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POPBCI – A POST-PROCESSOR FOR CALCULATING BEAM COUPLING IMPEDANCES IN HEAVILY DAMPED ACCELERATING CAVITIES (*)

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Abstract

An algorithm is presented, well suited for the calculation of the longitudinal and transverse shunt impedances of accelerating cavities connected to dissipative loads through waveguides. The algorithm is based on an eigenfunction expansion of the field inside the structure, which makes use of the resonating frequencies and of the modal fields of the lossless cavity obtained by replacing the loads with shorts. The algorithm has been implemented in the code POPBCI, which can process the data obtained by any electromagnetic solver capable of finding the resonances of lossless cavities.

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1 - Introduction

A stringent requirement in the design of accelerating cavities for high intensity storage rings is the prevention of bunch-to-bunch beam instabilities due to the excitation of Higher Order Modes (HOM) by the beam itself. The conditions for the onset of these instabilities can be formulated in terms of the frequency behavior of the longitudinal and transverse beam coupling impedances [1]. These conditions show that instabilities are highly probable when the impedances exhibit closely spaced sharp resonance peaks, like those due to the HOMs of the accelerating cavities. The primary cure for preventing coupled bunch instabilities consists in eliminating or smoothing these peaks, without lowering the longitudinal impedance of the fundamental (accelerating) mode. In existing machines this is accomplished by providing the accelerating cavities with HOM dampers, most of them consisting of absorbing loads connected to - the cavity through waveguide sections and suitable irises or loops. The waveguides are dimensioned in order to place their cutoff frequency between the frequency of the fundamental mode and the frequency of the first HOM. In this way-the dampers do not appreciably affect the Q-factor of the fundamental mode, whereas the Q-factors of HOMs are lowered to different extents, depending on their coupling with the propagating waveguide mode(s).

For these reasons a very effective damping of HOM can be achieved only by a substantial coupling between the waveguides and the cavity, which, in turn, requires that they are connected together through large apertures. Unfortunately, the design of the overall structure, and in particular the calculation of the coupling impedances at the beam harmonics, is a difficult task. In fact, the coupling impedances cannot be obtained straightforwardly from the most popular electromagnetic packages, like ARGUS and MAFIA, which are used for the analysis of three-dimensional resonators, since they are limited to Dirichlet and Neumann boundary conditions, and cannot take into account the presence of the dissipative loads at the end of the waveguides. Following a procedure recently proposed by Kroll and Yu [2], it is possible to obtain, at least for the first HOMs, the reduction of the Q-factors due to the loading by the waveguide(s). This is accomplished by considering, for each mode, the shift of its resonance when the waveguide(s) are short-circuited at different distances from the cavity. In cases where the field pattern inside the resonator does not change much when the shorts are replaced by absorbing termination, the reduction of the Q's can give the reduction in the coupling impedances. Unfortunately, the higher the coupling (and thus the more effective the damping), the harder it is to justify the above assumption, thus preventing the effective use of Kroll's method to determine the impedance in heavily damped cavities.

HOM-free accelerating structure have been studied at the Department of Electronics of the University of Pavia (Italy) during the last two years, originally in the framework of a consulting agreement with the Soc. SINCROTRONE TRIESTE. In a recent paper [3], a strategy was suggested for realizing non conventional resonators (named "Single Trapped Mode Resonators", or STMR), which are more similar to a symmetric waveguide junction terminated by absorbing loads, than to a cavity with dampers. The central region of these structures, where the beam interaction takes place, communicates with the waveguides through very large apertures. With a careful dimensioning of the whole structure it is possible that the central region traps a single high-Q resonating mode (used for the acceleration) whereas, above the frequency of this mode, no other high-Q resonance is possible due to the strong coupling to the loads. As a result, the longitudinal and transverse beam coupling impedances are very smalf and smooth at any frequency, except for the single peak of the longitudinal impedance at the resonating frequency of the trapped mode.

A detailed analysis of a specific kind of STMRs has been carried out, and will be published in a forthcoming paper [4]. It refers to cylindrical (two-dimensional) resonators (see fig. 1), consisting of a circular inner body connected through very large apertures to three rectangular waveguides, having the same height of the central body, radially oriented and symmetrically placed around the z-axis. The structure has a 3-fold symmetry and exhibits three symmetry planes passing through the z-axis, and one symmetry plane perpendicular to the same axis. The key point of that analysis is the development of an algorithm which yields the transverse and longitudinal beam coupling impedances as a function of the frequency, starting from the resonating frequencies and from the modal fields of the resonator obtained replacing the absorbing termination with shorts. Although the algorithm was developed with particular reference to the case of planar resonators, the theoretical approach is quite general, and can also be applied to different, more complicated geometries, such as those of the accelerating structure recently proposed for the asymmetric LBL-SLAC B-Factory [5].

This report is primarily intended to describe the modifications to the existing algorithm needed to take into account the three-dimensional geometry of the structure. It also presents the code POPBCI (POst Processor for Beam Coupling Impedances calculation), which implements the algorithm. POPBCI can be used to post-process the data from any three-dimensional electromagnetic code able to calculate the first resonating frequencies and modal fields of the lossless cavity obtained replacing the waveguide absorbing terminations with shorting plates. It gives the real part of the beam coupling impedances at any frequency up to a maximum value of 0.7 - 0.8 times the frequency of the highest resonance considered. Since the effect of the apertures between the accelerating structure and the beam pipe is neglected, the obtained results are meaningful only up to the cutoff frequency the beam pipe; however, at higher frequencies, the HOM suppression is enhanced by the additional energy leakage through the beam pipe.

Sec. 2 is devoted to the evaluation of the axial component of the electric field induced by the beam, which is used in Sec. 3 and 4 to derive the expressions of the longitudinal and transverse impedances. The code POPBCI is described in details in Sec. 5, and some numerical tests are reported in Sec. 6.

Using POPBCI it is possible to optimize the performances of the accelerating structure: in fact, as it is shown by the examples reported in Sec. 6, it is possible to design the whole structure, in particular the shape of the region between the central body and the waveguides, in order to obtain the best trade-off between the HOM damping and the impairment of the longitudinal shunt impedance of the accelerating mode due to the insertion of the waveguides.

