Γ. In evaluating the graph 2 we have used a regulator which preserves the gauge symmetry. The graph corresponds to an anomalous contribution of a mixed anomaly between the gauge group and the auxiliary connection Γ.[9] In which current the mixed anomaly appears depends on the regulator chosen. Here we have chosen a regulator which manifestly preserves the gauge symmetry.

It is straightforward to show that for the example (7) eq. (8) is exactly recovered by eq. (11). Furthermore, one can show generically that the appearance of a non-holomorphic h_I can always be understood via eqs. (9)–(11).[3] This identifies the appearance of non-harmonic gauge couplings as an infrared phenomenon. The graph 2 can also be evaluated with the general metric coupling of eq. (10) as the external leg. This introduces a non-local term in S_Γ which is of the form[10]

$$\mathcal{L}_{\rm nl} = \tfrac{1}{4}\tfrac{1}{16\pi^2} \sum_R c_R \int d^4\theta W^\alpha W_\alpha \, \frac{1}{\Box} \, D^2 \, \ln\det \hat{g}_R \; + \text{h.c.} \qquad . \tag{12}$$

($\hat{g}_R$ denotes the metric of the massless subsector of the matter fields ρ in the representation R.) When expanded to first order in T^I eq. (12) reduces to the local chiral integral considered above with h_I given by (11). The component expansion of (12) contains the local term $\sum_R F_{\mu\nu}F^{\mu\nu} \ln\det\hat{g}_R$ but no local terms proportional to $F\tilde{F}$. Thus we recover exactly the situation described below eq. (8). The gauge coupling constant is well defined and the contribution of the massless fields to the one loop threshold correction are proportional to $\ln\det\hat{g}$ and therefore non-harmonic. The generic expression for the threshold corrections of S_Γ reads*

$$\Delta = 16\pi^2 \,\mathrm{Re} f^{\mathrm{1loop}} - \sum_R 2\, c_R \, \ln\det \hat{g}_R \tag{13}$$

where $\mathrm{Re} f^{\mathrm{1loop}}$ is induced by the massive modes of the theory via eq. (6). However, $\mathrm{Re} f^{\mathrm{1loop}}$ is ambiguous since eqs. (12),(13) show that holomorphic redefinitions of the massless charged fields shift the f-term by a holomorphic piece. This is the supersymmetric version of the well known ambiguity in the θ-angle if massless fermions are in the spectrum.[11]

Now it is clear that the threshold corrections of the Wilson action are just given by $\mathrm{Re} f^{\mathrm{1loop}}$ in eq. (13). The contribution of the graph 2 has to be excluded from S_W. The precise form of $\mathrm{Re} f^{\mathrm{1loop}}$ does depend on the regulator chosen.

Let us summarize what we have learned in this section. Non-harmonic gauge couplings can appear at the one loop level in S_Γ due to massless particles generating a non-local term of the form (12). The Wilson action S_W which by definition cannot include a term like (12) has gauge couplings g_W which are harmonic, θ_W is well defined and $f_W = 1/g_W^2 - i\theta_W/8\pi^2$ is holomorphic.

The distinction between g_Γ and g_W was recently stressed in an important paper by Shifman and Vainshtein.[5] They make the observation that at two loops g_Γ is generically non-harmonic due to wave function renormalization. Again, if one introduces the Wilson action, then f_W can be shown to be holomorphic. In fact, Shifman and Vainshtein argue that f_W receives no contributions beyond the one loop level whereas higher loop corrections of g_Γ arise entirely due to infrared contributions. A similar observation was made by Nilles in the context of string theory.[12] His argument relied heavily on the holomorphic nature of f. From the discussion of this section and the observations by Shifman and Vainshtein it appears that f_W does satisfy this property and the argument outlined in ref. 12 should apply to f_W.

⋆ If the theory has a field dependent f at the tree level this formula is modified. Such a situation is discussed in the next section.

The nonlocal term (12) can be transformed into a local term (with similar properties) if a linear multiplet is in the theory.[6]† (See J. P. Derendinger's talk at this conference.) The same paper also noted the intimate connection of nonharmonic gauge couplings with sigma-model anomalies.

Moduli Dependence of the Gauge Coupling Constant for N = 1 Orbifolds

In this section we summarize our knowledge about the gauge couplings in superstring theory. At the string tree level the form of f_a is well known to be[13–17]

$$f_a^{\rm tree} = k_a S \tag{14}$$

where S is the (complex) dilaton superfield and k_a denotes the level of the Kac-Moody algebra of the gauge group. (For simplicity we will put $k_a = 1$ henceforth.)

