# DISCRETIZED LIGHT-CONE QUANTIZATION: FORMALISM FOR QUANTUM ELECTRODYNAMICS* 

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#### Abstract

A general non-perturbative method for solving quantum field theories in three space and one time dimensions, Discretized Light-Cone Quantization, is outlined and applied to quantum electrodynamics. This numerical method is frame independent and can be formulated such that ultraviolet regularization is independent of the momentum space discretization. In this paper we discuss the construction of the light-cone Fock basis, ultraviolet regularization, infrared regularization, and the renormalization techniques required for solving QED as a light-cone Hamiltonian theory.


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## 1. INTRODUCTION

Perhaps the most outstanding problem in quantum field theory is to compute the bound state spectrum and the relativistic wavefunctions of hadrons at strong coupling. In quantum chromodynamics one needs a practical computational method which not only determines the hadronic and exotic spectra, but also can provide non-perturbative hadronic matrix elements of the operator product expansion, weak decay amplitudes, structure functions, and distribution amplitudes. In general, the computation of hadronic scattering amplitudes requires knowledge of the bound state wavefunctions at arbitrary four-momentum. Lattice gauge theory has provided important tools for analyzing the lowest hadronic states of QCD, but detailed wave function information has been very difficult to obtain.

Even in the case of abelian quantum electrodynamics, very little is known about the nature of the bound state solutions in the large $\alpha$, strong coupling, domain. The Bethe-Salpeter formalism has been the central method for analyzing hydrogenic atoms in QED, providing a completely covariant procedure for obtaining bound state solutions. However, calculations using this method are extremely complex and appear to be intractable much beyond the ladder approximation. It also appears impractical to extend this method to systems with more than a few constituent particles.

The most intuitive approach for solving relativistic bound-state problems would be to solve the Hamiltonian eigenvalue problem for field theories

$$
\begin{equation*}
H|\psi\rangle=\sqrt{\vec{P}^{2}+M^{2}}|\psi\rangle \tag{1.1}
\end{equation*}
$$

for the particle's mass, $M$, and wavefunction, $|\psi\rangle$. Here, one imagines that $|\psi\rangle$ is an expansion in multi-particle occupation number Fock states and that the opera-
tors $H$ and $\vec{P}$ are second quantized Heisenberg picture operators. Unfortunately, this method, as described by Tamm and Dancoff ${ }^{1}$, is severely complicated by its non-covariance and the necessity to first understand its complicated vacuum eigensolution over all space and time. The presence of the square root operator also presents severe mathematical difficulties. Even if these problems could be solved, the eigensolution is only determined in its rest system; determining the boosted wavefunction is as complicated as diagonalizing $H$ itself. Fortunately, "light-cone" quantization offers an elegant avenue of escape. The square root operator does not appear in light-cone formalism, and as we will see explicitly in Section 2 , the structure of the vacuum does not play an important role in QED since there is no spontaneous creation of massive fermions in the light-cone quantized vacuum.

There are, in fact, many reasons to quantize relativistic field theories at lightcone time. Dirac ${ }^{2}$, in 1949 , showed that a maximum number of Poincare generators become independent of the dynamics in the "front form" formulation, including the required Lorentz boosts. In fact, unlike the traditional equal-time Hamiltonian formalism, quantization on the light-cone can be formulated without reference to the choice of a specific Lorentz frame; the eigensolutions of the light-cone Hamiltonian thus describe bound states of arbitrary four-momentum, allowing the computation of scattering amplitudes and other dynamical quantities. However, the most remarkable feature of this formalism is the simplicity of the light-cone vacuum. The vacuum state of the free Hamiltonian is the vacuum eigenstate of the total lightcone Hamiltonian. The Fock expansion constructed on this vacuum state provides a complete relativistic many-particle basis for diagonalizing the full theory.

In this paper we will quantize quantum electrodynamics on the light-cone in a discretized form which allows practical numerical solutions for obtaining its spec-

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trum and wavefunctions at arbitrary coupling strength $\alpha$. Hopefully, these techniques will be applicable to non-Abelian gauge theories, including quantum chromodynamics in physical space-time. In this paper we discuss the ultraviolet and infrared regularization of the theory which renders it finite. In addition to the momentum space regularization, we also discuss a covariant approximately gaugeinvariant particle number truncation of the Fock basis which is useful both for - computational purposes and physical approximations. In this method, "Discretized Light-Cone Quantization," (DLCQ) ${ }^{3}$ ultraviolet and infrared regularizations are kept independent of the discretization procedure, and are identical to that of the continuum theory. One thus obtains a finite discrete representation of the gauge theory which is faithful to the continuum theory and is completely independent of the choice of Lorentz frame. ${ }^{4}$

- In a second paper ${ }^{5}$ we will discuss the numerical methods which can be used to solve the DLCQ system and present initial results for the positronium spectrum in $\operatorname{QED}(3+1)$ at moderate values of $\alpha$.

The possibility of quantizing on the light cone was first discovered by Dirac. ${ }^{2}$ The initial applications to gauge theory were given by Casher, ${ }^{6}$ Chang, Root, and Yan, ${ }^{7}$ Lepage and Brodsky, ${ }^{8}$ Brodsky and Ji, ${ }^{9}$ Lepage, Brodsky, Huang, and Mackenzie, ${ }^{10}$ and McCartor. ${ }^{11}$ Casher gave the first construction of the light-cone Hamiltonian for non-Abelian gauge theory and gave an overview of important considerations in light-cone quantization. Chang, Root, and Yan demonstrated the equivalence of light-cone quantization with standard covariant Feynman analysis. There has also been important work on light-cone quantization by Franke, ${ }^{12,13,14}$ Karmanov, ${ }^{15,16}$ and Pervushin. ${ }^{17}$ Detailed rules for QCD, a discussion of the Fock basis, and applications to exclusive processes were provided by Brodsky and Lep-
age. They also present a table of light-cone spinor properties in their Appendix A. A summary of the light-cone perturbation theory rules for QED in light-cone gauge and their derivation is given in Appendix B of Ref. 9 and Appendix A of Ref. 10. The renormalization of light-cone wavefunctions and the calculation of physical observables is also discussed in these papers. The notation used in this paper will follow that used in these two references and is given in Table 1. A comparison of light-cone quantization with equal-time quantization is shown in Table 2.

McCartor ${ }^{18}$ has discussed how to handle the light-cone boundary at $x^{-}=\infty$, and shows for massive theories that the energy and momentum derived using lightcone quantization are not only conserved, but also are equivalent to the energy and momentum one would normally write down in an equal-time theory. A recent summary of QCD in light-cone quantization can be found in Brodsky ${ }^{19}$ and Brodsky and Lepage. ${ }^{20}$

A mathematically similar but conceptually different approach to light-cone quantization is the "infinite momentum frame" formalism. This method involves observing the system in a frame moving past the laboratory close to the speed of light. The first developments were given by Weinberg. ${ }^{21}$ It should be noted that though light-cone quantization is similar to infinite momentum frame quantization, it differs since no reference frame is chosen for calculations and is thus manifestly Lorentz covariant. The only aspect that "moves at the speed of light" is the quantization surface. Other works in infinite momentum frame physics include Drell, Levy, and Yan, ${ }^{22}$ Susskind and Frye, ${ }^{23}$ Bjorken, Kogut, and Soper, ${ }^{24}$ and Brodsky, Roskies, and Suaya. ${ }^{25}$ This last reference presents the infinite momentum frame perturbation theory rules for QED in Feynman gauge, calculates one-loop radiative corrections, and demonstrates renormalizability.

In order to capitalize on the features of light-cone quantization, Pauli and Brodsky ${ }^{3}$ developed the method of discretized light-cone quantization and applied it to solving for the mass spectrum and wavefunctions of Yukawa theory, $\bar{\psi} \psi \phi$, in one space and one time dimensions. This success lead to further applications including $1+1$ QED and the Schwinger model by Eller, Pauli, and Brodsky, ${ }^{26} \phi^{4}$ theory in $1+1$ dimensions by Harindranath and Vary, ${ }^{27}$ and $1+1$ QCD for $N_{C}$ ${ }^{-}=2,3,4$ by Hornbostel, Brodsky, and Pauli. ${ }^{28}$ In each of these applications, the mass spectrum and wavefunctions were successfully obtained, and all results agree with previous analytical and numerical work, where they were available. Recently, Hiller ${ }^{29}$ has used DLCQ and the Lanczos algorithm for matrix diagonalization method to compute the annihilation cross section, structure functions and form factors in $1+1$ theories.

- The initial successes of DLCQ provide the hope that one can use this method for solving $3+1$ theories. The application to higher dimensions is much more involved due to the need to introduce ultraviolet and infrared regulators, and invoke a renormalization scheme consistent with gauge invariance and Lorentz invariance. This is in addition to the work involved implementing two extra dimensions with their added degrees of freedom. In this paper, we will present the application of DLCQ to $3+1$ dimensional QED.

