SLAC–PUB–5510
May 1991
(A)

ON TWO SOLVABLE MODELS OF COHERENT BEAM-BEAM INTERACTION

D. PESTRIKOV*

Institute for Nuclear Physics†
630090 Novosibirsk, USSR.

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

The spectra of linear coherent oscillations due to beam-beam interaction are calculated in the paper for two model distributions in unperturbed bunches. In both cases the calculations predict the splitting of coherent spectra of any multipole mode onto generally infinite amount of submodes with well defined ground state. Among other general features, the influence of Landau damping on the stability of coherent beam-beam oscillations is discussed.

Submitted for Publication

⋆ Work supported in part by US Department of Energy contract DE–AC03–76SF00515 and in part by the Institute for Nuclear Physics, Novosibirsk, USSR.

† Permanent address.

1. INTRODUCTION

Better understanding of limitations due to the interaction of colliding bunches seems to be of primary importance for the realization of coming B Factories. Among others, this manifold problem includes the analysis of the behavior of strong-strong bunches and, as the first step, the performance of the calculations related to collective stability of colliding bunches. Numerous experimental and theoretical publications (many references on this subject can be found, for instance, in Refs. [1–3]) indicate the permanent interest to this field. First of all this problem is of practical importance due to possible applications for the diagnostic of colliding beams and limitations on the position of the working point of a collider.

The second field of interest can be considered as more theoretical and is associated with the construction of the theory, describing kinetic processes in colliding bunches, their heating (emittance blow-up) due to beam-beam instability and associated energy exchange between particles and coherent oscillations or fluctuations Ref. [4]. All these applications require the most comprehensive knowledge of beam-beam coherent spectra.

Straightforward calculations of collective spectra for beam-beam interaction typically meet serious mathematical difficulties. Therefore, the solvable models, which can provide the simplified description of the main properties of coherent beam-beam oscillations, becomes very desirable. There are two widely known approaches to this problem. One is based on the so-called rigid bunch model, when coherent oscillations of bunches are described by dipole momenta only (see, for instance, in Refs. [1,2] and references mentioned there). The other is based on the solution of one dimensional Vlasov equations assuming so-called water-bag distributions in colliding bunches (see, for instance, in Ref. [3]). In both cases simple solutions can be obtained analytically, and collective spectra, containing only one line predicted. In principle, it may happen, but usually a very strong degradation of the spectrum caused by the model itself is indicated. Even a brief examination of integral equations, which have to be solved, indicates the splitting of modes with the given multipole number into generally infinite sets of submodes Ref. [4]. Since the perturbed part of the solution of the Vlasov equation is proportional to the gradient (in oscillation amplitudes) of unperturbed distribution functions, one can expect that coherent oscillations are mainly located around the bunch width, with strong decay in the core and tail regions of the bunch. However, if the spectrum of oscillations contains higher states, these can penetrate into the core, or tail regions, disturbing particles there.

In this paper we shall consider two models, when such an analysis can be extended far enough.

2. General Equations

The most general description of coherent beam-beam oscillations is provided by the technique based on the Vlasov equation. In this paper for the sake of simplicity we shall consider the cases, when only betatron coherent oscillations are excited in bunches. Then, if betatron oscillations of individual particles are described by the action-phase variables $(I_\alpha, \psi_\alpha; \alpha = x, z)$, these equations read:

$$\begin{aligned} \left(\frac{\partial}{\partial\vartheta_s} + \vec{\nu}_1 \frac{\partial}{\partial\vec{\psi}}\right) f^{(1)} + \frac{\partial L_{1,2}}{\partial\vec{\psi}} \quad \frac{\partial f^{(1)}}{\partial\vec{I}} &= 0 \quad , \\ \left(\frac{\partial}{\partial\vartheta_s} + \vec{\nu}_2 \frac{\partial}{\partial\vec{\psi}}\right) f^{(2)} + \frac{\partial L_{2,1}}{\partial\vec{\psi}} \quad \frac{\partial f^{(2)}}{\partial\vec{I}} &= 0 \quad . \end{aligned} \tag{1}$$

Here $L_{1,2}$ (respectively, $L_{2,1}$) is the Lagrangian, describing the interaction of particles from the bunch 1 with fields of the counter-moving bunch 2; $f^{(1,2)}$ are distribution functions of colliding bunches. For relativistic and counter-charged particles ($e_{\bar{1}}e_2 = -e^2; \gamma = E/Mc^2 \gg 1$, E is the particle energy) one can write:

$$L_{1,2} = \frac{N_2 e^2}{c} \int \frac{d^2k}{\pi k^2} \int d\Gamma_2 exp\left(i\vec{k}[\vec{r}_\perp^{(1)} - \vec{r}_\perp^{(2)}]\right) \tag{2}$$
$$\delta_T\left(\vartheta_s + \frac{\Delta\vartheta_1 - \Delta\vartheta_2}{2}\right) f^{(2)}(\vec{I}, \vec{\psi}, \vartheta_s) \ ,$$
$$k^2 = k_x^2 + k_z^2 \quad , \quad d\Gamma = d^3p d^3r,$$

where $\Delta\vartheta(\vec{I}, \vec{\psi}, \vartheta_2) = \vartheta - \vartheta_s$, $\vartheta = s/R_0$ is the particle azimuth $\vartheta_s = \omega_s t$ is azimuth of the synchronous particle, $2\pi R_0$ is the orbit perimeter, $\delta_T(\vartheta)$ is the periodic δ-function:

$$\delta_T(\vartheta) = \sum_{n=-\infty}^{\infty} \exp(in\vartheta)/2\pi \quad .$$

As usual, to calculate both stability criteria and the spectra of small coherent oscillations one may linearize Eqs. (1) near the stationary state, which by the definition is described by distribution functions, independent of phase variables:

$$f^{(1,2)} = f_0^{(1,2)}(\vec{I}) + \tilde{f}^{(1,2)}(\vec{I}, \vec{\psi}, \vartheta_s) \quad ,$$
$$f_0^{(1,2)}(\vec{I}) \gg \tilde{f}^{(1,2)}(\vec{I}, \vec{\psi}, \vartheta_s) \quad . \tag{3}$$

In fact, for colliding bunches such a structure of the distribution function assumes that particle oscillations in the stationary state are stable and therefore the working point of the machine is placed outside the stopbands of the incoherent beam-beam instability.