2 - Evaluation of the field generated by the beam

Let us consider the structure shown in fig. 2: it consists of a central body connected through large apertures to three waveguides, symmetrically placed around the z-axis. The ports S_1 , S_2 , S_3 are defined on the waveguides. The structure has a 3-fold symmetry and exhibits three symmetry planes passing through the z-axis. The field will be studied in the volume V bounded by the conducting walls and by the ports S_1 , S_2 , S_3 . The effect of the apertures between the accelerating structure and the beam tube is neglected, so that the obtained results are meaningful only up to the cutoff frequency the beam tube. In the volume V the current density of a beam harmonic is represented by

$$J = I_o f(x,y) e^{-jhz} u_z$$
⁽¹⁾

where: *h* is the wavenumber of the harmonic ($h = \omega/velocity$ of particles); u_z is the unit vector of the z-axis; I_o is the current intensity flowing across the z = 0 plane; *f* is a function of the transverse coordinates differing from zero only near the z-axis, in the small region crossed by particles. We have:

$$\int_{S} f \, dx \, dy = 1 \qquad \int_{S} x \, f \, dx \, dy = \overline{x} \qquad \int_{S} y \, f \, dx \, dy = \overline{y} \tag{2}$$

where *S* is the section of the beam pipe and \overline{x} , \overline{y} are the coordinates of the beam axis, which are very small with respect to the transverse dimension of our structure. These coordinates will be considered zero in order to find the fields useful for the evaluation of the longitudinal impedance, whereas they will differ from zero in the case of the calculation of the transverse impedance.

According to the theory of cavity resonators (see [6] for instance) and assuming perfectly conducting walls, the electric field E and the magnetic field H in the bounded region V can be represented by the eigenvector expansions

$$E = -\frac{\eta}{jk} \sum_{i} f_{i} \int_{V} f_{i} \cdot \mathcal{J} \, dV +$$

-
$$\sum_{i} \frac{jk\eta \int_{V} \mathcal{E}_{i} \cdot \mathcal{J} \, dV + k_{i} \sum_{l=1}^{3} \int_{m} \int_{S_{m}} n \times E \cdot \mathcal{H}_{i} \, dS_{m}}{k_{i}^{2} - k^{2}}$$
(3a)

$$H = -\frac{1}{jk\eta} \sum_{i} g_{i} \sum_{l}^{3} \sum_{m} \int_{S_{m}} n \times E \cdot g_{i} dS_{m} + \sum_{i} \frac{k_{i} \int_{V} \mathcal{E}_{i} \cdot \mathcal{I} dV - \frac{jk}{\eta} \sum_{l}^{3} \sum_{m} \int_{S_{m}} n \times E \cdot \mathcal{H}_{i} dS_{m}}{k_{i}^{2} - k^{2}} \mathcal{H}_{i}$$
(3b)

where: vectors f_i are the irrotational electric eigenvectors; vectors g_i are the irrotational magnetic eigenvectors; vectors E_i and H_i are the electric and the magnetic divergenceless eigenvectors respectively; k_i is the resonating wavenumber of the *i*-th mode of the ideal cavity obtained when the ports are shorted, numbered in the non decreasing order; $\eta = 377 \Omega$ is the characteristic impedance of vacuum; n is the outward unit vector normal to a port. Eigenvectors are normalized according to

$$\int_{V} \boldsymbol{f}_{i} \cdot \boldsymbol{f}_{i}^{*} dV = 1; \quad \int_{V} \boldsymbol{g}_{i} \cdot \boldsymbol{g}_{i}^{*} dV = 1; \quad \int_{V} \boldsymbol{\mathcal{E}}_{i} \cdot \boldsymbol{\mathcal{E}}_{i}^{*} dV = 1; \quad \int_{V} \boldsymbol{\mathcal{H}}_{i} \cdot \boldsymbol{\mathcal{H}}_{i}^{*} dV = 1 \quad (3c)$$

It is remembered that, apart from the normalization, \boldsymbol{E}_i and \boldsymbol{H}_i coincide with the electric and the magnetic field of the *i*-th resonating mode of the ideal cavity of volume *V*.

Expressions (3) show that the field induced by the beam inside the volume V is determined by J (the current density exciting the structure) and by $n \times E$ at the ports, i.e. the electric field transverse to the waveguides. This field, in turn, depends on the beam current and on the boundary condition due to the absorbing terminations.

The field in the waveguides can be considered as superimposition of *TE* and *TM* guided modes. For this reason the transverse electric and magnetic field at the *m*-th port $E_T^{(m)}$ and $H_T^{(m)}$ can be represented as follows:

$$H_{T}^{(m)} = \sum_{0}^{P} I_{p}^{(m)} h_{p} \qquad n \times E_{T}^{(m)} = \sum_{0}^{P} V_{p}^{(m)} h_{p} \qquad (4)$$

where h_p denotes the magnetic modal vector of the *p*-th mode, satisfying the ortho-normalization condition

$$\int_{S_m} h_p \cdot h_q^* \, dS_m = \delta_{pq} \qquad (\delta_{pq} = Kroneker's \, symbol) \tag{5}$$

and $I_{p}^{(m)}, V_{p}^{(m)}$ represent the mode current and the mode voltage for the *p*-th mode at the *m*-th port (note that throughout this report superscripts refer to the ports, whereas subscripts are usually modal indices). In the summations (4)

TE and TM modes are labelled using a single index, and are numbered according to the non-decreasing order of their cut-off frequency. Though the summations should consist of an infinite number of terms, the first P terms only are retained, corresponding to the modes which propagate or are only a little below cutoff at the maximum frequency of interest. In fact, these are the only modes that have a significant amplitude at the ports.