The basic background for light-cone quantization and DLCQ is shown in Refs. 3, 26, and Sections 2 and 3 of Ref. 30. The light-cone Hamiltonian for $3+1$ dimensional QED is given in Section 2, ultraviolet regularization in Section 3, and infrared regularization in Section 4. Section 3 also introduces a new method for maintaining tree-level gauge invariance of the ultraviolet regulator. It is important to maintain gauge invariance and covariance when truncating the Fock space. A

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general method for preserving these symmetries while truncating the Fock space basis is presented in Section 5. Renormalization in this truncated space is discussed in Section 6. The question of self-induced inertias and the equivalence of Feynman rules and light-cone perturbation theory results for one-loop mass counterterms is also presented in Section 6. A number of mathematical details are given in the various appendices.

## 2. LIGHT-CONE QUANTIZATION OF QED

The derivation of the light-cone Hamiltonian, $H_{L C}$, from the $3+1$ dimensional QED Lagrangian

$$
\begin{equation*}
\mathcal{L}=\frac{i}{2}\left[\bar{\psi} \gamma^{\mu} \partial_{\mu}-\left(\partial_{\mu} \bar{\psi}\right) \gamma^{\mu}\right] \psi-m_{e} \bar{\psi} \psi-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-g \bar{\psi} \gamma^{\mu} \psi A_{\mu} \tag{2.1}
\end{equation*}
$$

can be carried out in the light-cone gauge $A^{+}=A^{0}+A^{3}=0$ using the standard methods of canonical quantization with (anti) periodic boundary conditions. The procedure and notation closely follow the quantization of QCD in one-space and one-time. See Ref. 28. Details of this derivation for $\operatorname{QED}(3+1)$ are given in Section 4 of Ref. 30. In a general frame, we write

$$
P^{-}=\frac{H_{L C}+P_{\perp}^{2}}{P^{+}}
$$

so that the eigenvalues of $H_{L C}$ give the invariant mass spectrum $M^{2}$. The result after using the classical equations of motion to eliminate the dependent fields, $\psi_{-}$ and $A^{-}$, imposing canonical commutation relations on the independent fields, $\psi_{+}$
and $\vec{A}_{\perp}$, and finally discretizing these two fields by expanding in plane waves and imposing boundary conditions is

$$
\begin{gather*}
H_{L C}=H_{0}+H_{1}+H_{2}  \tag{2.2}\\
H_{1}=V_{\text {flip }}+V_{\text {noflip }}, \quad H_{2}=V_{\text {instphot }}+V_{\text {instferm }}
\end{gather*}
$$

$$
\begin{align*}
H_{0}= & \sum_{\lambda, \underline{p}} \frac{K}{p}\left[\left(\frac{p_{\perp} \pi}{L_{\perp}}\right)^{2}+\lambda^{2}\right] a_{\lambda, \underline{p}}^{\dagger} a_{\lambda, \underline{p}}  \tag{2.3}\\
& +\sum_{s, \underline{n}} \frac{K}{n}\left[\left(\frac{n_{\perp} \pi}{L_{\perp}}\right)^{2}+m_{e}^{2}\right]\left[b_{s, \underline{n}}^{\dagger} b_{s, \underline{n}}+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right]
\end{align*}
$$

$$
\begin{align*}
V_{\text {flip }}= & \frac{K m_{e}}{2 \sqrt{\pi} L_{\perp}} \sum_{s} \sum_{\underline{p}, \underline{\underline{m}, \underline{n}}} \frac{1}{\sqrt{p}} \\
\{ & +a_{2 s, \underline{p}} b_{s, \underline{m}}^{\dagger} b_{-s, \underline{n}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)}\left(\frac{1}{n}-\frac{1}{m}\right)+\text { h.c. }  \tag{2.4}\\
& -a_{2 s, \underline{p}} d_{s, \underline{m}}^{\dagger} d_{-s, \underline{n}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)}\left(\frac{1}{n}-\frac{1}{m}\right)+\text { h.c. } \\
& \left.+a_{2 s, \underline{p}}^{\dagger} b_{s, \underline{m}} d_{s, \underline{n}} \delta_{\underline{n}+\underline{m}, \underline{p}}^{(3)}\left(\frac{1}{n}+\frac{1}{m}\right)+\text { h.c. }\right\}
\end{align*}
$$

$$
\begin{align*}
& V_{\text {noflip }}=g \sqrt{\frac{\pi}{2}} \frac{K}{L_{\perp}^{2}} \sum_{s} \sum_{\underline{p}, \underline{m}, \underline{\underline{n}}} \frac{1}{\sqrt{p}} \\
& \left\{+a_{2 s, \underline{p}} b_{s, \underline{m}}^{\dagger} b_{s, \underline{n}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)} \vec{\epsilon}_{2 s}^{\prime} \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{n}_{\perp}}{n}\right)+\right.\text { h.c. } \\
& +a_{-2 s, \underline{p}, \underline{m}} b_{s, \underline{m}}^{\dagger} b_{s, \underline{\underline{n}}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)} \vec{\epsilon}_{-2 s}^{\perp} \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{m}_{\perp}}{m}\right)+\text { h.c. } \\
& -a_{2 s, \underline{p}} d_{s, \underline{\underline{m}}}^{\dagger} d_{s, \underline{\underline{n}}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)} \vec{\epsilon}_{2 s}^{\perp} \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{n}_{\perp}}{n}\right)+\text { h.c. }  \tag{2.5}\\
& -a_{-2 s, \underline{p}} d_{s, \underline{m}}^{\dagger} d_{s, \underline{\underline{n}}} \delta_{\underline{n}+\underline{p}, \underline{m}}^{(3)} \vec{\epsilon}_{-2 s}^{\perp} \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{m}_{\perp}}{m}\right)+\text { h.c. } \\
& -a_{2 s, \underline{p}}^{\dagger} b_{s, \underline{m}} d_{-s, \underline{\underline{n}}} \delta_{\underline{n}+\underline{m}, \underline{p}}^{(3)} \vec{e}_{2 s}^{*} \stackrel{\perp}{ } \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{n}_{\perp}}{n}\right)+\text { h.c. } \\
& \left.-a_{-2 s, \underline{\underline{p}}}^{\dagger} b_{s, \underline{m}} d_{-s, \underline{n}} \delta_{\underline{n}+\underline{m}, \underline{\underline{c}}}^{(3)} \vec{\epsilon}_{-2 s}^{\perp} \cdot\left(\frac{\vec{p}_{\perp}}{p}-\frac{\vec{m}_{\perp}}{m}\right)+\text { h.c. }\right\},
\end{align*}
$$

$$
\begin{align*}
& V_{\text {instphot }}=g^{2} \frac{K}{2 \pi L_{\perp}^{2}} \sum_{s, t} \sum_{\underline{k}, l, \underline{m}, \underline{n}} \\
& \left\{-b_{s, \underline{k}}^{\dagger} b_{t, \underline{l}}^{\dagger} b_{s, \underline{m}} b_{t, \underline{\underline{n}}} \delta_{\underline{k}+\underline{l}, \underline{m}+\underline{n}}^{(3)} \frac{1}{2}[k-m \mid-l+m]\right. \\
& -d_{s, \underline{k}}^{\dagger} d_{t, \underline{\underline{l}}}^{\dagger} d_{s, \underline{\underline{m}}} d_{t, \underline{n}} \delta_{\underline{k}+\underline{l}, \underline{m}+\underline{n}}^{(3)} \frac{1}{2}[k-m \mid-l+m] \\
& -b_{s, \underline{\underline{k}}}^{\dagger} d_{-s, \underline{l}}^{\dagger} b_{t, \underline{\underline{m}}} d_{-t, \underline{n}} \delta_{\underline{k}+\underline{l}, \underline{m}+\underline{n}}^{(3)}[k+l \mid m+n]  \tag{2.6}\\
& +b_{s, \underline{\underline{k}}}^{\dagger} d_{-t, \underline{l}}^{\dagger} b_{s, \underline{\underline{m}}} d_{-t, \underline{n}} \delta_{\underline{k}+\underline{l}, \underline{\underline{m}}+\underline{\underline{n}}}^{(3)}[k-m \mid-l+n] \\
& +d_{s, \underline{k}}^{\dagger} d_{t, \underline{l} \underline{\underline{l}}} d_{s, \underline{\underline{m}}} b_{-t, \underline{\underline{n}}} \delta_{\underline{k}, \underline{l}+\underline{m}+\underline{n}}^{(3)}[k-m \mid l+n]+\text { h.c. } \\
& \left.+b_{s, \underline{k}}^{\dagger} b_{t, \underline{\underline{l}}} b_{s, \underline{m}} d_{-t, \underline{n}} \delta_{\underline{k}, \underline{l}+\underline{m}+\underline{n}}^{(3)}[k-m \mid l+n]+\text { h.c. }\right\},
\end{align*}
$$