If the contributions from the stationary part of the perturbation ($L_0 = L[f_0]$) is included in tunes $\vec{\nu}_{1,2}$ and β-functions of the ring, Eq.(3) yields the system of equations:

$$\left(\frac{\partial}{\partial\vartheta_s} + \vec{\nu}_1 \frac{\partial}{\partial\vec{\psi}}\right) \tilde{f}^{(1)} + \frac{\partial \tilde{L}_{1,2}}{\partial\vec{\psi}} \frac{\partial f_0^{(1)}}{\partial\vec{I}} = 0 \quad ,$$

$$\left(\frac{\partial}{\partial\vartheta_s} + \vec{\nu}_2\frac{\partial}{\partial\vec{\psi}}\right)\tilde{f}^{(2)} + \frac{\partial\tilde{L}_{2,1}}{\partial\vec{\psi}}\ \frac{\partial f_0^{(2)}}{\partial\vec{I}} = 0 \quad , \tag{4}$$

describing coherent oscillations of colliding bunches. For practical use these equations must be supplied by formulae, describing unperturbed oscillations of particles and generating the canonical transformation from the variables $(\vec{r},\vec{p})$ to action-phase variables. For the sake of simplicity we shall assume here no dispersion at the interaction point (IP), and that colliding bunches are rather short $\sigma_{\|} \ll \beta^\star$, where $\sigma_{\|}$ is the length of each bunch, $\beta^\star$ is β-function at the IP. Then betatron oscillations of particles are described by the formulae:

$$(x,z) = \sqrt{J\beta_{x,z}}\cos\left(\psi + \chi(\vartheta)\right)_{x,z'}\, p_{x,z} = \frac{p_s}{R_0}\frac{d}{d\vartheta}(x,z) \equiv \frac{p_s}{R_0}(x',z')\ , \tag{5}$$

$$\psi'_{x,z} = \nu_{x,z}(J_x,J_z), \quad \chi'_\alpha + \nu_\alpha = R_0/\beta_\alpha, \quad \alpha = x,z$$

$$I_{x,z} = p_s J_{x,z}/2\ .$$

With these suggestions one can write

$$\tilde{L}_{1,2} = N_2 e^2 \delta_T(\vartheta_s) U_{1,2}/c \quad ,$$

$$U_{1,2} = \int \frac{d^2k}{\pi k^2}\int d\Gamma_2 exp\left(i\vec{k}[\vec{r}_\perp^{(1)} - \vec{r}_\perp^{(2)}]\right)\tilde{f}^{(2)}(\vec{I},\vec{\psi},\vartheta_s) \quad . \tag{6}$$

Substituting these expressions in Eqs. (4) and using there the Fourier transformation

$$\tilde{f}^{(1,2)}(\vec{I},\vec{\psi},\vartheta_s) = \sum_{\vec{m}} f_{\vec{m}}^{(1,2)}(\vec{I},\vartheta_s)exp(i\vec{m}\vec{\psi}) \quad ,$$

one can rewrite (4) as follows,

$$\frac{\partial f_{\vec{m}}^{(1)}}{\partial\vartheta_s} + i\vec{m}\vec{\nu}_1 f_{\vec{m}}^{(1)} = -i\vec{m}\frac{\partial f_0^{(1)}}{\partial\vec{I}}\frac{N_2e^2}{c}\delta_T(\vartheta_s)[U_{1,2}]_{\vec{m}} \quad ,$$

$$\frac{\partial f^{(2)}_{\vec{m}}}{\partial \vartheta_s} + i\vec{m}\vec{\nu}_2 f^{(2)}_{\vec{m}} = -i\vec{m}\frac{\partial f^{(2)}_0}{\partial \vec{I}}\frac{N_1 e^2}{c}\delta_T(\vartheta_s)[U_{2,1}]_{\vec{m}} \quad . \tag{7}$$

To find the eigensolutions of these equations following the papers Refs. [3,5] let us note that after the crossing the IP amplitudes $f_{\vec{m}}$ get the variations

$$\delta f^{(1)}_{\vec{m}} = f^{(1)}_{\vec{m}}(0+) - f^{(1)}_{\vec{m}}(0-) = -i(\vec{m}\partial f_0/\partial \vec{I})\frac{N_2 e^2}{c}[U_{1,2}]_{\vec{m}} \quad ,$$

$$\delta f^{(2)}_{\vec{m}} = -i(\vec{m}\partial f_0/\partial \vec{I})\frac{N_1 e^2}{c}[U_{2,1}]_{\vec{m}} \, . \tag{8}$$

Here $f_{\vec{m}}(0+)$ and $f_{\vec{m}}(0-)$ are the amplitudes just after and before the IP. Between collisions particles execute free oscillations $f_{\vec{m}}(\vartheta_s) \sim \; exp(-i\vec{m}\vec{\nu}\vartheta_s)$. Hence, due to the periodicity of coefficients in Eq. (7) one can write for eigensolutions

$$f_{\vec{m}}(0+) = \lambda e^{2\pi i \vec{m}\vec{\nu}} f_{\vec{m}}(0-) \quad . \tag{9}$$

As Eq. (9) must be valid at any turn, for amplitudes $f_{\vec{m}}$ spaced by n subsequent turns it obviously yields:

$$f_{\vec{m}}(2\pi n + 0) \;=\; \lambda^n e^{2\pi i \vec{m}\vec{\nu} n} f_{\vec{m}}(0-).$$

Therefore coherent oscillations will be stable, if all eigenvalues of the problem λ satisfy the criterion $|\lambda| \leq 1$. The integral equations for eigenfunctions $f_{\vec{m}}$ will be obtained after the substitution of Eq. (9) in Eqs. (8):

$$f^{(1)}_{\vec{m}} = \frac{N_2 e^2}{c} \; \frac{i(\vec{m}\partial f^{(1)}_0/\partial \vec{I})[U_{1,2}]_{\vec{m}}}{1 - \lambda exp(2\pi i \vec{m}\vec{\nu}_1)} \quad ,$$

$$f^{(2)}_{\vec{m}} = \frac{N_1 e^2}{c} \; \frac{i(\vec{m}\partial f^{(2)}_0/\partial \vec{I})[U_{1,2}]_{\vec{m}}}{1 - \lambda exp(2\pi i \vec{m}\vec{\nu}_2)} \quad . \tag{10}$$

These equations indicate nontrivial behavior of solutions in the close vicinity of resonances (see in Refs. [1], [4] for details):

$$\vec{m}_1\vec{\nu}_1 - \vec{m}_2\vec{\nu}_2 = n \quad , \tag{11}$$

and

$$\vec{m}_1\vec{\nu}_1 = n, \ \vec{m}_2\vec{\nu}_2 = n \ , \ n = \pm 1, \pm 2, \ \ldots \tag{12}$$