At the string loop level, ref. 18 verified eq. (3) and derived an expression for Δ_a as an integral over the fundamental domain Γ on the worldsheet

$$\begin{aligned} \tilde{\Delta}_a &= \int_\Gamma \frac{d^2\tau}{\tau_2}(B_a(\tau,\bar\tau) - b_a) \\ B_a &= |\eta|^4 \sum_{\text{even}\, s} (-)^{s_1+s_2} \frac{dZ_\Psi}{2\pi i d\bar\tau}\; Tr_{s_1}(Q_a^2 (-)^{s_2 F} q^{H-\frac{11}{12}} \bar q^{\bar H - \frac{3}{8}})_{\rm int} \quad . \end{aligned} \tag{15}$$

Without delving into the details of the definition of B_a the reader should merely note that B_a encodes information about the full string spectrum. $\tilde\Delta_a$ differs from the Δ_a in eq. (3) by a universal piece Y which is independent of the gauge group and which was not calculated in ref. 18. ($\Delta_a = \tilde\Delta_a + Y$.) In deriving eq. (15) a choice of the unification scale $M_{\rm GUT}$ had to be made. It seems natural to define $M_{\rm GUT}$ to be proportional to the string scale α': $M_{\rm GUT}^2 \sim 1/\alpha'$. This definition has the merit that $M_{\rm GUT}$ does not depend on any vacuum expectation value (vev) of scalar fields in the theory. Thus, all field dependence of $1/g_a^2$ induced by string loop corrections will appear in Δ_a.

† The models we considered in this section do not contain a linear multiplet.

In light of the discussion in the last section one should note that eq. (15) calculates the physical threshold corrections which appear in S_Γ. The string graph used to derive (15) does not allow for the separation between massless and massive modes.

$N = 1$ supersymmetric orbifolds are examples of string vacua where the full spectrum is known and eq. (15) can be evaluated. Ref. 2 calculates the dependence of $\tilde{\Delta}_a$ on the untwisted moduli T^I of $N = 1$ orbifolds. It was found that a nontrivial moduli dependence arises if and only if the orbifold point group G contains a subgroup G' that by itself would produce an orbifold with $N = 2$ space-time supersymmetry. This is equivalent to requiring that G' leaves one two-torus (or one complex plane) untwisted. (As an example consider the orbifold Z_4 generated by the twist $\Theta = (i, i, -1)$ acting on the three tori. In this case G' is the Z_2 generated by $\Theta' = (-1,-1,1) = \Theta^2$ which leaves the third torus untwisted.) For each of the three tori there can be two complex moduli - one corresponding to deformations of the complex structure and one to the Kähler class. We will denote them collectively by T^I since both types of moduli enter $\tilde{\Delta}_a$ in a symmetric way. Ref. 2 also assumed that the (auxiliary) $N = 2$ orbifold associated with G', can be obtained as a toroidal compactification of a vacuum with $N = 1$ supersymmetry in six space-time dimensions.⋆

Without presenting the details of the calculation we quote the result:[2] †

$$\tilde{\Delta}_a = -\sum_I \frac{b_a'^I |G'^I|}{|G|} \left(\ln(|\eta(iT^I)|^4) + \ln \mathrm{Re} T^I \right) + \text{constant} \qquad (16)$$

where η is the Dedekind η-function and $|G|$ is the order of G. T^I are the untwisted moduli of the particular torus left untwisted by G'^I. (There can be different G''s for different tori.) $b_a'^I$ denotes the one loop β-function of the auxiliary $N = 2$ orbifold

⋆ Not all $N = 2$ vacua satisfy this criterion.

† In order to make contact with standard supergravity notation we have changed the definition of T compared with that in ref. 2 by interchanging the real and imaginary parts of T.

generated by G'. Generally it does not coincide with the β-function of the $N=1$ orbifold generated by G for which eq. (16) describes the threshold corrections.