$$
\begin{align*}
& V_{\text {instferm }}=g^{2} \frac{K}{4 \pi L_{\perp}^{2}} \sum_{s} \sum_{\underline{p}, \underline{q}, \underline{m}, \underline{\underline{n}}} \frac{1}{\sqrt{p q}} \\
& \left\{+a_{-2 s, \underline{,}}^{\dagger} a_{-2 s, \underline{\underline{q}}} b_{s, \underline{\underline{m}}}^{\dagger} b_{s, \underline{\underline{n}}} \delta_{\underline{p}+\underline{m}, \underline{q}+\underline{n}}^{(3)}\{p+m \mid q+n\}\right. \\
& -a_{2 s, \underline{p}}^{\dagger} a_{2 s, \underline{q}} b_{s, \underline{m}}^{\dagger} b_{s, \underline{\underline{n}}} \delta_{\underline{p}+\underline{m}, \underline{q}+\underline{n}}^{(3)}\{p-n \mid q-m\} \\
& +a_{-2 s, \underline{p}}^{\dagger} a_{-2 s, \underline{q}} d_{s, \underline{\underline{m}}}^{\dagger} d_{s, \underline{\underline{n}}} \delta_{\underline{p}+\underline{m}, \underline{q}+\underline{n}}^{(3)}\{p+m \mid q+n\} \\
& -a_{2 s, \underline{p}}^{\dagger} a_{2 s, \underline{q}} d_{s, \underline{\underline{m}}}^{\dagger} d_{s, \underline{\underline{n}}} \delta_{\underline{p}+\underline{m}, \underline{q}+\underline{n}}^{(3)}\{p-n \mid q-m\}  \tag{2.7}\\
& -a_{2 s, \underline{p}}^{\dagger} a_{-2 s, \underline{q}}^{\dagger} b_{s, \underline{m}} d_{-s, \underline{n}} \delta_{\underline{p}+\underline{q}, \underline{m}+\underline{n}}^{(3)}\{p-m \mid-q+n\}+\text { h.c. } \\
& -a_{2 s, \underline{p}}^{\dagger} a_{2 s, \underline{q}} b_{-s, \underline{m}} d_{s, \underline{n}} \delta_{\underline{p}, \underline{q}+\underline{m}+\underline{n}}^{(3)}\{p-n \mid q+m\}+\text { h.c. } \\
& +a_{2 s, \underline{p}}^{\dagger} a_{2 s, \underline{q}} b_{s, \underline{m}} d_{-s, \underline{n}} \delta_{\underline{p}, \underline{q}+\underline{m}+\underline{n}}^{(3)}\{p-m \mid q+n\}+\text { h.c. } \\
& -a_{-2 s, \underline{p}} a_{2 s, \underline{q}} \underline{q}_{s, \underline{m}}^{\dagger} b_{s, \underline{n}} \delta_{\underline{m}, \underline{p}+\underline{q}+\underline{n}}^{(3)}\{p+n \mid-q+m\}+\text { h.c. } \\
& \left.-a_{-2 s, \underline{p}} a_{2 s, \underline{q}} d_{s, \underline{\underline{m}}}^{\dagger} d_{s, \underline{n}} \delta_{\underline{m}, \underline{p}+\underline{q}+\underline{\underline{n}}}^{(3)}\{p+n \mid-q+m\}+\text { h.c. }\right\} \text {. }
\end{align*}
$$

$V_{\text {flip }}$ is the spin-flip amplitude for a (anti-) fermion to emit (absorb) a photon and $V_{\text {noflip }}$ is the no spin-flip amplitude for this process. The familiar three-point Dirac QED vertex is just the sum of these two amplitudes. Two other types of vertices appear in light-cone quantization: a four-point instantaneous photon exchange, $V_{\text {instphot }}$, and a four-point instantaneous fermion exchange, $V_{\text {instferm }}$. These are just the graphs needed to reproduce the usual covariant Feynman S-matrix results for scattering amplitudes. An example of this for Møller scattering ( $e^{-} e^{-} \rightarrow e^{-} e^{-}$) is shown in Appendix A. One can think of the instantaneous photon exchange graph in light-cone gauge as being analogous to the Coulomb exchange graph in Coulomb gauge. All the interactions conserve $k^{+}$and $\vec{k}_{\perp}$, as they must, and are shown schematically in Fig. 1.

In the above expression for $H_{L C}, g$ is the coupling constant, $2 L_{\perp}$ is the size
of the transverse box, $\lambda$ is an artificial photon mass which is ultimately set equal to zero and, as will be explained shortly, $K$ is related to the value of $P^{+}$. The expression has also been normal-ordered to remove vacuum values and self-induced inertias (more on these in Section 6). The integers $p, q, m, n, \ldots$ are allowed to take on the values

$$
\begin{align*}
p^{i}, q^{i}, k^{i}, l^{i}, m^{i}, n^{i} & =0, \pm 1, \pm 2, \ldots, \quad i=1,2 \\
p, q & =2,4,6, \ldots,  \tag{2.8}\\
k, l, m, n & = \begin{cases}2,4,6, \ldots & \text { (periodic b.c.) } \\
1,3,5, \ldots & \text { (anti-periodic b.c.) }\end{cases}
\end{align*}
$$

[ $n \mid m]$ and $\{n \mid m\}$ were first defined in Ref. 26. A modified version using a method suggested by Hamer ${ }^{31}$ based on the form of the Lagrangian, Eq. (2.1), leads to

$$
\begin{gather*}
{[n \mid m]= \begin{cases}\frac{1}{n^{2}} \delta_{n, m} & n, m \neq 0 \\
\kappa & n \text { and } m=0 \\
0 & \text { otherwise }\end{cases} }  \tag{2.9}\\
\{n \mid m\}= \begin{cases}\frac{1}{n} \delta_{n, m} & n, m \neq 0 \\
0 & n \text { or } m=0\end{cases}
\end{gather*}
$$

Details can be found in Section 4 and Appendix B of Ref. 30. Since the gauge invariant cut-off introduced in Section 3 eliminates all occurrences of $[0 \mid 0]$, the value of the unknown constant, $\kappa$, can be set equal to zero.

The free fermion and photon spinors are ${ }^{32}$

$$
\begin{align*}
& \chi(\uparrow)=\frac{1}{\sqrt{2}}\left(\begin{array}{l}
1 \\
0 \\
1 \\
0
\end{array}\right), \quad \chi(\downarrow)=\frac{1}{\sqrt{2}}\left(\begin{array}{c}
0 \\
1 \\
0 \\
-1
\end{array}\right),  \tag{2.10}\\
& \vec{\epsilon}_{\perp}(\uparrow)=\frac{-1}{\sqrt{2}}(1, i), \quad \vec{\epsilon}_{\perp}(\downarrow)=\frac{1}{\sqrt{2}}(1,-i)
\end{align*}
$$

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and the fermion and photon fields have discrete momenta in the $x^{-}$and $x^{i}, i=1,2$ directions due to the boundary conditions,

$$
\begin{align*}
& \text { photons : } \quad k^{i}=\frac{p^{i} \pi}{L_{\perp}}, \quad p^{i}=0, \pm 1, \pm 2, \ldots \\
& \qquad k^{+}=\frac{p \pi}{L}, \quad p=2,4,6, \ldots, \\
& \text { fermions : } \quad k^{i}=\frac{n^{i} \pi}{L_{\perp}}, \quad n^{i}=0, \pm 1, \pm 2, \ldots  \tag{2.11}\\
& \qquad k^{+}=\frac{n \pi}{L}, \quad \bar{n}= \begin{cases}2,4,6, \ldots & \text { (periodic b.c.) } \\
1,3,5, \ldots & \text { (anti-periodic b.c.) } .\end{cases}
\end{align*}
$$

We have chosen periodic boundary conditions for the photon field, $\vec{A}_{\perp}$, in the $x^{-}$and $\vec{x}_{\perp}$ directions, and periodic boundary conditions for the fermion field, $\psi_{+}$, in the $\vec{x}_{\perp}$ directions. $\psi_{+}$may have periodic or anti-periodic boundary conditions in the $x^{-}$direction. In the rest of this paper, anti-periodic conditions will be used. Note that only positive $k^{+}$are allowed. This is because the mass shell condition,

$$
\begin{equation*}
k^{-}=\frac{k_{\perp}^{2}+m^{2}}{k^{+}} \tag{2.12}
\end{equation*}
$$

only allows for $k^{+}$and $k^{-}$both positive or both negative. As one does in equaltime considerations, the modes with negative energy (in our case, negative $k^{-}$) are re-defined to be anti-particles (the photon is its own anti-particle). The result is that in light-cone quantization, one only has states with both positive $k^{+}$and positive $k^{-}$.

The above expression for $H_{L C}$ is still incomplete due to the need to include fermion mass renormalization counterterms (see Section 6). We also note that $H_{L C}$ is independent of the longitudinal box size, $L$. This last result arises because $P^{+}$ is proportional to $1 / L$ and $P^{-}$is proportional to $L$.