Nonresonant oscillations are stable and have tunes close to unperturbed ones

$$\lambda = e^{-2\pi i \nu} \ , \ \nu \simeq \vec{m}_1\vec{\nu}_1 \ , \ \nu \simeq \vec{m}_2\vec{\nu}_2 \quad .$$

3. Flat Colliding Bunches with Lorentz Radial Distribution

The direct solution of the system (10) for arbitrary stationary distributions in colliding bunches is still too complicated. To simplify this problem here we shall consider the special case, when colliding bunches move in the same ring and have identical intensities as well as identical stationary distribution functions:

$$f_0^{(1,2)} = \frac{\theta(I_{0z} - I_z)}{(2\pi)^2 I_{0z}} \quad \frac{1}{I_{0x} + I_x} \quad , \tag{13}$$

where $\theta(x) = 1$, if $x \geq 0$, and $\theta(x) = 0$, if $x < 0$. These distributions cannot be normalized and, therefore, still are singular. However, distribution functions in Eq. (13) describe the bunches concentrated within the region,

$$|z| \leq \sqrt{\epsilon_z \beta_z^\star} \ , \ |x| \leq \sqrt{\epsilon_x \beta_x^\star} \ , \ (I_0)_{x,z} = p\epsilon_{x,z}/2 \quad .$$

This circumstance turns out to be very important for the properties of collective spectra.

The strength of the beam-beam interaction for both coherent and incoherent phenomena can be specified by the value of so-called beam-beam parameters. For the bunches described by the distribution functions Eq. (13) these can be written in the form,

$$\xi_{x,z} = \frac{4}{\pi} \quad \frac{Ne^2\beta^{\star}_{x,z}}{2\pi pc(\sigma_{x,z})\sigma_x} \quad \sigma_x \gg \sigma_z .$$

To simplify the calculations, below we shall assume that the beams are flat $I_{0z} \ll I_{0x}$ and have the typical ratio between partial beam-beam parameters $\xi_z \gg \xi_x$. As will be seen, this enables one to consider vertical coherent oscillations as the most dangerous and therefore, to make the calculations assuming that only vertical coherent oscillations of bunches are excited.

For these modes one has $\vec{m} = \{0, m, 0\}$ and the substitution of distributions functions from Eq. (13) in Eq. (10) yields

$$\begin{aligned} f^{(1)}_{\vec{m}} &= \frac{-2Ne^2}{pc\epsilon_z} \quad \frac{im\delta\left(I_z - I^{(1)}_{0z}\right)}{1-\lambda exp(2\pi im\nu_z)} \quad \frac{[U_{1,2}]_{m,0}}{I_{0x}+I_x} \quad , \\ f^{(2)}_{\vec{m}} &= \frac{-2Ne^2}{pc\epsilon_z} \quad \frac{im\delta\left(I_z - I^{(2)}_{0z}\right)}{1-\lambda exp(2\pi im\nu_z)} \quad \frac{[U_{2,1}]_{m,0}}{I_{0x}+I_x} \quad , \end{aligned} \tag{14}$$

where now,

$$U_m = \int \frac{d^2k}{\pi k^2} J_m(k_z a_z) J_0(k_x a_x) \int d\Gamma_2 exp\left([-i\vec{k}\vec{r}^{(2)}_{\perp}]\right) \tilde{f}(\vec{I}, \vec{\psi}, \vartheta_s) \quad , \tag{15}$$

$J_m(x)$ are Bessel functions Ref. [6] and

$$a_{x,z} = \sqrt{J\beta}|_{x,z} \quad .$$

Using the substitutions,

$$f^{(1,2)}_{\vec{m}} = \delta\left(I_z - I_{0z}\right) w^{(1,2)}_m(a_x) \ , \ I_x = \frac{pa_x^2}{2\beta^{\star}_x} \quad , \tag{16}$$

and

$$w^{(1,2)}(k_x) = \int_0^\infty du u J_0(k_x u) w^{(1,2)}(u) \quad ,$$

$$\sigma_{x,z}^2 = (\epsilon\beta^\star)_{x,z} \ , \ u = a_x/\sigma_x \quad , \tag{17}$$

as well as Eq. (15) one can transform Eq. (14) into the following system of integral equations

$$[1 - \lambda exp(2\pi i m \nu_z)] w_m^{(1)}(k) = -4\pi i \xi_z m \sum_{m'} M_{m,m'} \int_0^\infty dk' K(k,k') w_{m'}^{(2)}(k') \quad ,$$

$$[1 - \lambda exp(2\pi i m \nu_z)] w_m^{(2)}(k) = -4\pi i \xi_z m \sum_{m'} M_{m,m'} \int_0^\infty dk' K(k,k') w_{m'}^{(1)}(k'), \tag{18}$$

$$K(k,k') = \begin{cases} K_0(k) I_0(k'), & k \geq k' \ , \\ I_0(k) K_0(k'), & k \leq k' \ , \end{cases} \tag{19}$$

$$M_{m,m'} = \begin{cases} 0, & |m| + |m'| = 2l+1, \\ \frac{4}{\pi} \int_0^\infty \frac{dk}{k^2} J_{|m|}(k) J_{|m'|}(k), & |m| + |m'| = 2l, \end{cases} \tag{20}$$

$$M_{m,m'} = \frac{(-1)^{|m|-l+1}}{[4(|m|-1)^2 - 1][l^2 - 1/4]} \quad , \quad |m| + |m'| = 2l \quad .$$

Here $I_0(x)$ and $K_0(x)$ are modified Bessel functions [6].