From (4) and (5) we have:

$$I_{p}^{(m)} = \int_{S_{m}} H_{T}^{(m)} \cdot h_{p} \, dS_{m} \tag{6a}$$

$$V_{p}^{(m)} = \int_{S_{m}} (\mathbf{n} \times \mathbf{E}_{T}^{(m)}) \cdot \mathbf{h}_{p} \, dS_{m}$$
(6b)

The modal currents and voltages, due to the absorbing terminations, must be related by the following relations:

$$\frac{I_{p}^{(m)}}{V_{p}^{(m)}} = Y_{p}$$
(7)

where Y_p is the admittance of the termination for the *p*-th mode. Using (3b) to express $H_T^{(m)}$ in (6a), and taking into account exp. (1) and (4) we can write:

$$I_{p}^{(m)} = -\sum_{l=1}^{3} \sum_{n=1}^{P} Y_{pq}^{(mn)} V_{q}^{(n)} + T_{p}^{(m)} I_{o}$$
(8)

where:

$$Y_{pq}^{(mn)} = \frac{1}{jk\eta} \sum_{i} \int_{S_{m}} \mathbf{g}_{i} \cdot \mathbf{h}_{p} \, dS_{m} \int_{S_{n}} \mathbf{g}_{i} \cdot \mathbf{h}_{q} \, dS_{n} + \frac{jk}{\eta} \sum_{i} \frac{\int_{S_{m}} \mathbf{H}_{i} \cdot \mathbf{h}_{p} \, dS_{m} \int_{S_{n}} \mathbf{H}_{i} \cdot \mathbf{h}_{q} \, dS_{n}}{k_{i}^{2} - k^{2}}$$
(9a)

$$T_{p}^{(m)} = \sum_{i} \frac{k_{i} \int_{S_{m}} \mathcal{H}_{i} \cdot h_{p} \, dS_{m}}{k_{i}^{2} - k^{2}} \int_{V} u_{z} \cdot \mathcal{E}_{i} f \, e^{-jhz} \, dV \qquad (9b)$$

Introducing the boundary condition (7) into (8) we obtain:

$$Y_{p} Z_{p}^{(m)} + \sum_{l=1}^{3} \sum_{n=1}^{P} Y_{pq}^{(mn)} Z_{q}^{(n)} = T_{p}^{(m)}$$
(10)

where

$$Z_{p}^{(m)} = V_{p}^{(m)} / I_{o}$$
(11)

is the transimpedance between the beam current and the mode voltage of the p-th mode at the m-th port. Writing eq. (10) for all the ports and the modes we obtain a system consisting of $3 \times P$ equations from which we determine the $3 \times P$ unknown transimpedances. Once the transimpedances are known, it is possible to obtain the electric field generated at the ports by a given beam current and thus, making use of exp. (3), to determine the electromagnetic field everywhere. The z-component of the electric field is important for the calculation of the beam coupling impedances; it is given by:

$$E_{z} = -\frac{\eta I_{o}}{jk} \sum_{i} u_{z} \cdot f_{i} \int_{V} u_{z} \cdot f_{i} f e^{-jhz} dV -$$

$$I_{o} \sum_{i} u_{z} \cdot \mathbf{E}_{i} \frac{jk\eta \int_{V} u_{z} \cdot \mathbf{E}_{i} f e^{-jhz} dV + k_{i} \sum_{I}^{3} \sum_{m} \sum_{I}^{P} Z_{p}^{(m)} \int_{S_{m}} \mathbf{H}_{i} \cdot h_{p} dS_{m}}{k_{i}^{2} - k^{2}} \qquad (12)$$

The slow convergence of the summations involving the index *i*, both in (12) and in (9a,b), makes the use of the algorithm described so far almost impractical, since it would be necessary to evaluate numerically a large number of irrotational and divergenceless eigenvectors. In [4] a procedure is presented, which transforms the above-mentioned summations into much more rapidly converging ones, well suited to the implementation of an efficient numerical code.

The procedure basically consists in extracting, from the slowly-converging series, their limits when $k \rightarrow 0$. These limots are then re-expressed in a more convenient way. Following that procedure, not reported here for brevity, exp. (9a,b) can be approximated as follows:

$$Y_{pq}^{(mn)} \approx jB_p \,\delta_{mn} \,\delta_{pq} + \frac{jk^3}{\eta} \sum_i \frac{\int_{s_m} \mathcal{H}_i \cdot h_p \,dS_m \int_{s_m} \mathcal{H}_i \cdot h_q \,dS_n}{k_i^2 \,(k_i^2 - k^2)}$$
(13a)

$$T_{p}^{(m)} \approx k^{2} \sum_{i} \frac{\int_{S_{m}} \mathcal{H}_{i} \cdot h_{p} \, dS_{m}}{\sum_{V} \frac{\int_{V} \mathcal{H}_{i} \cdot h_{p} \, dS_{m}}{k_{i} \, (k_{i}^{2} - k^{2})}}$$
(13b)

where:

$$jB_{p} = \begin{cases} \frac{1}{\eta} \left(\frac{k_{cp}}{jk} + \frac{jk}{2k_{cp}} \right) & (if \ p-th \ mode \ is \ a \ TE \ mode) & (14a) \\ \\ j\frac{k}{\eta k_{cp}} & (if \ p-th \ mode \ is \ a \ TM \ mode) & (14b) \end{cases}$$

In exp. (14) k_{cp} is the cut-off wavenumber of the *p*-th mode. It can be shown that exp. (13) are very accurate if the lengths of the lateral waveguides are $\geq \pi/k_{cl}$.