$\tilde{\Delta}_a$ is not the real part of a holomorphic function (non-harmonic) due to the term proportional to $\ln \mathrm{Re}T^I$. In analogy with the previous section we will now show that the appearance of the $\ln \mathrm{Re}T$ term can be understood as an infrared phenomenon and recalculate it entirely from the massless spectrum of the orbifold vacua. To do so we have to evaluate the graph 2 using the tree level effective supergravity action for the orbifold vacua. Supergravity introduces additional couplings compared to the previous calculation. In addition to the metric coupling of eq. (10) also couplings of the Kähler potential and Kähler connection have to be included. From ref. 1 or 19 we collect the terms which contribute to $\Delta_a(T)$. They are most easily displayed in components where the moduli T^I couple through the covariant derivatives of the (Weyl) fermions.

$$\begin{aligned}
D_\mu \chi^i &= \partial_\mu \chi^i + \Gamma^i{}_{Jk}\partial_\mu T^J \chi^k - K_\mu \chi^i + \dots \\
D_\mu \bar{\chi}^{\bar{i}} &= \partial_\mu \bar{\chi}^{\bar{i}} + \Gamma^{\bar{i}}{}_{\bar{J}\bar{k}}\partial_\mu \bar{T}^{\bar{J}} \bar{\chi}^{\bar{k}} + K_\mu \bar{\chi}^{\bar{i}} + \dots \\
D_\mu \lambda &= \partial_\mu \lambda + K_\mu \lambda + \dots \\
D_\mu \bar{\lambda} &= \partial_\mu \bar{\lambda} - K_\mu \bar{\lambda} + \dots \\
K_\mu &= \tfrac{1}{4}(K_J \partial_\mu T^J - K_{\bar{J}} \partial_\mu \bar{T}^{\bar{J}}) + \dots \\
\Gamma^i{}_{Jk} &= g^{i\bar{l}} g_{k\bar{l},J} \quad .
\end{aligned} \tag{17}$$

χ^i denotes the charged fermions, λ represents the gauginos. The details of the calculation of the graph 2 can be found in ref. 4 (similar calculations also appear in ref. 6,20). We linearize the gravitational background and find to lowest order in the gravitational field the non-local term

$$\mathcal{L}_{\mathrm{nl}} = \tfrac{1}{8}\tfrac{1}{16\pi^2} \int d^4\theta W^\alpha W_\alpha \frac{1}{\Box} D^2 \left(c_V\, K + \sum_R c_R \left[2 \, \ln \det g_R - K\right] \right) + \text{h.c.} \quad . \tag{18}$$

Thus eq. (13) is replaced by

$$\Delta = 16\pi^2 \,\mathrm{Re} f^{1\mathrm{loop}} - c_V K - \sum_R c_R \left(2 \,\ln\det g_R - K\right) \quad . \tag{19}$$

(We dropped the 'hat' on g.) So far formulas (18) and (19) are quite general. Even though we only gave the couplings of the moduli T in eq. (17) they are easily generalized for other fields in the spectrum. One exception is the dilaton field S which has the unique property that it couples to the gauge sector at the tree level via eq. (14). Without presenting any details let us state that due to these extra couplings Δ_a receives a contribution proportional to the dilaton which is of the form $b_a \ln(S+\bar{S})$. In terms of the string calculation this dilaton dependence can be understood as a redefinition of M_{GUT} from $1/\sqrt{\alpha'}$ to M_{Pl} ($M_{\mathrm{Pl}}^2 \sim (S+\bar{S})/\alpha'$). The standard $N=1$ supergravity Lagrangian whose couplings we used above is written in terms of M_{Pl} whereas the string calculation used the string scale α'. Thus the dilaton dependent piece is perfectly consistent with the string calculation.

To make contact with eq. (16) we have to evaluate eq. (19) for the orbifolds considered in ref. 2. This can be done with the help of ref. 21 where the dependence of K and g on the untwisted moduli is calculated. For simplicity we will discuss only the orbifolds of gauge group $E_8 \otimes E_6 \otimes U(1)^2$.* Their Kähler potential reads

$$K = \sum_I -\ln(T+\bar{T})^I + \sum_i g_i(T,\bar{T})\phi^i e^V \bar{\phi}^{\bar{i}} + O((\phi\bar{\phi})^2) + \ldots \tag{20}$$

where

$$g_i(T^I,\bar{T}^I) = \prod_I \left[T^I + \bar{T}^{\bar{I}}\right]^{-q_i^I} \quad . \tag{21}$$

The exponents q_i^I depend on the particular matter field ϕ^i as well as on the modulus T^I in question and are calculated in ref. 21. Inserting eqs. (20),(21) into (18) and

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⋆ It is a straightforward extension to treat (2,2) orbifolds such as Z_3 and Z_4 which have larger gauge groups than $U(1)^2$.[4]