One can construct the functions $w_m^{(1,2)}(k)$ as expansions

$$w_m^{(a)}(k) = \sum_j C_{mj}^{(a)} w_j(k) \ , \ a = 1,2 \tag{21}$$

in eigenfunctions of the following integral equation

$$\Lambda_j w_j(k) = \int_0^\infty dk' K(k,k') w_j(k') \quad . \tag{22}$$

Simple calculations (see in Appendix) yield

$$\begin{aligned} & w_j(k) = \sqrt{2} e^{-k} L_j(2K) \ , \ \Lambda_j = \frac{1}{1+2j} \quad , \\ & \int_0^\infty dk w_j(k) w_{j'}(k) = \delta_{j,j'} \ , j = 0,1,\ldots \quad , \end{aligned} \tag{23}$$

where $L_j(x)$ are the Laguerre polynomials Ref. [6]. The substitution of expansions (21) in Eq. (18) yields the system of algebraic equations for coefficients $C_{mj}^{(1,2)}$

$$\begin{aligned} \left[1 - \lambda exp(2\pi i m \nu_z)\right] C_{mj}^{(1)} &= -4\pi i \xi_z m \Lambda_j \sum_{m'} M_{m,m'} C_{m'j}^{(2)} \quad , \\ \left[1 - \lambda exp(2\pi i m \nu_z)\right] C_{mj}^{(2)} &= -4\pi i \xi_z m \Lambda_j \sum_{m'} M_{m,m'} C_{m'j}^{(1)} \quad . \end{aligned}$$

This system separates the combinations

$$C_{mj}^{\pm} = C_{mj}^{(1)} \pm C_{mj}^{(2)} \quad ,$$

which satisfy the equations

$$\left[1 - \lambda exp(2\pi i m \nu_z)\right] C_{mj}^{\pm} = \pm 4\pi i \xi_z m \Lambda_j \sum_{m'} M_{m,m'} C_{m'j}^{\pm} \quad . \tag{24}$$

Using the same arguments as in the paper Ref. [3] we may concentrate subsequent analysis on the π-modes, which are described by C_{mj}^{-}. Near the particular resonance $\nu_z \simeq n/m$ one can write

$$1 - \lambda exp(2\pi i m \nu_z) \simeq 2\pi i(\nu - m\Delta),\ \Delta = \nu_z - n/m$$

and therefore expect that only diagonal items in Eq. (24) are important. This yields the dispersion equation

$$1 = \frac{4\xi\Delta\Lambda_{mj}}{(\nu/m)^2 - \Delta^2} \quad , \quad \Lambda_{mj} = \frac{1}{2j+1} \quad \frac{1}{m^2 - 1/4} \quad . \tag{25}$$

The requirement of real eigenfrequencies of this equation:

$$\nu_{m,j} = \pm\left\{\Delta^2 + 4\xi\Delta\Lambda_{m,j}\right\}^{1/2} \tag{26}$$

yields for modes with particular numbers m and j both the positions and the widths of the stopbands. Since the eigennumbers $\Lambda_{m,j}$ decrease with j, for the oscillations with the given multipole number m the width of the stopband is determined by the ground state mode ($j = 0$):

$$-\frac{4\xi}{m^2 - 1/4} \leq \Delta \leq 0 \quad . \tag{27}$$

In fact, the position of the upper border of the stopband must be found by combining Eq. (27) and the stability criterion for incoherent oscillations (see, for instance in Ref. [3]). Outside the stopbands coherent oscillations are stable and the measurements of their spectra can be used for the beam diagnostic. Equation (26) indicates (see also in Fig. 1) that these spectra have well defined the ground-state lines ($j = 0$), whereas the distances between higher-state lines ($j \gg 1$) decreases like $1/j^2$. Qualitatively this behavior of the collective spectra for colliding buches does not contradict to the results of measurements (see, for instance, in Ref. [7]). Here the splitting of the spectrum near the given harmonic $m\nu_z$ onto the series of lines describes the dependence of the solution on amplitudes of radial oscillations. More generally it is specific for coherent oscillations of bunches with smooth distributions in amplitudes of oscillations.

4. Flat Colliding Beams with Gaussian Radial Distributions

The models, which adopt as the stationary distributions the step-like functions, in fact describe coherent oscillations of quasi-monochromatic beams. Once in this case coherent oscillations propagate along the bunch surface in its phase space

$$f_{\vec{m}}(I) \sim \delta(I - I_0) \quad ,$$

nonlinearities of incoherent betatron oscillations only modify unperturbed tunes, but does not cause Landau damping of coherent modes. Here we shall consider the model, which enables one to incorporate into calculations the dispersion of betatron oscillations in a more or less straightforward way and, thus, to make some predictions about the influence of Landau damping on coherent beam-beam oscillations.

The model, which will be discussed in this section is based on the analysis of the properties of radial oscillations ($z = 0$) in colliding bunches, which have the following stationary distribution functions:

$$f_0^{(1,2)}(I_x, I_z) = \delta(I_z)\frac{exp[-I_x/Io]}{(2\pi)^2 I_0}, \quad I_0 = p_s\epsilon. \tag{28}$$

Some important values referring to incoherent beam-beam interaction for bunches with such distributions can be found in Refs. [4,8]. For instance, the incoherent tune shift of radial oscillations calculated within the framework of this model reads:

$$\Delta\nu_x(a_x) = \frac{2\xi}{\alpha^2}\Big[1 - exp(-\alpha^2/2)\Big], \; \alpha = a_x/\sigma \quad ,$$

$$\xi = \frac{Ne^2}{2\pi pc\epsilon} \quad .$$

Let us make now the calculations for horizontal coherent oscillations. Assuming the working point of the ring to be close to some particular resonance $m_x \nu_x \simeq n$ and $\lambda = exp(-2\pi i\nu)$, one can rewrite Eqs. (10) in the form (the subscript x will be omitted for short)

$$\chi_m^{(1,2)} = (f_m + f_{-m})^{(1,2)} \ , \ m > 0 \quad ,$$

$$\chi_m^{(1)} = \frac{2m^2\Delta(x)\xi}{\nu^2 - m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x,x') exp(-x'^2/2)\chi_m^{(2)} \quad ,$$

$$\chi_m^{(2)} = \frac{2m^2\Delta(x)\xi}{\nu^2 - m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x,x') exp(-x'^2/2)\chi_m^{(1)} \quad , \tag{29}$$

$$M_m(x,x') = 2\int_0^\infty \frac{dk}{k} J_m(kx) J_m(kx') = \frac{2}{m} \begin{cases} (x/x')^m \ , & x \le x' \quad , \\ (x'/x)^m \ , & x \ge x' \quad , \end{cases} \tag{30}$$

$$\Delta(x) = \nu_x(x) - n/m = \Delta_0 + \Delta\nu_x(x) \quad .$$

As previously, the system Eq. (29) separates the modes

$$\chi_m^{\pm} = \chi_m^{(1)} \pm \chi_m^{(2)} \quad ,$$

and again we shall consider π-mode. It satisfies the equation

$$\chi_m^{-} = \frac{2m^2\Delta(x)\xi}{\nu^2 - m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x,x') exp(-x'^2/2)\chi_m^{-} \quad . \tag{31}$$