Since the electric field is to be determined in the axial region only, exp. (12) can be transformed as follows:

$$E_{z} = E_{qs_{z}} - j\eta k^{3} I_{o} \sum_{i} \frac{u_{z} \cdot \mathbf{E}_{i}}{k_{i}^{2} (k_{i}^{2} - k^{2})} \int_{V} u_{z} \cdot \mathbf{E}_{i} f e^{-jhz} dV - k^{2} I_{o} \sum_{i} \frac{u_{z} \cdot \mathbf{E}_{i}}{k_{i} (k_{i}^{2} - k^{2})} \sum_{I}^{3} m \sum_{I}^{P} \sum_{p} Z_{p}^{(m)} \int_{S_{m}} \mathbf{H}_{i} \cdot \mathbf{h}_{p} dS_{m}$$
(15)

where E_{qs_z} , which represents the quasi-static approximation of E_z , is given by:

$$E_{qs_z} = -\frac{\eta I_o}{jk} \sum_i u_z \cdot f_i \int_V u_z \cdot f_i f e^{-jhz} dV - jk\eta I_o \sum_i \frac{u_z \cdot \boldsymbol{\mathcal{E}}_i}{k_i^2} \int_V u_z \cdot \boldsymbol{\mathcal{E}}_i f e^{-jhz} dV \quad (16)$$

Using exp. (13-16) in place of (9),(10),(12) is far more convenient: in fact the series in (13) and (15) converge much faster than the original ones, because their terms go to zero as fast as k_i^3 or k_i^4 . This permits one to truncate them, retaining only a limited number of terms. Moreover, the knowledge of the irrotational magnetic eigenvectors is not needed, whereas irrotational electric eigenvectors contribute to E_{qs_z} only. On the other hand, the quasi-static field E_{qs_z} in the axial region is practically unaffected by the waveguides: therefore, as discussed in detail in [4], it can be evaluated approximately, without making use of (16), considering a simpler structure where the apertures connecting the central body to the waveguides are replaced with electric walls. As a matter of fact, in the following sections it will be shown that E_{qs_z} contributes to the coupling impedances only through a slowly-varying reactive term, and therefore it is not of prime concern in the evaluation of the effect of HOM on coupled bunch instability, since the condition for the beam instability [1] involves only the real part of the coupling impedances.

3 - Calculation of the longitudinal beam coupling impedance

The longitudinal beam coupling impedance is defined as

$$Z_{||} = -\frac{1}{|I_o|^2} \int_V E \cdot J^* dV = -\frac{1}{|I_o|} \int_V E_z f e^{jhz} dV \qquad [\Omega]$$
(17)

where *E* is the electric field generated by a centered beam. Substituting (15) into (17), and denoting by \mathcal{E}_i the z-component of \mathbf{E}_i , we have:

$$- Z_{||} = jX_{qs_{||}} + jk^{3}\eta \sum_{i} \frac{\left| \int_{V} \mathcal{E}_{i} f e^{jhz} dV \right|^{2}}{k_{i}^{2} (k_{i}^{2} - k^{2})} +$$

$$k^{2} \sum_{i} \frac{\int_{V} \mathcal{E}_{i} f e^{jhz} dV}{k_{i} (k_{i}^{2} - k^{2})} \sum_{l}^{3} \sum_{m} \sum_{l}^{P} Z_{p}^{(m)} \int_{S_{m}} \mathcal{H}_{i} \cdot h_{p} dS_{m}$$
(18a)

where

$$X_{qs_{||}} = \frac{j}{I_o} \int_V E_{qs_z} f e^{jhz} dV$$
(18b)

Note that, according to exp. (16), $X_{qs_{||}}$ is a real quantity, so that its contribution to the impedance is purely reactive.

Let us consider the volume integrals in (18a). They appear in rapidly converging series, where only the first resonating modes are involved. Due to this fact, it is possible to assume that the transverse variations of \mathcal{E}_i are smooth in the beam region so that, in calculating the above integral, it is possible to use the approximation:

$$\mathcal{E}_{i} \approx (\mathcal{E}_{i})_{o} + x (\partial_{x} \mathcal{E}_{i})_{o} + y (\partial_{y} \mathcal{E}_{i})_{o}$$

where $\partial_u \equiv \partial/\partial u$ and the subscript "*o*", from now on, denotes quantities evaluated at x = y = 0. Using this approximation and remembering the properties of the function *f* it is easily shown that:

$$\int_{V} \mathcal{E}_{i} f e^{jhz} dV = \int_{0}^{L} (\mathcal{E}_{i})_{0} e^{jhz} dz + \overline{x} \int_{0}^{L} (\partial_{x} \mathcal{E}_{i})_{0} e^{jhz} dz + \overline{y} \int_{0}^{L} (\partial_{y} \mathcal{E}_{i})_{0} e^{jhz} dz$$
(19)

where *L* represents the length of the interaction region. For a centered beam $\overline{x} = \overline{y} = 0$, and we have

$$\int_{V} \mathcal{E}_{i} f e^{jhz} dV = \int_{0}^{L} \left(\mathcal{E}_{i} \right)_{o} e^{jhz} dz$$
(20)

Exp. (9), (10), (13), (14), (18a) and (20) permit to evaluate the longitudinal shunt impedance (a part from the quasi-static term). In practice, the series involving cavity modes are truncated, retaining the first M terms. The rapid convergence of these series assures that the calculation is very accurate up to a frequency about 0.7 times the resonating frequency of the highest mode considered.

In the case of a centered beam it is possible to reduce exp. (10),(13) and (18a)to simpler ones. In fact, as discussed in more detail in [4], the modes of the short-circuited structure can be classified according to their symmetry with respect to the axis and to one of the symmetry planes passing through the waveguides. We have "symmetric" modes (having a three-fold symmetry with respect to the z-axis) and "asymmetric" modes (all other modes); in each one of these two categories we can further distinguish modes exhibiting either a magnetic wall condition ("even" modes) or an electric wall condition ("odd" modes) on the symmetry plane. Therefore anyone of the modes can be classified into one of the following four categories: SE, SO, AE, AO. The only modes having $(\mathcal{E}_i)_0 \neq 0$ are the SE modes; thus only terms corresponding to SE modes can be retained in the modal series in (13) and (18a). When this is done, the integrals over the ports in (13) and (18a), for a given pair of cavity and waveguide modes (involving the *i*-th cavity mode and the *p*-th waveguide mode), yield the same value, independently of the port where they are evaluated. Moreover the transimpedances $Z_{p}^{(m)}$ for the *p*-th mode do not depend on *m*, and we have:

$$Z_{p}^{(1)} = Z_{p}^{(2)} = Z_{p}^{(3)} \equiv Z_{p}$$

Therefore, taking into account exp (20), it is possible to transform exp. (18a) as follows:

$$Z_{||} = jX_{qs_{||}} + jk^3\eta \sum_{l=i}^{M} \frac{|D_i|^2}{k_i^2(k_i^2 - k^2)} + 3k^2 \sum_{l=i}^{M} \frac{D_i^*}{k_i(k_i^2 - k^2)} \sum_{l=i}^{P} S_{ip} Z_p \quad (21a)$$

where

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$$D_i = \int_0^\infty \left(\mathcal{E}_i \right)_0 e^{-jhz} dz \tag{21b}$$

$$S_{ip} = \int_{S_I} \mathcal{H}_i^{SE} \cdot \boldsymbol{h}_p \, dS_I \tag{21c}$$