Other conserved quantities in the theory include the charge, ${ }^{33}$ light-cone momentum, and transverse momentum. The expressions for these in light-cone quantum mechanics after-normal-ordering to remove vacuum values are

$$
\begin{align*}
Q & =g \sum_{s, \underline{n}}\left[b_{s, \underline{n}}^{\dagger} b_{s, \underline{n}}-d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right] \\
P^{+} & =\sum_{\lambda, \underline{p}} k^{+} a_{\lambda, \underline{p}}^{\dagger} a_{\lambda, \underline{p}}+\sum_{s, \underline{n}} k^{+}\left[b_{s, \underline{n}}^{\dagger} b_{s, \underline{n}}+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right]  \tag{2.13}\\
P^{i} & =\sum_{\lambda, \underline{p}} k^{i} a_{\lambda, \underline{p}}^{\dagger} a_{\lambda, \underline{p}}+\sum_{s, \underline{n}} k^{i}\left[b_{s, \underline{n}}^{\dagger} b_{s, \underline{n}}+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right]
\end{align*}
$$

The last two equations are just statements of $k^{+}$and $\vec{k}_{\perp}$ momentum conscrvation: $P^{+}$is just the sum of the individual $k^{+}{ }_{s}$ and $\vec{P}_{\perp}$ is just the sum of the individual $\vec{k}_{\perp}$ s. These expressions are especially simple, and since they are already diagonal, the wavefunction, $|\psi\rangle$, can immediately be chosen as an eigenstate of them. For convenience we can choose $P^{+}=2 m_{e}$ and $\vec{P}_{\perp}=\overrightarrow{0}_{\perp}$ corresponding to the positronium center of mass and obtain

$$
\begin{gather*}
\left\{\sum_{\lambda, \underline{p}} p a_{\lambda, \underline{p}}^{\dagger} a_{\lambda, \underline{p}}+\sum_{s, \underline{n}} n\left[b_{s, \underline{n}}^{\dagger} b_{s, \underline{n}}+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right]\right\}|\psi\rangle=\frac{2 \pi m_{e} L}{\pi}|\psi\rangle=K|\psi\rangle \\
\left\{\sum_{\lambda, \underline{p}} p^{i} a_{\lambda, \underline{p}}^{\dagger} a_{\lambda, \underline{p}}+\sum_{s, \underline{n}} n^{i}\left[b_{s, \underline{\underline{n}}}^{\dagger} b_{s, \underline{n}}+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\right]\right\}|\psi\rangle=0|\psi\rangle \\
p=2,4,6, \ldots, \quad n=1,3,5, \ldots \quad \text { (anti-periodic b.c.) } \\
p^{i}, n^{i}=0, \pm 1, \pm 2, \ldots \tag{2.14}
\end{gather*}
$$

From now on, only those expansion states satisfying these equations need be considcred. In the first cxpression, the integer $K$ is defined to be the eigenvalue $P^{+}$ times $L / \pi$,

$$
\begin{equation*}
P^{+}=\frac{K \pi}{L} \tag{2.15}
\end{equation*}
$$

In Refs. 3 and 26, $K$ is called the "harmonic resolution."
Finally observe that because of $k^{+}$momentum conservation and positivity of $k^{+}$, there are no interactions involving spontaneous creation or annihilation of a fermion pair and a photon from the vacuum. Because of this fact, the Fock state vacuum (the state with no particles) is an eigenstate of the light-cone Hamiltonian with mass zero,

$$
\begin{equation*}
H_{L C}|0\rangle=0|0\rangle \tag{2.16}
\end{equation*}
$$

This immensely simplifies solving for bound states because it removes the need to constantly recalculate the vacuum.

We now focus on the positronium bound state problem. As in normal quantization we can apply the creation operators on the vacuum to create a complete light-cone Fock basis. The eigensolutions for the bound states will have the form

$$
\begin{align*}
|\psi\rangle & =\sum_{n} \psi_{n}\left(x, \vec{k}_{\perp}\right)|n\rangle  \tag{2.17}\\
& =\sum \psi_{e^{+} e^{-}}\left|e^{+} e^{-}\right\rangle+\psi_{e^{+} e^{-} \gamma}\left|e^{+} e^{-} \gamma\right\rangle+\ldots
\end{align*}
$$

so that $\left\langle e^{+} e^{-} \gamma \mid \psi\right\rangle=\psi_{e^{+} e^{-} \gamma}\left(x, k_{\perp}, \lambda\right)$. The labelling of the parton momenta for the positronium $e^{+} e^{-} \gamma$ Fock state is shown explicitly in Figure 2. The sum is over all Fock states $|n\rangle$ with constituent momenta $x_{i}$ and $\vec{k}_{\perp i}$. In general all Fock states are needed to describe the bound-system; we will discuss the errors introduced by a truncation later. The Fock states are eigenstates of $P^{+}, \vec{P}_{\perp}$, and $H_{0}$. The $\vec{k}_{\perp i}$ and $x_{i}$ are internal relative coordinates and are independent of the total momentum. The formalism is thus independent of the choice of reference frame. For calculational convenience, one can make the choice $P^{+}=2 m_{e}$, and
$\vec{P}_{\perp}=\overrightarrow{0}_{\perp}$, for which

$$
\begin{align*}
P^{+}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle & =\frac{K \pi}{L}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle \\
\vec{P}_{\perp}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle & =\overrightarrow{0}_{\perp}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle  \tag{2.18}\\
H_{0}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle & =\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}\left|n: k_{i}^{+}, \vec{k}_{\perp i}\right\rangle
\end{align*}
$$

Because we are working with a discrete representation, the light-cone bound state equation,

$$
\begin{equation*}
H_{L C}|m\rangle=M^{2}|m\rangle \tag{2.19}
\end{equation*}
$$

can be converted into a matrix equation for the eigenvalues, $M^{2}$, and eigenvectors, $\psi_{n}$, by projecting out the $n$th component,

$$
\begin{equation*}
\sum_{m}\langle n| H_{L C}|m\rangle \psi_{m}\left(x_{i}, \vec{k}_{\perp i}, \lambda_{i}\right)=M^{2} \psi_{n}\left(x_{i}, \vec{k}_{\perp i}, \lambda_{i}\right) \tag{2.20}
\end{equation*}
$$

For our case of positronium, the matrix equation is

Here, $H_{L C}$ has been split into an interacting piece, $V$, and a non-interacting piece, $H_{0}=\sum_{i}\left(k_{\perp i}^{2}+m_{i}^{2} / x_{i}\right) . m_{i}$ is the mass of the $i$ th constituent particle. For the case of positronium, it is either the fermion mass or the photon mass. Diagonalization of this equation can now be done on a computer (after implementing ultraviolet

-     - 

and infrared regulators) to reveal the complete spectrum of positronium states and multi-particle scattering states with the same quantum numbers, along with their corresponding wavefunction expansion coefficients, $\psi_{\boldsymbol{n}}$. Solving field theory has now been reduced to obtaining the solution to this fairly simple equation.

In summary, the discretized light-cone quantization procedure is straightforward. The light-cone Hamiltonian is derived from the Lagrangian by a procedure very similar to standard canonical quantization. The commuting operators, the light-cone momentum $P^{+}=K \pi / L$, transverse momentum $\vec{P}_{\perp}$, and light-cone Hamiltonian $H_{L C}$ are constructed by expanding in Fock states and are simultaneously diagonalized. The expressions for $P^{+}$and $\vec{P}_{\perp}$ are already diagonal if one expands in plane waves. The system is discretized by requiring periodic or antiperiodic boundary conditions in the light-cone spatial dimensions, and the system is quantized by imposing canonical commutation relations between the independent fields and their canonical momenta. The bound state equation $H_{L C}|\psi\rangle=M^{2}|\psi\rangle$ is diagonalized to obtain the invariant mass spectrum and wavefunctions. Both of these quantities are independent of $L$. To recover the continuum theory, one lets $K$ and $L_{\perp}$ approach infinity (this is equivalent to letting $L, L_{\perp} \rightarrow \infty$ ).

## 3. COVARIANT ULTRAVIOLET REGULATOR

Before continuing, a method of regulating the $\vec{k}_{\perp}$ Fock space and other ultraviolet divergences is necessary. The Fock space is naturally finite in $k^{+}$because the total $k^{+}$is just the sum of the individual, constituent $k^{+}$s. Combining the fact that all the individual $k^{+} \mathrm{s}$ are positive, non-zero integers with the fact that there are only a finite number of ways of summing a set of positive, non-zero integers to form a given positive number demonstrates finiteness of the $k^{+}$space. As an example, a Fock state with one electron and two photons with $K=9$ can have the following quantum numbers (anti-periodic boundary conditions),

| Fock State | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Electron | 1 | 1 | 1 | 3 | 3 | 5 |
| Photon 1 | 2 | 4 | 6 | 2 | 4 | 2 |
| Photon 2 | 6 | 4 | 2 | 4 | 2 | 2 |.