The substitution (upperscripts ($-$) can be omitted below)

$$\chi_m = \frac{4m^2\Delta(x)\xi}{\nu^2 = m^2\Delta^2(x)} X(x) \tag{32}$$

transforms Eq. (31) into the following integral equation:

$$X(x) = \frac{x^{-m}}{m} \int_0^x dx' x'^{M+1} V_m(x') X(x) + \frac{x^m}{m} \int_x^\infty dx' x'^{1-m} V_m(x') X(x') \quad , \tag{33}$$

where

$$V_m(x) = \frac{4\Delta(x)\xi}{(\nu/m)^2 - \Delta^2(x)} \, e^{-x^2/2} \quad . \tag{34}$$

One more substitution $X = w(x)/\sqrt{x}$ and subsequent double differentiation of Eq. (33) transforms it into the following differential equation:

$$w'' + \left[2V_m(x) - \frac{m^2 - 1/4}{x^2}\right] w(x) = 0 \quad . \tag{35}$$

2. It is obvious that the choice of Gaussian function as the stationary distribution is not principal for these calculations. The same differential equation will describe horizontal coherent oscillations of any beam with the distribution function $f_0(x)$ after the substitution:

$$exp(-x^2/2) \rightarrow -2\partial f_0/\partial x^2 \quad .$$

Let us consider first some cases, when Eq. (35) has more or less simple solutions. For instance, if $f_0(x)$ is the step function $f_0 \sim \theta(x^2 - 1)$, one can write:

$$V_m(x) = \frac{1}{\Lambda_m}\delta(x-1) \ , \ (1/\Lambda_m) = \frac{4m^2\Delta(x)\xi}{\nu^2 - m^2\Delta^2(x)} \tag{36}$$

and Eq. (35) yields the following eigenfunctions and eigenvalues:

$$w(x) = \begin{cases} x^{1/2+m} & , x \leq 1 \\ x^{1/2-m} & , x \geq 1 \end{cases} \qquad \Lambda_m = 1/m \tag{37}$$

as well as the dispersion equation (see also in Ref. [3])

$$1 = \frac{4\Delta(1)\xi/m}{(\nu/m)^2 - \Delta^2(1)} , \ \xi \ll 1 \quad .$$

Another example presents the histogram distribution

$$f_0(x) = f_j \ , \ x_{j-1} \le x \le x_j \ , \ j = 1,\ 2,\ \ldots,\ q,\ x_0 = 0 \quad , \tag{38}$$

corresponding to

$$V_m(x) = \sum_{k=1}^{q} V_{mj}\delta(x - x_j)/x_j,\ V_{mj} = \frac{4\Delta_j\xi}{(\nu/m)^2 - \Delta_j^2}(f_j - f_{j-1}) \quad . \tag{39}$$

In this case solutions of Eq. (35) can be written in the form

$$w_j(x) = B_j x^{1/2+m} + C_j x^{1/2-m} \ , \ x_{j-1} \le x \le x_j \quad . \tag{40}$$

Using the boundary conditions

$$w_j(x_j) = w_{j+1}(x_j) \ , \ (w'_{j+1} - w'_j)\,\big|_{x=x_j} = \frac{2V_{mj}}{x_j}\ w_{j+1}(x_j)$$

one can find the recurrence equations for coefficients B_j and C_j

$$\begin{aligned} C_j &= C_{j+1}\left(1 - \frac{V_{mj}}{m}\right) - \frac{V_{mj}}{m}\ B_{j+1}x_j^{2m} \quad , \\ B_j &= B_{j+1} + (C_{j+1} - C_j)x_j^{-2m} \end{aligned} \tag{41}$$

with obvious initial conditions

$$C_1 = B_{q+1} = 0,\ C_{q+1} = 1 \quad .$$

This enables one to find all coefficients B_j, C_j as well as the dispersion equation

$$\sum_{j=1}^{a}(C_{j+1} - C_j) = 1 = \sum_{j=1}^{q}\frac{V_{mj}}{m}(C_{j+1} + B_{j+1}x_j^{2m}) \quad , \tag{42}$$

but results in cumbersome expressions, which can be useful probably only for numerical analysis. However, these expressions indicate a very important general result. If for the sake of simplicity we shall neglect the tune spread in Eq. (42), and introduce the values

$$1/\Lambda_m = \frac{4\delta\xi}{(\nu/m)^2 - \Delta^2} \quad ,$$

one can see that Eq. (42) is a polynomial of $1/\Lambda_m$ of the q-th order which therefore has exactly q roots. In the contrast with single step distribution function the spectrum of beam with histogram distribution Eq. (38) will consist of q lines. With some care such histogram distributions can be used for simulation of oscillations in bunches with smooth distribution functions.

3. The direct solution of Eq. (35) with smooth distributions f_0 still is very difficult. However, many general properties of eigenfunctions and spectra can be predicted using the analogy of Eq. (35) and the Schrödinger equation in the quantum mechanic, which is written for a particle with zero energy moving in effective potential well

$$U_{eff}(x) = \frac{m^2 - 1/4}{x^2} - 2V_m(x) \tag{43}$$

Let us again start with the discussion of properties of solutions for monochromatic beams ($\Delta(x) \to \Delta_0$). Defining as previously

$$1/\Lambda_m = \frac{2\Delta_0\xi}{(\nu/m)^2 - \Delta_0^2} \quad , \tag{44}$$

we shall rewrite Eq. (35) in the form

$$w'' - U_{eff}(x)w(x) = 0 \quad , \tag{45}$$

$$U_{eff}(x) = \frac{2}{\Lambda_m}e^{-x^2/2} - \frac{m^2 - 1/4}{x^2} \quad . \tag{46}$$

Equation (45) may have nontrivial solutions corresponding to the discrete, nondegenerate spectrum, if in some regions of x the function U_{eff} becomes negative (see, for instance, in Fig. 2). Adopting this for minimum value of U_{eff} and replacing the function $2x^2e^{-x^2/2}$ by its maximum value $4/e$, one can estimate Λ_m for the ground state mode at least by

$$\Lambda_m < \frac{4/e}{m^2 - 1/4} \quad . \tag{47}$$

Qualitatively, this behaviour can be illustrated by Figs. 2–4. If $j = 0, 1, \ldots$ marks the normal modes of Eq. (45), one may expect that Λ_{mj} decreases at least like $1/j^2$, when j increases, (this circumstance can be verified, for instance, by the direct calculation of the sum

$$\sum_{j=0}^{\infty} \Lambda_{mj} = \frac{1}{m} \quad ,$$

which can be done using Eq. (33). Since solutions of Eq. (45) have asymptotes

$$w(x) + \begin{cases} x^{1/2+m} \ , & x \to 0 \\ \\ x^{1/2-m} \ , & x \to \infty \end{cases} \ ,$$

they are mainly concentrated between roots of $U_{eff}(x)$, which correspond to stop-points of the mechanical problem

$$U_{eff}(x_{1,2}) = 0 \ , \ \text{or} \ 2\left(x^2e^{-x^2/2}\right)_{1,2} = \Lambda_{mj}(m^2 - 1/4) \quad . \tag{48}$$