Transimpedances Z_p are determined solving the following set of equations, which are derived from (10) and (13):

$$(Y_{p} + jB_{p}) Z_{p} + 3 \frac{jk^{3}}{\eta} \sum_{l}^{P} \sum_{q}^{M} \sum_{l}^{M} \frac{S_{ip} S_{iq}}{k_{i}^{2}(k_{i}^{2} - k^{2})} Z_{q} = k^{2} \sum_{l}^{M} \frac{S_{ip} D_{i}}{k_{i}(k_{i}^{2} - k^{2})} \qquad (p = 1, 2, ..., P)$$
(22)

Note that in exp. (21a) and (22), thanks to symmetry, the summation over the ports has been replaced by the multiplicative factor of 3. This reduces the order of the system (22) from $3 \times P$ to P, thus saving computer time and storage.

4 - Calculation of the transverse beam coupling impedance

The transverse momentum acquired by a unit charge travelling across the interaction structure at the transverse coordinates x,y is equal to cv_{\perp} (*c*=velocity of light), where the vector v_{\perp} is the so-called *deflecting voltage*, given by [7]:

$$\boldsymbol{v}_{\perp} = \boldsymbol{v}_{\perp}(x,y) = -\frac{1}{jk} \int_{0}^{L} \boldsymbol{\nabla}_{\perp} E_{z} e^{jhz} dz$$

(∇_{\perp} denotes the gradient transverse to z). The average value of the deflecting voltage due to the interaction between the beam and the accelerating structure is:

$$V_{\perp} = \int_{S} f v_{\perp} dx dy = -\frac{1}{jk} \int_{V} f \nabla_{\perp} E_{z} e^{jhz} dV$$

where E_z is the z-component of the electric field generated by the beam itself.

In [4] it is shown that, provided the function f is symmetric with respect to the point $(\overline{x}, \overline{y})$, the symmetry of the structure guarantees that a symmetric field is generated if the beam is centered $(\overline{x} = \overline{y} = 0)$, thus yielding a zero deflecting voltage. An off-axis beam produces a deflecting voltage which can be consid-

ered as a linear combination of \overline{x} and \overline{y} , provided the beam displacement is small. Moreover, thanks to the fact that *AE* and *AO* modes (the only modes having a non-zero $\nabla_{\perp} E_z$ in the axial region and then contributing to V_{\perp}) occur in degenerate pair, V_{\perp} is aligned with the displacement of the beam. Therefore, as for the more usual axisimmetric cavities, also in this case it is possible to write:

$$\mathbf{V}_{\perp} = -j \, Z_{\perp} \, I_o \left(\overline{\mathbf{x}} \, \mathbf{u}_{\mathbf{x}} \, + \, \overline{\mathbf{y}} \, \mathbf{u}_{\mathbf{y}} \right) \tag{23}$$

where u_x and u_y are the unit vectors in the directions of the *x*- and *y*-axis. The scalar coefficient Z_{\perp} represents the "transverse beam coupling impedance", which can be calculated using the formula

$$Z_{\perp} = -\frac{1}{k I_o \bar{x}} \int_V f \, \partial_x E_z \, e^{jhz} dV \qquad [\Omega/m] \qquad (24)$$

where E_z now represents the *z*-component of the field generated by a beam displaced in the *x* direction ($\overline{y} = 0$)

Introducing (15) into (24) we obtain an expression containing rapidly converging series whose terms depend on the integrals

$$\int_{V} f \,\partial_{x} \mathcal{E}_{i} e^{jhz} dV \approx \int_{0}^{L} (\partial_{x} \mathcal{E}_{i})_{o} e^{jhz} dz$$

which differ from zero only for the *AE* modes. Therefore only these modes are involved in the calculation of (24). Since the electric field at the *z*-axis is zero for these modes, according to (19) we can write

$$\int_{V} f \mathcal{E}_{i} e^{-jhz} dV \approx \overline{x} \int_{0}^{L} (\partial_{x} \mathcal{E}_{i})_{o} e^{-jhz} dz$$

Using the last two equations, upon substitution of (15) into (24), we have:

$$Z_{\perp} = jX_{qs_{\perp}} + jk^{2}\eta \sum_{i=1}^{M} \frac{\left| \int_{0}^{L} (\partial_{x}\mathcal{E}_{i})_{o} e^{jhz} dz \right|^{2}}{k_{i}^{2} (k_{i}^{2} - k^{2})} + k\sum_{i=1}^{M} \frac{\int_{0}^{L} (\partial_{x}\mathcal{E}_{i})_{o} e^{jhz} dz}{k_{i} (k_{i}^{2} - k^{2})} \sum_{i=1}^{3} \sum_{j=1}^{P} z_{p}^{(m)} \int_{S_{m}} \mathcal{H}_{i}^{AE} \cdot h_{p} dS_{m}$$
(25)

where:

$$X_{qs_{\perp}} = \frac{j}{k I_o \bar{x}} \int_V f \ \partial_x E_{qs_z} e^{jhz} dV$$

As in the case of the longitudinal impedance, the quantity $X_{qs_{\perp}}$ is real, and then the quasi-static term in (25) is a pure reactance. Quantities $z_{p}^{(m)}$ – representing $Z_{p}^{(m)}/\bar{x}$ – are solutions of a system of equations derived from (10), which can be manipulated, together with exp. (25), in order to take into account the symmetry. Following an argument similar to that used in the previous section, it is seen that, due to the symmetry, we have:

$$z_p^{(2)} = z_p^{(3)} \qquad ; \qquad \qquad \int_{S_2} \mathcal{H}_i^{AE} \cdot h_p \, dS_2 = \int_{S_3} \mathcal{H}_i^{AE} \cdot h_p \, dS_3$$