(19) we find

$$\begin{aligned}\mathcal{L}_{\rm nl} &= \tfrac{1}{8}\tfrac{1}{16\pi^2}\sum_a\sum_I\ \alpha_a^I\ \int d^4\theta (W^\alpha W_\alpha)_a\ \frac{1}{\Box}\ D^2\ \ln(T+\bar{T})^I\ +\ \text{h.c.}\\ \Delta_a(T,\bar{T}) &= 16\pi^2\ \text{Re}f_a^{\rm 1loop}\ -\ \sum_I\ \alpha_a^I\ \ln(T+\bar{T})^I\end{aligned} \qquad (22)$$

where

$$\alpha_a^I = -(c_V)_a - \sum_i \Big(c_i(2q_i^I - 1)\Big)_a \quad . \qquad (23)$$

This result differs from the $\tilde{\Delta}_a$ calculated in eq. (16) by the universal piece Y. The T dependence of the non-harmonic piece is identical which requires the coefficients α to satisfy

$$\alpha_{a_1}^I - \alpha_{a_2}^I \doteq \frac{|G'^I|}{|G|}\,(b'^I_{a_1} - b'^I_{a_2}) \quad . \qquad (24)$$

So far we have no direct proof of this equation but all examples we checked satisfied eq. (24). It has to hold in order for the two calculations to be consistent.

For the gauge group E_6 one can simplify eq. (23) by using the fact that string theory is (gauge) anomaly free. In particular it has no mixed $E_6 \otimes U(1)^2$ anomaly. The vanishing of the mixed anomaly constrains the sum of the $U(1)$ charges of the **27** and $\overline{\mathbf{27}}$ which are related to the q_i^I in way specified in ref. 21. This constraint allows us to rewrite[4]

$$\alpha_{E_6}^I = -c_V - c_F\ \left(\frac{\chi}{6} - 2 - 2\sum \mathcal{N}_{27}^{N=2}\right) \qquad (25)$$

where $\mathcal{N}_{27}^{N=2}$ denotes the number of **27**'s in the $N=2$ sectors of the orbifold and it is being summed over all $N=2$ sectors. χ is the Euler number of the orbifold.

To summarize, in this section we have outlined how the non-harmonic piece in eq. (16) can be recalculated from the massless spectrum and its couplings alone. Furthermore, we also have inferred the non-harmonic dependence of the universal piece Y.

Modular invariance

Eq. (16) is invariant under target space modular transformations belonging to $SL(2,\mathbf{Z})$, which are of the form

$$T^I \rightarrow \frac{aT^I - ib}{icT^I + d} , \qquad ad - bc = 1 , \qquad a,b,c,d \in \mathbf{Z} \quad . \tag{26}$$

This symmetry property of eq. (16) is expected since it is manifest in the string amplitude.[22] It also agrees with the fact that the physical gauge coupling constant is modular invariant. The contribution of the massless modes we just calculated can be understood as an anomaly in modular transformations. The canonically normalized fermions of the theory transform under modular transformations according to

$$\begin{aligned} \lambda &\rightarrow \lambda \; e^{-\frac{1}{4}(F-\bar{F})} \\ \chi^i &\rightarrow \chi^i \; e^{-\frac{1}{4}\sum_I (2q_i^I - 1)(F^I - \bar{F}^I)} \end{aligned} \tag{27}$$

where

$$F = \sum_I F^I = \sum_I \ln(icT^I + d) \quad . \tag{28}$$

Note that this is a local phase transformation of the fermions even though the transformation of T (eq. (26)) is discrete. The connections Γ and K_μ in eq. (17) gauge the modular transformations (27) of the fermions. The 'modular anomaly' is obtained by variation of $\mathcal{L}_{\rm nl}$ in eq. (22) under the transformations (26). One finds

$$\delta\mathcal{L}_{\rm nl} = \tfrac{1}{2}\tfrac{1}{16\pi^2}\sum_a\sum_I \alpha_a^I \int d^2\theta (W^\alpha W_\alpha)_a F^I(T) + {\rm h.c.} \quad . \tag{29}$$

This variation can be canceled by the counterterm

$$\begin{aligned} \mathcal{L}_{ct} &= \tfrac{1}{4}\sum_a \int d^2\theta \, f_a \, (W^\alpha W_\alpha)_a \; + \; {\rm h.c.} \\ f_a &= \frac{-2}{16\pi^2} \; \sum_I \alpha_a^I \; \ln \eta^2(iT^I) \end{aligned} \tag{30}$$

where we have used the transformation $\eta^2(iT^I) \rightarrow \eta^2(iT^I)e^{F^I}$. The contribution of $\mathcal{L}_{ct}$ to Δ_a is of course equivalent to the first term in eq. (16) (including the universal piece Y). Thus we have completely recovered eq. (16) by using the massless spectrum, its couplings and a 'stringy' symmetry (modular transformations). In terms of the distinction between the two gauge coupling constants outlined above we conclude that the one loop contribution to f_W coincides with the f_a given in eq. (30).