The behaviour of U_{eff}, which is shown in Fig. 1 indicates that while the localization of the ground state mode only slightly moves outside the bunch, when m increases, the higher modes ($j \gg 1$) can penetrate inside bunches ($x_1 \to 0, \Lambda_{mj} \to 0$). The total amount of modes, corresponding to the given value of Λ_m can be estimated by the phase space volume of the potential well

$$\Delta j \simeq \frac{1}{\pi} \int_{x_1}^{x_2} dx \sqrt{-U_{eff}(x)} \ , \ U_{eff}(x_{1,2}) = 0 \quad . \tag{49}$$

4. The sensitivity of eigensolutions to the tune spread due to beam-beam interaction can be studied in a similar way. The direct inspection of the effective potential curves, calculated for $\nu = \pm i\delta$ (see in Figs. (5–8) definitely indicates the existence of eigenmodes within the stopbands, which are slightly displaced but have roughly the same widths as the corresponding stopbands for monochromatic bunches. Analogous stopbands can be found out by the inspection of corresponding curves for quadrupole ($m = \pm 2$), sextupole ($m = \pm 3$) and other oscillations. Such influence of Landau damping on coherent beam-beam instability could be expected beforehand (see in Ref. [9]) — for resonant coherent instabilities the nonlinearity of betatron oscillations rather helps to form buckets in the phase space, than to damp the oscillations.

As Eq. (35) presents the particular case of more general Sturm-Liouville equation, its spectra can be estimated using the general relationships. For instance, exact solutions $w(x) = w_j(x)$ minimize the integral

$$I[w] = \int_0^\infty dx \left[(w')^2 + \frac{m^2 - 1/4}{x^2} w^2 - 2V_m w^2 \right] \quad . \tag{50}$$

This yields the normalization condition for the eigensolutions

$$\int_0^\infty dx \left[(w_j')^2 + \frac{m^2 - 1/4}{x^2} w_j^2 \right] = 2 \quad ,$$

and the dispersion equation of the problem:

$$\int_0^\infty dx |w_j|^2 \frac{4\Delta(x)\xi}{(\nu/m)^2 - \Delta^2(x)} e^{-x^2/2} = 1 \ , \ Im\nu > 0 \quad . \tag{51}$$

If the eigenfunctions w_j are not known, both they and spectra can be found using the minimization routines starting from more or less suitable set of probe functions $w(x)$.

Acknowledgements.

It is my pleasure to thank SLAC and its B factory team for their hospitality. I also would like to thank for their useful discussions and numerous comments N. Dikansky, P. Ivanov, E. Perevedentsev, V. Strakhovenko from INP (Novosibirsk) as well as A. Hutton and V. Ziemann from SLAC. K. Bane, S.A. Heifets, S.Kheifets. P. Chen, R. Siemann and J. Tennyson from SLAC are acknowledged for their encouriging interest in this paper.

Appendix

Integral equation (22)

$$\Lambda w(k) = K_0(k) \int\limits_0^k dk' I_0(k') w(k') + I_0(k) \int\limits_k^\infty dk' K_0(k') w(k') \qquad (A.1)$$

can be solved by transformation into a differential equation. After double differentiation over k Eq. (A.1) gets the form

$$\Lambda w''(k) = K_0''(k) \int\limits_0^k dk' I_0(k') w(k') + \\ I_0''(k) \int\limits_k^\infty dk' K_0(k') w(k') + w(k)(K_0' I_0 - I_0' K_0) \quad . \qquad (A.2)$$

Using here the differential equation for modified Bessel functions of the 0-th order Ref. [6]:

$$K_0'' = K_0 - K_0'/k \ , I_0'' = I_0 - I_0'/k$$

and

$$I_0' = I_1 \ , K_0' = -K_1 \ , \ K_1 I_0 + I_1 K_0 = 1/k \quad ,$$

one can transform Eq. (A.2) into

$$w'' + \frac{w'}{k} + \left[\frac{1}{\Lambda k} - 1\right] w = 0 \quad . \tag{A.3}$$

Then the substitution

$$z = 2k \ , \ w = exp(-z/2)v(z)$$

yields the following equation for $v(z)$

$$v'' + (1-z)v' + \frac{1-\Lambda}{2\Lambda} v = 0 \quad . \tag{A.4}$$

Solution of this equation will coincide with Laguerre polynomials Ref. [6]

$$v(z) = c_j L_j(z)$$

and, therefore, w will decay, when $z \to \infty$, provided

$$1/\Lambda - 1 = 2j, \ j = 0, \ 1, \ \ldots \tag{A.5}$$

The normalization condition

$$\int_0^\infty dk w_j(k) w_{j'}(k) = \delta_{jj'} \frac{|c_j|^2}{2} \int_0^\infty dz exp(-z) L_j^2(z) = \delta_{jj'} \frac{|c_j|^2}{2} = 1$$

yields eigenfunctions Eq. (23).

References

1. N. S. Dikansky, D. V. Pestrikov. Collective beam-beam phenomena. Proc. of the 3rd Advance—d ICFA Beam Dynamic Workshop on Beam-Beam Effects in Circular Colliders.

2. K. Hirata. Phys. Rev. D–37, (1988), 1307.

3. A. Chao, R. Ruth. Part. Acc., 16, (1985), 201.

4. See, for instance, in the book: N. S. Dikansky, D. V. Pestrikov. Physics of intense beams in storage rings. Nauka, Novosibirsk 1989; (in Russian).