(ports 2 and 3 are the two ports symmetrically placed with respect to the xzplane, see fig. 2). Then exp. (25) becomes:

$$Z_{\perp} = jX_{qs_{\perp}} + jk^{2}\eta \sum_{l=1}^{M} \frac{|d_{i}|^{2}}{k_{i}^{2}(k_{i}^{2}-k^{2})} + k\sum_{l=1}^{2} m (1+\delta_{m2}) \sum_{l=1}^{M} \frac{d_{i}^{*}}{k_{i}(k_{i}^{2}-k^{2})} \sum_{l=1}^{P} A_{ip}^{(m)} z_{p}^{(m)}$$
(26a)

where

$$d_{i} = \int_{0}^{L} \left(\partial_{x} \mathcal{E}_{i}\right)_{0} e^{-jhz} dz$$
(26b)

$$A_{ip}^{(m)} = \int_{S_m} \mathcal{H}_i^{AE} \cdot h_p \, dS_m \tag{26c}$$

The following set of equations, derived from (10) and (13), determines the $2 \times P$ quantities $z_p^{(m)}$:

$$(Y_{p} + jB_{p}) z_{p}^{(m)} + \frac{jk^{3}}{\eta} \sum_{l=n}^{2} (1 + \delta_{n2}) \sum_{l=q}^{P} \sum_{l=q}^{M} \sum_{l=i}^{M} \frac{A_{ip}^{(n)} A_{iq}^{(n)}}{k_{i}^{2} (k_{i}^{2} - k^{2})} z_{q}^{(n)} = k^{2} \sum_{l=i}^{M} \frac{A_{ip}^{(m)} d_{i}}{k_{i}^{-} (k_{i}^{2} - k^{2})} \qquad (p = 1, 2, ..., P; \ m = 1, 2)$$
(27)

Up to now the effect of wall losses has been neglected. It can be taken into account using the same approximation which is made in the study of forced oscillation of cavity resonators. This approximation consists in making the following substitution everywhere in the exp. (21a), (22), (26a) and (27):

$$\frac{1}{k_i^2 - k^2} \rightarrow \frac{1}{k_i^2 + j\frac{kk_i}{O_i} - k^2}$$
(28)

where Q_i is the quality factor of the i-th mode.

5 - The code POPBCI

The algorithm described in the previous sections has been implemented in the computer code POPBCI (POst Processor for Beam Coupling Impedances calculation). POPBCI is a post-processor which can use the results of any electromagnetic solver, capable of finding the resonances of lossless cavities, to evaluate the longitudinal and transverse coupling impedances at specifyed frequencies.

At first the electromagnetic solver is used to analyze the short-circuited structure; the length of the waveguides is immaterial, provided they are not shorter than half their cut-off wavelength. In the case of rectangular waveguides this criterion leads to waveguides whose lengths are greater than or

equal to their widths. The quantities of interest are those appearing in (21), (22), (26), (27) and (28). More specifically the code used for the modal analysis must provide, for each mode:

i) the resonating frequency;

ii) the *Q*-factor;

iii) the distribution along the beam axis of the z-component of the electric field (or of its *x*-derivative, for the transverse impedance calculation);

iv) the distribution of the magnetic field at the short-circuited ports.

All these quantities are readily obtainable from the electromagnetic solvers commonly used to analyze resonant cavities. Since almost all solvers may exploit symmetries, taking into account the considerations in Sec. 3 and 4, it is convenient analyze only half of the structure, imposing a magnetic wall boundary condition on the xz symmetry plane. This assures that "even" mode only are calculated. If the solver does not permit the calculation of "symmetrical" or "asymmetrical" modes separately, it is necessary to select the two kinds of modes in order to pass to POPBCI the correct data for either longitudinal or transverse impedance calculation. This can be done very easily, looking at the behavior of E_z near the axis.

The present version of POPBCI assumes that the central body is connected to rectangular waveguides. Thus the code computes analytically the cut-off wavenumber k_{cp} and the mode vectors h_p for the first P waveguide modes, in order to evaluate (14), (21c) and (26c). Anyway, the code can be easily modified in order to consider circular waveguides, or even arbitrarily shaped waveguides, provided that k_{cp} and h_p are calculated numerically.

The code consists of two separate programs, suited to calculate either longitudinal or transverse shunt impedances. The general features of the two programs are very similar, and they share many common routines. POPBCI is written in VAX-Fortran 77, and runs under VAX-VMS operating system. It includes some graphic module, based on Tektronix PLOT-10 routines, used to show plots of the impedances versus frequency on Tektronix 41xx, 42xx graphic terminals or compatible equipments. To assure the portability of the code between different operating systems and to permit the use of different graphic packages, the input-output and graphic routines are well separated from the code that performs the actual calculation. This allows them to be easily replaced with other functionally equivalent routines, if the code has to be converted in order to run inside different environments. The tasks performed by the most significant routines are shortly described in the following, with reference to the program for the longitudinal impedances only. More details can be found on the comments included in the source code.

MAIN - The main program has the very simple task of dimensioning some arrays used by subsequent routines and to call in sequence the routines to read the input data (*READ_INPUT_DATA*), to define a table of waveguide modes (*DEFINE_WG_MODES*), to calculate the frequency independent coefficients S_{ip} from exp. (21c) (*CALC_SMAT*) and finally to calculate and plot the longitudinal impedance (*CALC_PLOT*). All the dimensioning of the arrays used throughout the code depend on few parameters, whose values are defined in the MAIN program. They are:

- the maximum number of resonances of the short circuited structure;

- the maximum number of points used to sample E_z ;

- the maximum number of points used to sample \mathcal{H}_i ;

- the maximum number of waveguide modes;

- the maximum number of frequency points in a plot;

All these parameters are checked by the subsequent routines and suitable error messages are issued if an "out of bound" condition is detected.