In contrast to $k^{+}$, the Fock space is naturally infinite in $\vec{k}_{\perp}$ because $\vec{k}_{\perp}$ can take values that are positive or negative. An ultraviolet regulator must therefore be introduced.

We will discuss scveral possibilities for the frame-independent ultraviolet truncation of the light-cone Fock state. In the first method, which we refer to as the "global cut-off", we restrict the sum of the light-cone energies $\left(k_{\perp}^{2}+m^{2}\right) / x$ of the particles of each Fock state to be less than a cut-off value, $\Lambda^{2}$ (see Ref. 8),

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}} \leq \Lambda^{2} \tag{3.1}
\end{equation*}
$$

The left hand side of this equation is just the invariant mass (for a single particle state, the invariant mass is the rest mass) squared of the Fock state, $M^{2}=P^{+} P^{-}$
$P_{\perp}^{2}$. It is also the value of the light-cone Hamiltonian at zero coupling. Thus the global cutoff simply requires the invariant mass squared of the individual Fock states to be less than $\Lambda^{2}$. Since the invariant mass is frame independent, this regulator is Lorentz invariant. It should be emphasized that the variables $\vec{k}_{\perp i}$ and $x_{i}$ are relative internal coordinates, independent of the total momentum $P^{+}$and $\vec{P}_{\perp}$ of the bound state. The physical momentum of the particle in any given - Lorentz frame is $\vec{p}_{\perp i}=\vec{k}_{\perp i}+x_{i} \vec{P}_{\perp}$ and $p^{+}=x_{i} P^{+}$.

Each Fock state is off the light-cone energy shell by the amount

$$
\begin{align*}
\sum_{i} k_{i}^{-}-P^{-} & =\sum_{i}\left[\frac{\left(\vec{k}_{\perp i}+x_{i} \vec{P}_{\perp}\right)^{2}+m_{i}^{2}}{x_{i} P^{+}}\right]-\frac{P_{\perp}^{2}+M^{2}}{P^{+}}  \tag{3.2}\\
& =\frac{1}{P^{+}}\left[\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}-M^{2}\right] .
\end{align*}
$$

One sees immediately that the ultraviolet truncation given in Eq. (3.1) removes Fock states not by particle number, but because they are far off-shell. This is a reasonable procedure because far off-shell states give only a small contribution to a physical wavefunction. It is known from general considerations ${ }^{34}$ that the probability for high far-off-shell fluctuations of the renormalized wavefunction in a renormalizable theory are power-law suppressed, so that one expects convergence of all physical quantities as long as $\Lambda$ is taken larger than all relevant mass scales of the problem. In fact, one sees from Eqs. (2.20) and (2.21) that a typical wavefunction in QED will have the form

$$
\begin{equation*}
\psi_{n}\left(x_{i}, \vec{k}_{\perp i}, \lambda_{i}\right)=\frac{1}{M^{2}-\sum_{i}\left(k_{\perp i}^{2}+m_{i}^{2}\right) / x_{i}}(V \Psi) \tag{3.3}
\end{equation*}
$$

which tends to vanish as

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}-M^{2} \rightarrow \infty \tag{3.4}
\end{equation*}
$$

In principle, one must make $\Lambda$ infinite to recover the full theory. In practice, one can take moderate values of the cut-off and study the convergence of the spectrum and physical quantities as a function of $\Lambda$. In fact, since the binding energy is the relevant scale, it is more useful in practice to only restrict the kinetic part of the off-shell energy. We thus define the "kinetic cut-off":

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}-\min \left\{\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}\right\} \leq \Lambda^{2} \tag{3.5}
\end{equation*}
$$

where the minimum is taken over all allowed kinematic configurations. By using the kinetic cut-off, states with high momentum constituents are cut-off, but fermion pair states which play an important role in Compton amplitudes are not preferentially excluded.

Cutting off the photon's momentum $\vec{k}_{\perp}$ is clearly not compatible with gauge invariance because the various graphs involved in photon exchange are cut-off in a different way. That is, one can imagine a situation in Møller scattering ( $e^{-} e^{-} \rightarrow$ $e^{-} e^{-}$), for example, in which the exchange of a real, physical photon is cut-off (the relevant Fock state is the $e^{-} e^{-} \gamma$ intermediate state) but the exchange of an instantaneous photon is not (there is no intermediate state in this graph).

We can avoid this problem with gauge invariance by considering the instantaneous photon in the instantaneous photon exchange graph to have quantum numbers as if it were a real photon. One then cuts it off in a manner similar to
the Fock state cut-off for a real intermediate state. That is, one requires

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}} \leq \Lambda^{2} \tag{3.6}
\end{equation*}
$$

where the sum is over the individual particles in the Fock state plus the instantaneous photon. A similar procedure is taken for the instantaneous fermion interac-- tion so the correct Feynman S-matrix amplitudes are restored in this sector also. As a concrete example, consider the graphs involved in Møller scattering shown in Figure 3. Assume $k_{1}^{+}$is larger than $k_{3}^{+}$. In the first graph, the photon's momenta are fixed by momentum conservation, and the three particle intermediate state is cut-off by

$$
\begin{equation*}
\frac{k_{3 \perp}^{2}+m_{e}^{2}}{x_{3}}+\frac{k_{2 \perp}^{2}+m_{e}^{2}}{x_{2}}+\frac{q_{\perp}^{2}}{x_{q}} \leq \Lambda^{2} . \tag{3.7}
\end{equation*}
$$

In the second graph, one assigns momenta to the instantaneous photon, $q^{+}=$ $k_{1}^{+}-k_{3}^{+}, \vec{q}_{\perp}=\vec{k}_{1 \perp}^{-}-\vec{k}_{3 \perp}$, and then requires

$$
\begin{equation*}
\frac{k_{3 \perp}^{2}+m_{e}^{2}}{x_{3}}+\frac{k_{2 \perp}^{2}+m_{e}^{2}}{x_{2}}+\frac{q_{\perp}^{2}}{x_{q}} \leq \Lambda^{2} \tag{3.8}
\end{equation*}
$$

With this requirement, whenever the instantaneous photon exchange graph occurs, a corresponding graph with the exchange of a real, intermediate photon occurs because both graphs are now cut-off in exactly the same way. As shown in Appendix A, the sum of the graphs is simply the gauge invariant Feynman rules answer, $1 / q_{F}^{2}$. Thus, we see that this method maintains gauge invariance of the ultraviolet cut-off for tree-level diagrams. It is not clear if this conclusion can be carried over to loop diagrams. ${ }^{35}$

We have now completed the ultraviolet regularization of light-cone theory. All Fock states are cut-off by requiring the invariant mass squared to be less than $\Lambda^{2}$,

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}} \leq \Lambda^{2} \tag{3.9}
\end{equation*}
$$

Graphs involving an instantaneous photon or instantaneous fermion arc trcated as if they were real particles and cut-off in the same fashion. With this inclusion, the Fock space is finite and the ultraviolet regulation is both Lorentz invariant and (tree-level) gauge invariant. We also note that this regulation procedure is continuum regulator: the cut-off condition is not changed by discretization.

In principle, the global or kinetic cut-off can be used as the sole ultraviolet regulator needed to define the renormalized theory. However, these regulators have the disadvantage that at finite $\Lambda$ the renormalization constants will depend on the kinematics of the "spectator" particles in the Fock statc, rather than just the particles participating in the UV-divergent self-energy and vertcx subgraphs. However, one still has the option of introducing further UV regulation such as massive Pauli-Villars particles ${ }^{35}$ or massive supersymmetric partners to produce counterterms which render these subgraphs finite. We illustrate this method in Appendix C. Alternatively, one can also directly regulate the matrix elements of the interaction Hamiltonian such that ${ }^{36}\langle n| H_{L C}|m\rangle=0$ if

$$
\left|\sum_{i \in m} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}-\sum_{i \in n} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}}\right| \geq \Lambda^{2} .
$$

When using any of these "local" cut-offs, the mass counterterms can be defined independently of the bound-state wavefunction, as in the standard treatment of the Lamb Shift in QED. ${ }^{37}$ The counterterms at a specific renormalization scale
are chosen so that one obtains the physical values of the electron mass and photon mass when solving the light-cone equation of motion in the respective quantum number sector. ${ }^{38}$

## 4. COVARIANT INFRARED REGULATOR

There are a number of potential sources of infrared singularities and divergences in light-cone quantized QED. These are

1. Singularities in $H_{0}$ and the three-point interactions from fermions with $x=0$ $\left(k^{+}=0\right)$,
2. Singularities at $x=0$ and divergences near $x=0$ from photons in $H_{0}$ and the three-point interactions,
3. The singularity from the exchange of an instantaneous fermion at $x=0$, and
4. The singularity at $x=0$ and the divergence near $x=0$ from the exchange of an instantaneous photon.