5. Ya. S. Derbenev. Proc. of the 3rd All Union Part. Acc. Conference. Moscow 1972, Nauka (1973), v.1, 382.

6. H. Bateman, A. Erdelyi. Higher Transcendental Functions. v.2, McGraw-Hill (1953).

7. T. Ieiri, et al. NIM, A265, (1988), 364.

8. N. S. Dikansky, D. V. Pestrikov. Part. Acc., v.12, (1982), 27.

9. Ya. S. Derbenev, N. S. Dikansky. Proc. of the All Union Part. Acc. Conf., Moscow 1968, Moscow (1970), 391.

Figures

Fig. 1. Schematic dependence of collective response of bunches on the dimensionless frequency of the excitation, the damping rates in these calculations were assumed the same ($\delta_j = .1$) for all modes; arbitrary units.

Fig. 2. Effective potential for dipole ($m = \pm 1$) oscillations; from top to bottom $\Lambda_m = 2, .5, .2$.

Fig. 3. Effective potential for quadrupole ($m = \pm 2$) oscillations; from top to bottom $\Lambda_m = 2, .5, .2$.

Fig. 4. Effective potential for sextupole ($m = \pm 3$) oscillations; from top to bottom $\Lambda_m = 2, .5, .2$.

Fig. 5. Effective potential for dipole $(m = \pm 3)$ oscillations; $\Delta_0 = -.05$, $\xi = .05$, from top to bottom $-i\nu/m = .1, .075, .05, .025$.

Fig. 6. Effective potential for dipole $(m = \pm 1)$ oscillations; $\xi = .05$, $\nu = i.01$, for the top line $\Delta_0 = 0$, then from the bottom $\Delta_0 = -.125, -.025, -.375$.

Fig. 7. Effective potential for dipole $(m = \pm 1)$ oscillations; $\xi = .05$, $\nu = i.01$, from top to bottom $\Delta_0 = 0, -.0125, -.015, -.02$.

Fig. 8 Effective potential for dipole $(m = \pm 1)$ oscillations; $\xi = .05$, $\nu = i.01$, from top to bottom $\Delta_0 = -.4, -.35, -.3, -.25$.