READ_INPUT_DATA - It reads an input file containing all the data passed to POPBCI by the electromagnetic solver. The file is a formatted one, in order to ease the data exchange between possibly different environments. A free format can be used, and is possible to include comment lines in the input file. The number of resonances and the number of the points used to sample E_z and \mathcal{H}_i are read at first, together with their coordinates; then the routine reads resonating frequencies and Q-factors, and, for each mode, the value of E_z and of \mathcal{H}_i . The magnetic field at the ports is sampled on a possibly irregular rectangular grid.

DEFINE_WG_MODES - This routine defines the indices and the cut-off frequencies of the first *TE* and *TM* modes of a rectangular waveguide. The mode generation is stopped when it reaches a cut-off frequency higher than the maximum resonating frequency of the calculated cavity mode, times a user definable coefficient. This coefficient is 1, at present. The modes are numbered according to the non decreasing order of their cut-off frequencies. In the case of the longitudinal impedance, the only modes that are considered are those having an "even" symmetry (the same symmetry exhibited by H_i). Modal indices, cut-off frequencies and a TE/TM identifier are stored in a table. This routine may be replaced by other functionally equivalent routines in order to consider waveguides of different cross-section.

- CALC_SMAT It calculates numerically the surface integrals (21c), using a simple trapezoidal rule. To obtain good accuracy, sufficiently closely spaced samples of the vectors \mathbf{H}_i must be provided. The values of the waveguide modal vectors \mathbf{h}_p are calculated analytically in the routine *MMFRWG*.
- *MMFRWG* This routine calculates the magnetic modal fields in a rectangular waveguide. The calculation is performed for the *p*-th mode, at a given point. It may be replaced by other functionally equivalent routines in order to consider waveguides of different cross-section.
- *CALC_PLOT* This is an interactive routine used to define the frequency range where the impedances are to be calculated. It permits also to set an accuracy flag used by the routine *CALC_ROUTINE*. This flag affects the accuracy in the evaluation of the impedance near the resonance peak(s). The higher the value of the accuracy flag, the closer is the spacing in frequency.
- *CALC_ROUTINE* The main task of this routine is to change adaptively the frequency step between successive impedance evaluation, in order to achieve a good accuracy near sharp resonance peaks and to save computing time away from resonance, where the impedance variations are smooth. As a result, the frequency values are not regularly distributed. The routine calls the function *ZL*, which returns the value of the impedance. Frequency and impedance values are stored for the subsequent plotting.
- ZL This function is used as a driver to the routines which perform the impedance calculation at a specific frequency: it calls CALC_DMAT to evaluate integrals (21b), ZSOLVER to solve the system (22), CALCZLM to calculate exp. (21a) (a part from the quasi-static term) and eventually returns the longitudinal impedance. The calculation of the quasi-static term (18b) can be added to this routine, if needed.

 $CALC_DMAT$ - It calculates numerically the integrals (21b), using a simple trapezoidal rule. To obtain a good accuracy, a sufficiently closely spaced samples of the vectors E_z must be provided.

ZSOLVER - It solves the complex linear system (22) in order to find the transimpedances Z_p. The values of Y_p and B_p are calculated in the routine *WGYIN* and *WGYQS* respectively. The solution of the system is performed using a standard routine of the *NAG* Mathematical Library. It is also possible
to consider the case of the short-circuited structure (loads replaced by shorts), by setting all the transimpedances to zero.

CALCZLM - This routine evaluates exp. (21a), apart from the quasi-static term (18b).

WGYIN - It calculates the quantity Y_p , i.e. the input admittance of the *p*-th mode, at a given frequency (below or above the cut-off of the mode). At present, the routine can model an infinite hollow waveguide, or a termination consisting of possibly lossy dielectric uniformly filling the waveguide, starting from a section at a given distance from the port. The effect of more complicated termination can be considered, or even measured data can be included, by performing suitable changes to the routine.

- WGYQS This routine calculates the quantity B_p , according to (14), at a given frequency.
- *PLOTTING_ROUTINE* This output routine permits the interactive plotting of the impedances calculated by *CALC_ROUTINE*. All the calls to the graphic package used in this code are in this routine.

6 - Numerical tests

The code POPBCI has been checked by running many test examples concerning different accelerating structures. Two codes were used in the modal analysis of the short-circuited structure: a general purpose 3-D electromagnetic solver (ARGUS) and a specialized 2-D package (PAGODA) developed at the Department of Electronics of Pavia [8]. In all numerical calculations the wall losses were taken into account, considering the conductivity of copper.

The first example concerns the Y-shaped resonator show in fig. 3: It consists of a pill-box connected to three rectangular waveguides through axial slots. The structure is two-dimensional, since its cross-section does not vary along the zaxis. For this reason it was analyzed by means of the 2-D code PAGODA. Different structures with increasing values of the slot width W have been analyzed, in order to consider increasing values of the coupling to the waveguides. The length of the structure was the same (L = 50 mm) for all the computations. Resonances up to 3 GHz were computed; up to this frequency only resonances with no axial variation may occur. Perfectly matched terminations were considered. Figures 4-8 show the plots of the longitudinal impedance versus frequency for W=40 mm, W=60 mm, W=100 mm, W=150 mm and W=200 mmrespectively. Actually, this last case refers to the case where the waveguides are connected directly to the pill-box, without any slot. In each figure the upper plot refers to the structure with loads replaced by shorts, the bottom plot to the structure connected to perfectly matched terminations. In all plots, impedances are reported in a logarithmic scale, normalized to 0.675 Mohm, i.e. to the longitudinal impedance of the fundamental mode of the pill-box. From figures 4-8 it can be seen that a very small HOM damping is obtained using the smallest slot, whereas the 60 mm slot yields an attenuation of about 15 db, and the 100 mm slot gives an attenuation of about30db. A little additional damping for the monopole modes is obtaining by further enlarging or even removing the slot completely (fig. 7,8). The same figures show the impairment of the longitudinal impedances of the fundamental modes due to the increased coupling. Fig. 9 summarize the results, giving the monopole HOM attenuation (in db) and the percentage lowering of the fundamental mode shunt impedance (referred to the plain pill-pox) as a function of the slot width, normalized to the waveguide width.