The singularity described in item 1 can be removed by requiring anti-periodic boundary conditions for the fermions in the $x^{-}$direction. Similarly, the singularity in item 3 is removed if the fermions obey anti-periodic boundary conditions and the photons periodic boundary conditions because the momentum exchange will never be zero. Recall that the instantaneous fermion interaction is proportional to $1 / q^{+}$where $q^{+}=k_{\text {outgoing photon }}^{+}-k_{\text {incoming fermion }}^{+}$.

The singularity arising from photons with $x=0$ (point 2 ) is eliminated by the cut-off described in the previous section if $\vec{q}_{\perp} \neq \overrightarrow{0}_{\perp}$ because the invariant mass

-     - 

squared of such a photon would be greater than any finite $\Lambda^{2}$. That is,

$$
\begin{equation*}
\frac{q_{\perp}^{2}}{x}>\Lambda^{2} \tag{4.1}
\end{equation*}
$$

for $q^{+}=0$. The case of $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ is dealt with below. The singularity from instantaneous photons at $x=0$ (point 4) and $\vec{q}_{\perp} \neq \overrightarrow{0}_{\perp}$ is eliminated because instantaneous photons are treated for purposes of the cut-off as if they were real photons. As a result, they are also eliminated because

$$
\begin{equation*}
\frac{q_{\perp}^{2}}{x}>\Lambda^{2} \tag{4.2}
\end{equation*}
$$

where $q^{+}$and $\vec{q}_{\perp}$ are assigned to the instantaneous photon according to momentum conservation as explained in Section 3. Again, the situation for $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ is described below.

If periodic boundary conditions had been chosen for the fermions instead of anti-periodic conditions, the singularities at $x=0$ for real and instantaneous fermions would be eliminated by the same reasoning as for real and instantaneous photons.

The divergence as $x$ approaches 0 for real and instantaneous photons is removed by invoking an infrared cut-off,

$$
\begin{equation*}
\frac{q_{\perp}^{2}}{x} \geq \epsilon \tag{4.3}
\end{equation*}
$$

All Fock states with real photons not satisfying this condition and all instantaneous photon interactions not meeting this criterion are removed. Once again, $q^{+}$and $\vec{q}_{\perp}$ for a real Fock state photon are taken to be their actual values; $q^{+}$and $\vec{q}_{\perp}$ for an instantaneous photon are assigned according to momentum conservation as if it were a real photon.

Note that if $\epsilon$ is chosen to be any value smaller than $\left(\pi / L_{\perp}\right)^{2}$ but greater than 0 , then the only effect of the infrared cut-off is to remove photons with $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$. Since the effect of the cut-off is identical for all $\epsilon$ less than $\left(\pi / L_{\perp}\right)^{2}$, one may as well take the limit $\epsilon \rightarrow 0$ right away. Since the point $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ has now been removed, the problem of the $x=0$ singularity for real and instantaneous photons with zero $\vec{q}_{\perp}$ described above has been taken care of. Another way of removing the point - $x=0$ when $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ is to imagine that the photon has a small mass $\lambda$. Then $x=0$ would be eliminated for all $\vec{q}_{\perp}$ by the ultraviolet cut-off, Eq. (3.1).

The infrared cut-off is only necessary for numerical reasons when one uses a discrete measure. In the continuum the spectrum and wavefunction of positronium has no infrared divergence. The numerical problem is illustrated in Figure 4 which shows the divergent behavior of the lowest energy level in a variational calculation $\mathrm{as}^{-} K$ is increased if one does not use an infrared cutoff. Details of this calculation are described in Ref. 30. An explanation for this behavior is that the integral that must be reproduced to obtain the ground state energy level,

$$
\begin{equation*}
\left\langle\psi_{0}\right| H_{L C}\left|\psi_{0}\right\rangle=M_{0}^{2} \tag{4.4}
\end{equation*}
$$

has an integrand that diverges like

$$
\begin{equation*}
\frac{1}{x\left(q_{\perp}^{2}+m_{e}^{2}\right)-q_{\perp}^{2}} \tag{4.5}
\end{equation*}
$$

for small $x, \vec{q}_{\perp}$. Of course, the integral itself is still finite. In the continuum, the points near $x=0, \vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ are a set of measure zero and give a finite contribution to the integral. Unfortunately, in the discrete case, any one Fock state has a finite measure since there are only a finite number of Fock states. Each ( $e^{+} e^{-} \gamma$ ) Fock
state contributes one point to the sum, Eq. (4.4). As a result, the Fock states with photon $x$ near zero and $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ give a contribution proportional to $1 / x \sim K$. Thus photons with $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ must be removed by an infrared cut-off such as Eq. (4.3) to keep the sum Eq. (4.4) finite as $K \rightarrow \infty$.

Another way to eliminate this difficulty is to add and subtract an appropriate term in the Hamiltonian which removes the discretized infrared divergence and replaces this term at small $q_{\perp}$ and $x$ by the appropriate continuum value. We will discuss this method in detail in Ref. 5.

In summary, an infrared regulator is included by requiring that all photons, real and instantaneous, have invariant mass squared greater that $\epsilon$,

$$
\begin{equation*}
\frac{q_{\perp}^{2}}{x} \geq \epsilon \tag{4.6}
\end{equation*}
$$

This Lorentz invariant, (tree-level) gauge invariant regulator ensures that all infrared divergences are well defined and cancel in a charge-zero system such as positronium. The numerical demonstration for this last statement is given in Ref. 5 Since the only effect of the cut-off is to remove photons with $\vec{q}_{\perp}=\overrightarrow{0}_{\perp}$ for any $0<\epsilon<\left(\pi / L_{\perp}\right)^{2}$, the limit $\epsilon \rightarrow 0$ can be taken immediately. Also note that this infrared regulator is a continuum condition: the cut-off requirement is unaffected by discretization.

## 5. TRUNCATED FOCK SPACE

The basic program for solving $3+1$ QED using DLCQ has now been given: The light-cone Hamiltonian and bound state equation are given in Section 2, ultraviolet regularization is described in Section 3, and infrared regularization in Section 4. There are several problems which need to be confronted.

One must choose a consistent scheme for truncating the Fock space in order to have a system with finite number of degrees of freedom. In the case of one-space and one-time theories the parameter K automatically provides this truncation. In the case of physical theories in three space and one time, the covariant global and kinetic cutoffs define in Section 3 provide a physically motivated cut-off. Unlike the Tamm-Dancoff ${ }^{1}$ truncation, there is no a priori fixed limit on the number of particles in this scheme. Such a Fock space truncation also provides a continuum regularization for renormalization. Unlike lattice gauge theory this cut-off can be performed independent of the discretization. Ideally one should use ultraviolet regulators such as dimensional regularization in $d^{2} k_{\perp}$ or a generalized Pauli-Villars scheme. ${ }^{10}$ The Fock space truncation of the regulated theory then has only a mild effect at higher $\Lambda^{2}$.

However, a more fundamental problem is that as of yet, no non-perturbative prescription is available for renormalization to all orders in closed form. This problem needs to be answered before the full $3+1$ QED light-cone Hamiltonian can be systematically diagonalized. An example of the construction of a nonperturbative counter-term is presented in the next section.

A simple non-trivial approximation to $\operatorname{QED}(3+1)$ which retains its all orders non-perturbative features is the Tamm-Dancoff ${ }^{1}$ truncation to just two classes of Fock states on the light cone. To be specific, for the charge-zero sector, the Fock
space will be limited to just ( $e^{+}, e^{-}$) and ( $e^{+}, e^{-}, \gamma$ ). For the charge-one sector the only Fock states will be $\left(e^{-}\right)$and $\left(e^{-}, \gamma\right)$. The number of interactions effectively allowed in this truncated Fock space is very much reduced from the full set shown in Fig. 1. All graphs involving pair creation are effectively removed because the truncated Fock space does not allow for extra fermion pairs (diagrams 3, 6, 9, 11, $12,17,18$, and 19). Diagrams $14,16,20$, and 21 are effectively removed because they involve two photons in flight. Finally, diagram 10 is eliminated when it occurs in the presence of a spectator photon because such a situation also has two photons in flight. Taking all these removals into account, the only diagrams that need be considered are $1,2,4,5,10,13$, and 15 .

Limiting the Fock space may bring gauge invariance into question. However, we have carefully made sure that everytime an intermediate state with real photons is removed, the corresponding intermediate state with instantaneous photons is also removed. This restores gauge invariance because photons are thus removed from the theory in gauge invariant sets. For example the interaction $e^{+} e^{-} \rightarrow \gamma \rightarrow e^{+} e^{-}$ is removed from consideration because the intermediate state with one real photon has been eliminated. To restore gauge invariance, we have been careful to drop diagram 9 which involves the same process, but through an instantaneous photon.