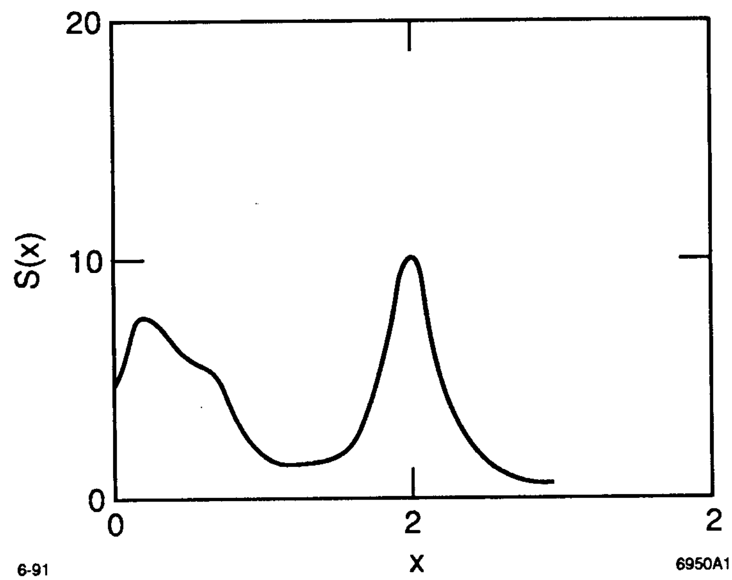

Fig. 1

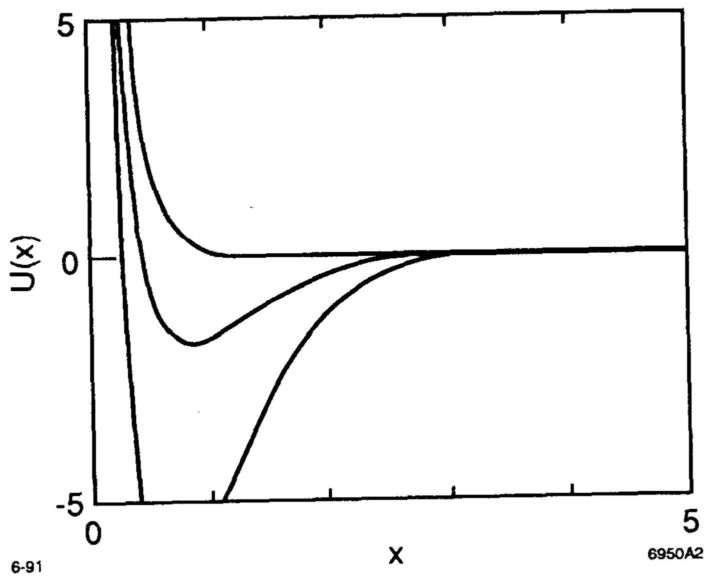

Fig. 2

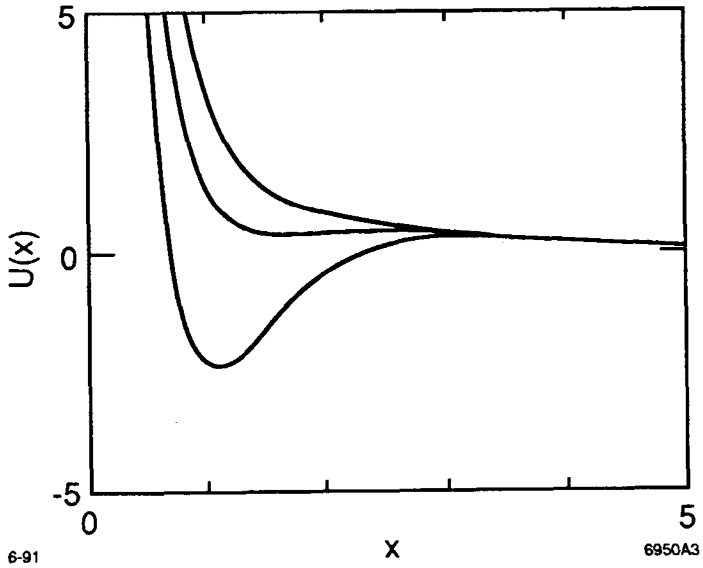

Fig. 3

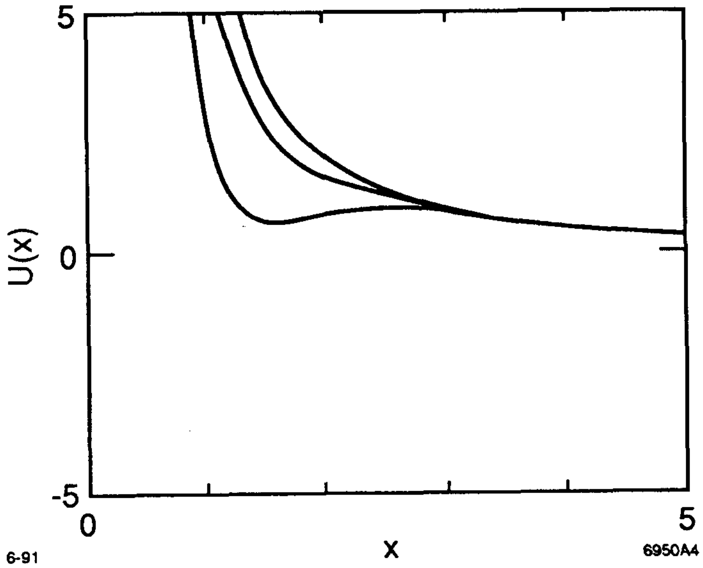

Fig. 4

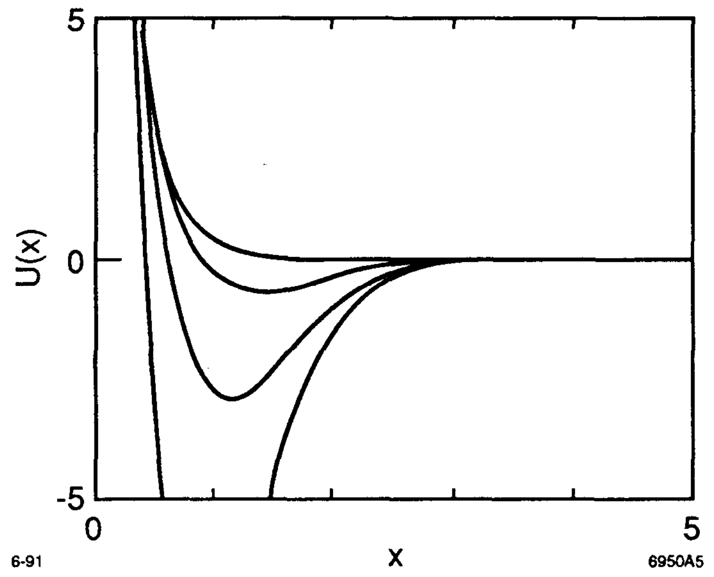

Fig. 5

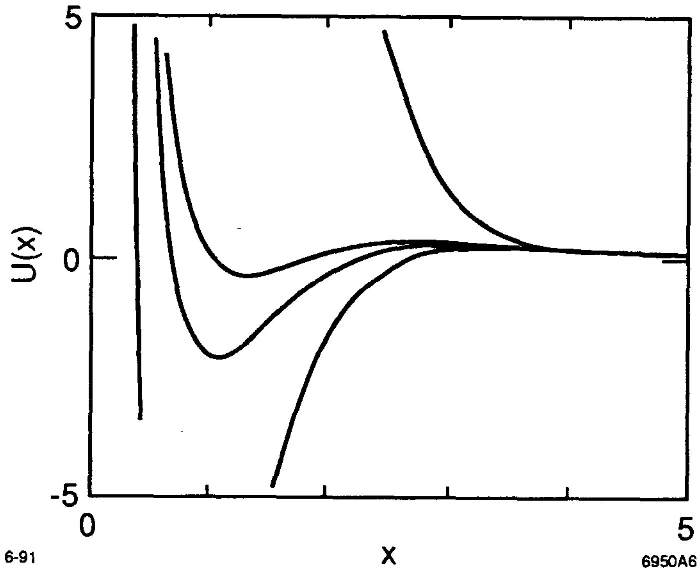

Fig. 6

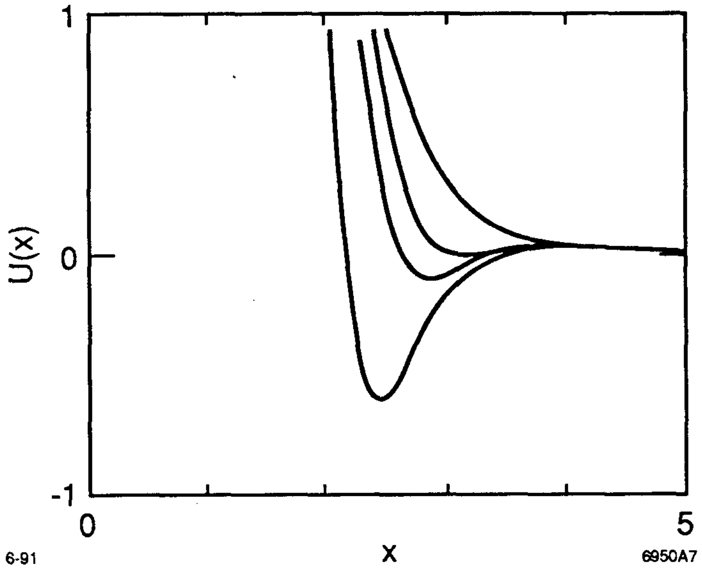

Fig. 7

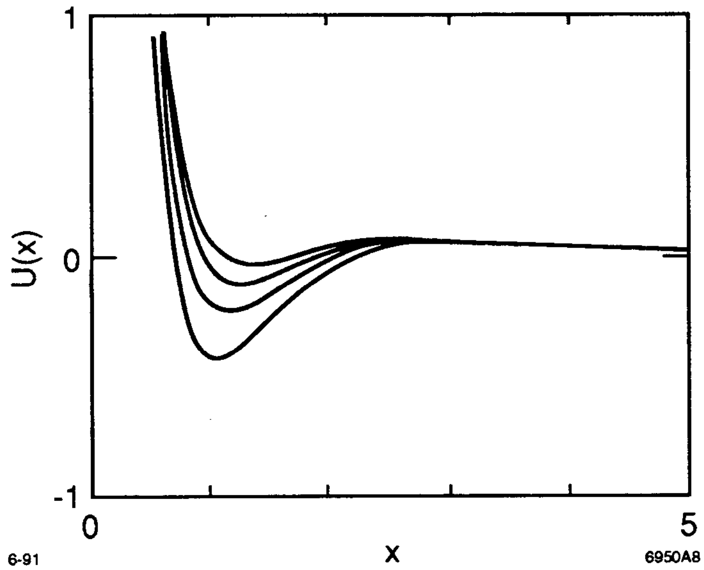

Fig. 8