The calculation just presented opens up the exciting possibility of calculating Δ_a for other compactifications or its dependence on different moduli of the orbifold without doing a full-fledged string calculation.

Gaugino Condensation

Finally, let us apply these considerations to the mechanism of gaugino condensation. For simplicity we will confine ourselves to the case of condensation in a hidden sector which is a pure Yang-Mills gauge sector, e.g. the unbroken E_8 of the orbifold vacua. Shifman and Vainshtein point out that there are two different gaugino condensates.[5] First, there is the physical condensate which can be estimated by an RG invariant scale Λ_c

$$\langle\lambda\lambda\rangle_\Gamma \sim \Lambda_c = M^3 \exp(\frac{-8\pi^2}{c_V g_\Gamma^2(M)}) . \tag{31}$$

$\langle\lambda\lambda\rangle_\Gamma$ is real and modular invariant. However, the quantity which appears in a non-perturbative superpotential of $N=1$ supergravity and is chiral is

$$U \sim M^3 \exp(\frac{-8\pi^2 f_W}{c_V}) \quad . \tag{32}$$

Indeed, eq. (32) can be understood in terms of an effective superpotential $W_{np}(U)$ of the composite chiral superfield $U(= W^\alpha W_\alpha)$ describing the dynamics of gaugino condensation[23,25,7]

$$
\begin{aligned}
W_{np} &= \frac{U}{4}\left(f_W(S,T) + \frac{c_V}{8\pi^2}\ln\frac{U}{M^3}\right) \\
f_W(S,T) &= S + \frac{c_V}{8\pi^2}\sum_I \ln\eta^2(iT^I) \quad .
\end{aligned}
\tag{33}
$$

W_{np} satifies all anomalous Ward identities. Under modular transformation W_{np} has to transform equivalently to the tree level superpotential $W \to We^{-F}$. This determines the transformation law of U to be of the same form $U \to Ue^{-F}$ as can be seen from the first term in eq. (33). (The dilaton does not transform under modular transformation.) The variation of the term proportional to $\ln U$ gives precisely the modular anomaly of the massless modes (eq. (29)) and Uf_W is the counterterm given in eq. (30).* The solution of the minimization condition $\frac{\partial W_{np}}{\partial U} = 0$ is eq. (32).

The relationship between the two gauge coupling constants depends on the scale at which they are compared. At the Planck scale one finds

$$
\frac{16\pi^2}{g_\Gamma^2(M_{\rm Pl})} = 16\pi^2 {\rm Re} f_W + c_V\sum_I \ln(T+\bar{T})^I + 3c_V\ln(S+\bar{S}) \tag{34}
$$

which used (22) and the dilaton dependent piece discussed above. This translates into the following relationship between the condensates (at $M_{\rm Pl}$)

$$
({\rm Re}S)\,\langle\lambda\lambda\rangle_\Gamma = |U|\; e^{K/2} \quad . \tag{35}
$$

The relationship between $1/g_\Gamma^2$ and f_W and the appropriate condensates at the two loop level is discussed in ref. 5.

⋆ Note that $\alpha^I = -c_V$ holds for E_8. This was observed in ref. 24 by using the large radius behavior of Δ.

Eq. (33) was previously derived in refs. 25,7 for the overall radius modulus. (See talks given by M. K. Gaillard and T. Taylor.) By inserting eq. (32) into (33) one arrives at the nonperturbative superpotential given in refs. 26,27. (See talk given by M. Cvetič.) These papers derived W_{np} by demanding modular covariance of the superpotential. Here we have stressed the connection with the modular anomaly discussed in the previous section and showed that both approaches are essentially identical when applied to gaugino condensation. We have also clarified the relationship with the string threshold corrections (16) obtained in ref. 2. The calculation of the modular anomaly and deducing its counterterm generalizes in some sense the approach of refs. 26,25,27,7 in that it can be discussed for all gauge groups and without referring to a gaugino condensation mechanism.

Hidden sectors including matter fields can similarly be discussed in the spirit of this section.[28,4]

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