It should be emphasized that though the Fock space is limited, the analysis remains non-perturbative because the allowed Fock states can be iterated as many times as one wishes. In particular, keeping only $\left(e^{+} e^{-}, e^{+} e^{-} \gamma\right)$ is similar to the ladder approximation in Bethe-Salpeter methods, which is an all orders calculation. Since this approximation has been solved in Bethe-Salpeter formalism for the spectrum of positronium, diagonalizing the light-cone QED Hamiltonian in this truncated Fock space must also reproduce the positronium spectrum. In the
following paper ${ }^{5}$ we show that the Bohr spectrum and the hyperfine splitting of positronium (actually, the muonium spectrum since the annihilation channel has been removed) at large $\alpha \sim 0.3$ is correctly reproduced.

## 6. RENORMALIZATION: SELF-INDUCED INERTIAS AND MASS COUNTERTERMS

Two issues are of concern regarding renormalization. First is the question of the self-induced inertias that appear in the theory if one does not normal-order the light-cone Hamiltonian. The second is whether the light-cone perturbation theory results for the one-loop radiative corrections agree with the usual Feynman S-matrix answers. Let us investigate the first question.

If one begins with a Hamiltonian that is not normal-ordered and proceeds to normal-order, one finds extra terms arising from interchanging operators in the instantaneous photon and instantaneous fermion interactions. These terms have been referred to in Refs. 3 and 26 as "self-induced inertias" and have been the source of much discussion concerning their role in light-cone physics. In 3+1 QED, these extra terms take the form

$$
\begin{equation*}
\frac{2 \alpha}{L_{\perp}^{2}} \sum_{\lambda, \underline{p}} a_{\lambda, \underline{\underline{p}}}^{\dagger} a_{\lambda, \underline{p}} J_{p}, \quad J_{p}=\frac{1}{2 p} \sum_{\underline{m}}[\{p-m \mid p-m\}-\{p+m \mid p+m\}] \tag{6.1}
\end{equation*}
$$

-     -         - 

for the photon and

$$
\begin{align*}
\frac{2 \alpha}{L_{\perp}^{2}} & {\left[\sum_{s, \underline{\boldsymbol{n}}} b_{s, \underline{\underline{n}}}^{\dagger} b_{s, \underline{n}}\left(I_{n}+K_{n}\right)+d_{s, \underline{n}}^{\dagger} d_{s, \underline{n}}\left(I_{n}+M_{n}\right)\right] } \\
I_{n} & =\frac{1}{2} \sum_{\underline{m}}\{[n-m \mid n-m]-[n+m \mid n+m]\}  \tag{6.2}\\
K_{n} & =\frac{1}{2} \sum_{\underline{q}} \frac{1}{q}\{n-q \mid n-q\} \\
M_{n} & =\frac{1}{2} \sum_{\underline{q}} \frac{1}{q}\{n+q \mid n+q\}
\end{align*}
$$

for the fermion. Remember that for fermion anti-periodic boundary conditions and photon periodic conditions,

$$
\begin{equation*}
p, q=2,4,6, \ldots, \quad m, n=1,3,5, \ldots \tag{6.3}
\end{equation*}
$$

The question then arises: should the self-induced inertias remain in the theory or should they be removed? Simply starting with a normal-ordered Hamiltonian eliminates these inertias. A satisfactory answer for the truncated Fock space we are considering is that they are not needed, i.e. they are replaced by the mass counterterms below. In the case of the fermion, this counterterm happens to have the same continuum limit as the original self-induced inertia in the limit $\Lambda \rightarrow \infty$. The correct procedure for all $\Lambda$ that properly renormalizes the fermion mass in the truncated Fock space requires mass counterterms equal to the one-loop lightcone perturbation theory mass counterterms. It should be noted that this result, which will be detailed below, only holds in the truncated space $\left(e^{+} e^{-}, e^{+} e^{-} \gamma\right)$ or $\left(e^{-}, e^{-} \gamma\right)$.

In our truncated Fock space, the full set of proper one-loop radiative corrections is shown in Figure 5 (improper graphs do not need to be renormalized). Again, there is no vacuum polarization because the Fock space does not allow an extra fermion pair to be created. Mass counterterms are needed to cancel these self-mass diagrams. The discretized counterterms are

$$
\begin{align*}
\delta H_{L C}^{(1)} & =-\frac{\varepsilon_{n, \overrightarrow{\vec{T}}_{\perp}}}{} \\
& =K \frac{2 \alpha}{L_{\perp}^{2}} \sum_{q, \vec{q}_{\perp}} \frac{\frac{1}{2 n(n-q)}\left[n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}\right]+\frac{n^{2}}{q^{2}}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}} \tag{6.4}
\end{align*}
$$

and

$$
\begin{align*}
& -\delta H_{L C}^{(2)}=-\sum_{N=2}^{\infty} \frac{\left.\left.s^{5} n^{2}\right\}, 2^{2}\right\}, 1+\{N\}}{n} \\
& =-K \frac{\beta_{f} \pi^{2}}{n L_{\perp}^{2}} \frac{\left[\frac{\alpha}{\pi^{2}} \sum_{q, \vec{q}_{\perp}} \frac{q}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}}\right]^{2}}{1+\frac{\alpha}{\pi} \sum_{q, \vec{q}_{\perp}} \frac{n-q}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}}} \tag{6.5}
\end{align*}
$$

where

$$
\begin{gather*}
\beta_{f}=\left(\frac{m_{e} L_{\perp}}{\pi}\right)^{2}, \quad \beta_{\gamma}=\left(\frac{\lambda L_{\perp}}{\pi}\right)^{2}, \quad \alpha=\frac{e^{2}}{4 \pi}, \\
n=1,3,5, \ldots \quad \text { (anti-periodic b.c.) }  \tag{6.6}\\
q=2,4,6, \ldots, \\
n^{i}, q^{i}=0, \pm 1, \pm 2, \ldots .
\end{gather*}
$$

$\left(n, \vec{n}_{\perp}\right)$ are the quantum numbers for the incoming fermion and $\lambda$ is a fake photon mass that is ultimately set equal to zero. The sum is over $2 \leq q \leq n-1$ and must
satisfy both the ultraviolet and infrared cut-offs,

$$
\begin{align*}
\frac{q_{\perp}^{2}+\beta_{\gamma}}{q}+\frac{\left(\vec{n}_{\perp}-\vec{q}_{\perp}\right)^{2}+\beta_{f}}{n-q} & \leq \frac{1}{K}\left(\frac{\Lambda L_{\perp}}{\pi}\right)^{2}-\sum_{\text {spec }} \frac{m_{\perp}^{2}+\beta}{m}  \tag{6.7}\\
\frac{q_{\perp}^{2}+\beta_{\gamma}}{q} & \geq \frac{1}{K}\left(\frac{L_{\perp}}{\pi}\right)^{2} \epsilon
\end{align*}
$$

The sum in the first equation is over the quantum numbers ( $m, \vec{m}_{\perp}$ ) of all the spectator particles (i.e. particles that go from the initial to final state without an interaction). The derivation of these results is given in Appendix B. Note that when one uses the Fock space truncation as a regulator, one must take into account the unavoidable dependence on the spectator kinematics for any finite cut-off.

Inclusion of these mass counterterms and diagonalizing the space ( $e^{-}, e^{-} \gamma$ ) reproduces the real electron mass to be one to 12 significant figures on an IBM 3090 running 64-bit (double precision) real variables and thus verifies that this is indeed the correct fermion mass renormalization prescription. The numerical results are shown in Section 7. If self-induced inertias are retained, the mass counterterm is modified to include -(self-induced inertias). This just cancels the original inertias and diagonalizing again reproduces the real electron mass $=1.000 \ldots m_{\epsilon}$.

Let us now return to the second question posed at the beginning of this section. The equivalence of the mass counterterms derived from the usual covariant Feynman theory and light-cone perturbation theory is discussed in Appendix C. It is shown there that the $\mathrm{TOPTh}_{\infty}$ and Feynman rules results for the one-loop fermion self-energy in Feynman gauge are identical if one is careful to do the $x$ integral first and interchange limit and integral only when allowed in the $\mathrm{TOPTh}_{\infty}$ calculation. If one takes the limit first, one obtains the non-Z graph as the complete answer, which agrees with the usual LCPTh answer for the one-loop fermion self-energy,
but disagrees with the Feynman answer. The discrepancy is found in a non-zero contribution from the Z-graph in TOPTh $_{\infty}$ near $x=0$. The LCPTh and Feynman rules answers for the one-loop fermion self-energy agree if an extra piece equal to the $\mathrm{TOPTh}_{\infty}$ Z-graph is added to the diagonal part of the light-cone Hamiltonian. This has to be done since the Z-graph contribution to the fermion self-energy is not obtained from the off-diagonal matrix elements of the light-cone Hamiltonian.

- However, since this piece is a self-energy, it is cancelled when one includes the corresponding mass counterterm.

In practice the extra piece from the Z-graph can thus be ignored. It should be emphasized that in the above deliberations, $\lambda$ is only included as an infrared regulator and is at the end taken to be zero. The conclusions do not carry over to theories with a true massive photon.