Fig. 10 and 11 show the transverse impedance of the structure of fig. 3, in the case of a slot width of 100 mm and 150 mm respectively. The reference impedance is 8 Kohm/mm, i.e. the transverse shunt impedance of the first dipole mode in the pill-box. Dipole HOM damping ranging between 24 and 30 db is obtained.

The effect of an imperfect match of the waveguides is reported in fig. 12, which refers to the structure with a 150 mm slot. The waveguides are filled with a lossy dielectric ($\varepsilon_r = 9$, tan $\delta = 0.01$ @ 3 GHz), starting from a section placed at a distance of 200 mm from the ports. Fig. 12a refers to the longitudinal impedance, whereas fig. 12b reports the transverse impedance. It is noted that the increase of the impedance values is fairly small.

Fig. 14 refers to the Y-shaped resonator of fig. 13a. It has the same cross-section of the structure used to obtain the plots of fig 8, but the length of the structure was increased to 200 mm. This is the shape originally considered in [3,4]. This structure was first analyzed using the 2-D code PAGODA, considering resonances up to 3 GHz. Fig. 14a reports the spectrum of the longitudinal impedance of the short-circuited structure, normalized to the value of the longitudinal impedance of the fundamental mode in the pill-box. Note the increase in the mode density, with respect to the corresponding plot of fig 7a, due to the increase of length (in this case many resonances with axial variations occur up to 3 GHz). The longitudinal impedance for the matched structure is plotted in Fig. 14b. The residual peaks are due to modes having one or more axial variations, but having a transverse magnetic field distribution equal to the one of the fundamental mode. These modes, as discussed in [4], remain trapped because they cannot couple to the loads due to the cylindrical shape of the structure. To damp these modes too it is necessary to perturb the cylindrical shape. Two possible solutions are shown in fig. 13b, where the pill-box is connected to waveguides which are slightly offset in the axial direction, and in fig 13c, where the waveguides have been bisected by a metal septum. Actually, in this last case the pill-box is connected to six waveguides, and the algorithm requires minor modifications. The modal analysis of these structures has been performed using the 3-D code ARGUS, looking for resonances up to 1 GHz only. This permits the investigation, in reasonable CPU times, of the damping of the first HOM still trapped in the matched cylindrical structure.

To compare the result of the two codes, ARGUS was used at first to analyze also the cylindrical structure of fig. 13a. The resulting longitudinal impedance, normalized to the value of the longitudinal impedance of the fundamental mode in the pill-box, is plotted in fig. 15a (short-circuited structure) and in fig. 15b (matched structure). Comparing these plots with the frequency behavior, up to 1 *GHz*, of the longitudinal impedances plotted in fig. 14, it is noted a fairly good agreement between the results using data from the two different codes.

The difference in the value of the longitudinal impedance of the accelerating mode (3.5 *Mohm* using the data form PAGODA, 4.37 *Mohm* using the data from ARGUS) derives from the fact that ARGUS yields a *Q*-factor for that mode which is about 1.25 times the value calculated by PAGODA. As a result, the longitudinal impedance calculated using the data from ARGUS is higher than the one of the simple pill-box, which is clearly an unrealistic result. Anyway, in both cases a sharp peak around 920 *MHz* survives in the longitudinal impedance of the matched structure, corresponding to the frequency of the first mode having an axial variation.

Fig. 16 shows the longitudinal impedance of the structure obtained by offsetting the waveguides by 30 mm (see fig. 13b). An attenuation of about 25 db for the 920 MHz mode can be observed comparing the plot in fig. 16a (short-circuited structure) and fig. 16b (matched structure). Though quite effective in damping the residual peak, the offset waveguides have the drawback of destroying the symmetry with respect to the plane perpendicular to the beam axis (see fig. 1). The structure of fig. 13c overcomes this drawback, as all the symmetries of fig. 1 are preserved. Its longitudinal impedance is plotted in fig. 17a (short-circuited structure); the insertion of the septum in the waveguides lowers the frequency of the first mode with an axial variation to about 900 MHz. An attenuation of about 25 db for this mode can be observed in the plot of fig. 17b (matched waveguides). Higher values could be obtained extending the septum closer to the center of the pill-box.

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Fig. 3 - Geometry of the cylindrical structure used in the examples of Fig. 4 - 11. Analyses were performed for different values of the slot width *W* . All dimensions are in mm.























Fig. 9 - Minimum monopole HOM attenuation (db, left scale) and reduction of the longitudinal impedance for the fundamental mode (percentage referred to the longitudinal impedance of the pill-box) versus normalized slot width for the structure of fig. 3.











Fig 12 - Longitudinal and transverse impedance of the structure of fig. 3, in the case W=150 mm. A lossy dielectric ($\epsilon_r = 9$, tan $\delta \approx 0.01$ @ 3GHz) fills the waveguides, starting from a section placed 200 mm away from the reference sections.



a

b

C





Fig. 13 - Geometry of the structures used in the examples of Fig. 14 - 17. A pill-box (radius = 200 mm, height = 200 mm) is connected to three waveguides (200 x 200 mm). a) cylindrical resonator. b) the waveguides are offset by 30 mm in the axial direction. c) the waveguides are bisected by a septum (thickness = 20 mm, distance from center = 230 mm).



Fig. 14 - Longitudinal impedance of the structure of fig. 13a (cylindrical resonator). a) Short-circuited ports; b) Matched ports. The modal analysis was performed using the two-dimensional solver PAGODA.



Fig. 15 - Longitudinal impedance of the structure of fig. 13a (cylindrical resonator). a) Short-circuited ports; b) Matched ports. The modal analysis was performed using the three-dimensional solver ARGUS.



Fig. 16 - Longitudinal impedance of the structure of fig. 13b (waveguides offset by 30 mm). a) Short-circuited ports; b) Matched ports. The modal analysis was performed using the three-dimensional solver ARGUS.



I

Fig. 17 - Longitudinal impedance of the structure of fig. 13c (waveguides bisected by a 20 mm thick septum starting at a distance of 230 mm from the center of the cavity). a) Short-circuited ports; b) Matched ports. The modal analysis was performed using the three-dimensional solver ARGUS.