- This completes the discussion of electron mass renormalization. Due to the absence of pair creation, there is no renormalization arising from vacuum polarization in the truncated Fock space consideration. This leaves just electron wavefunction renormalization, which is equivalent to simply stating that the real electron's wavefunction is normalized. The probability of finding the bare Fock electron inside the real electron is given by the expansion coefficient $\psi_{e^{-}}$for the single electron Fock state shown in Eq. (2.17). This coefficient is just the wavefunction renormalization constant $\sqrt{Z_{2}}$.

To summarize, there is no photon wavefunction renormalization (charge renormalization) in the truncated Fock space, $\left(e^{+} e^{-}, e^{+} e^{-} \gamma\right)$ or $\left(e^{-}, e^{-} \gamma\right)$. Electron wavefunction renormalization is automatic because the real electron's wavefunction is normalized. If one is careful about the behavior near the endpoints, $x=0,1$, the one-loop self-mass corrections in $\mathrm{TOPTh}_{\infty}$ and LCPTH agree with the an-
swer from S-matrix analysis. Mass renormalization is then done by inserting mass counterterms into $H_{L C}$ that exactly cancel the one-loop self-mass contributions. If one decides to keep the "self-induced inertias", these are also cancelled by mass counterterms. Since the self-mass endpoint corrections and self-induced inertias are just cancelled anyway, what one effectively does is start with a normal-ordered Hamiltonian (i.e. without self-induced inertias) and inserts the mass counterterms - given in Eqs. (6.4) and (6.5). Once again, this prescription is valid only in the truncated Fock space of one additional photon. If higher Fock states are included, a more general method is necessary which may in fact include the self-induced inertias in a crucial way.

Since only elementary particles require renormalization, no further renormalization needs to be done. That is, there is no positronium mass or wavefunction reñormalization. The full light-cone Hamiltonian given by Eqs.(2.2)-(2.7) plus mass counterterms given by Eqs. (6.4) and (6.5) is now ready to be diagonalized.

## 7. DIAGONALIZATION: CHARGE-ONE SPACE

The prescription for diagonalizing the QED light-cone bound state equation Eq. (2.20) is then the following. $H_{L C}$ is equal to $H_{0}+H_{1}+H_{2}+H_{\text {self }}$ where $H_{0}$, $H_{1}$, and $H_{2}$ were given in Eqs. (2.2)- (2.7) and $H_{\text {self }}$ is the mass counterterms given in Eqs. (6.4) and (6.5). The Fock space is generated by keeping all Fock states that satisfy

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}} \leq \Lambda^{2} \tag{7.1}
\end{equation*}
$$

-     -         - 

and have photons that satisfy

$$
\begin{equation*}
\frac{q_{\perp}^{2}}{x} \geq \epsilon \tag{7.2}
\end{equation*}
$$

These two cut-off conditions are also applied to the instantaneous fermion and photon interactions with the instantaneous particles treated as if they were real particles. Diagonalizing gives the full mass spectrum of states and their corresponding wavefunctions as a Fock state expansion,

$$
\begin{equation*}
|\vec{\psi}\rangle=\sum_{n} \psi_{n}\left(x, \vec{k}_{\perp}\right)|n\rangle \tag{7.3}
\end{equation*}
$$

In principle, the true continuum theory is recovered by taking the limits $K, L_{\perp}, \Lambda \rightarrow$ $\infty$ and $\epsilon \rightarrow 0$. Recall from Section 4 that the results are identical for any choice of $\epsilon$ less than $\left(\pi / L_{\perp}\right)^{2}$; therefore, one is allowed to take the limit $\epsilon \rightarrow 0$ immediately. In this paper, the Fock space is limited for various reasons discussed in Section 5 to just ( $e^{-}, e^{-} \gamma$ ) for charge one and ( $e^{+} e^{-}, e^{+} e^{-} \gamma$ ) for charge zero.

Diagonalizing the light-cone Hamiltonian in the charge-one space of ( $e^{-}, e^{-} \gamma$ ) for any value of $\alpha, K, L_{\perp}, \Lambda$ and $\epsilon$ reproduces

$$
\begin{equation*}
M^{2}=1.000 \ldots m_{e}^{2} \tag{7.4}
\end{equation*}
$$

for the ground state. Remember that as pointed out in Section 5, in this truncated Fock space consideration, diagram 14 must be dropped from the full set of lightcone diagrams in Fig. 1. The accuracy of this result is only limited by machine precision. On an IBM 3090 running 64 -bit real variables, this is 12 places behind the decimal point. This result numerically proves that fermion mass renormalization is being done correctly in the truncated space $\left(e^{-}, e^{-} \gamma\right)$ because the physical mass
of the fermion (i.e. the ground state mass, $M$ ) is equal to the bare fermion mass, $m_{e}$.

One also obtains the fermion's structure function by summing the ground state wavefunction over all modes with a fixed $x$,

$$
\begin{equation*}
f(x)=\sum_{n, \text { fixed } x}\left|\psi_{n}\left(x, \vec{k}_{\perp}\right)\right|^{2} \tag{7.5}
\end{equation*}
$$

A typical structure function for $\alpha=.3$ is shown in Figure 6. As expected, the structure function is peaked at $x=1$ and has a characteristic long radiative tail.

## 8. SUMMARY

Discretized Light-Cone Quantization (DLCQ) has been presented as a fully relativistic discrete representation of quantum field theories and has been demonstrated to work in principle for quantum electrodynamics in three space and one time dimensions. Covariant, (tree-level) gauge invariant ultraviolet and infrared regularization were presented in Sections 3 and 4 and a complete renormalization scheme in the truncated Fock space of $\left(e^{-}, e^{-} \gamma\right)$ or $\left(e^{+} e^{-}, e^{+} e^{-} \gamma\right)$ was outlined in Section 6. The numerical check of the renormalization method is the demonstration that the electron's bare mass is equal to its physical mass using diagonalization. This is presented in Section 7.

Most of the positronium spectrum is contained in this truncated Fock space: the Bohr levels, $L \cdot S$ coupling, the hyperfine interaction, and the part of the Lamb shift from the fermion self-energy diagram are all included (the results obtained in
this truncated Fock space will actually be for muonium because the annihilation potential is not present).

A possible method of extending this procedure to include the Fock state with two photons, $\left(e^{+} e^{-} \gamma \gamma\right)$, is to include mass counterterms for the fermion self-mass diagrams with two photons in flight. A subset of these are shown in Figure 7. Including this Fock state with two photons should reproduce the full Lamb shift - excluding the Uehling term from vacuum polarization. The Uehling term can be included by further extending the Fock state to include ( $e^{+} e^{-} e^{+} e^{-}$). This extension can be implemented by introducing photon mass counterterms for the graphs in Figure 8. As explained in Appendix D and E of Ref. 30, photon mass counterterms are necessary because we are using a non-subtractive ultraviolet regularization scheme. A test of whether this is done correctly is to check that the ground state has $M^{2}=0$. This would verify that the bare photon mass remains equal to the physical photon mass. Including this extra Fock state also puts back the annihilation potential needed to calculate true positronium levels.

The method of DLCQ has a number of important positive attributes:

1. The technique is straightforward, non-perturbative, fully relativistic, and can be applied to quantum field theories in general, the most obvious candidate being quantum chromodynamics. Even the truncated Fock space analysis is non-perturbative since the Fock states that are allowed are iterated an infinite number of times. ${ }^{39}$
2. Due to the positivity of $P^{+}$, there are no interactions in the theory that create particles out of the vacuum. As a result, the vacuum structure is simple: the perturbative vacuum is the Fock state vacuum is the true vacuum, and they are all eigenstates of $H_{L C}$ with $M^{2}=0$. The possibility that the light-cone
vacuum can carry topological quantum numbers in the Schwinger model and non-Abelian gauge theories is discussed in Ref. 39.
3. Diagonalization has the potential of giving the full spectrum of bound states and scattering states along with their respective wavefunctions. Unlike equal time theory, ${ }^{40}$ the structure functions and distribution amplitudes needed in calculations of high-energy scattering processes can be obtained directly from the light-cone wavefunctions.
4. The fermions are treated-in a natural way. There are no fermion determinants or fermion doubling.
5. In $A^{+}=0$ gauge, there are only two physical photon polarizations.

The DLCQ method clearly has many advantages for solving non-perturbative problems in field theory. Many technical problems have been solved on how to regulate and renormalize gauge the Hamiltonian form of gauge theories quantized on the light. The true test of this procedure will be in the numerical applications. ${ }^{41}$ Results for the spectrum and bound-state wavefunctions for positronium at large $\alpha$ will be presented in a separate paper. ${ }^{5}$

## APPENDIX A

In this appendix the tree-level Møller scattering amplitude ( $e^{-} e^{-} \rightarrow e^{-} e^{-}$) derived using Feynman's $S$ matrix approach is shown to be identical to that derived from light-cone perturbation theory (LCPTh). The rules for LCPTh are given in Appendix B in Ref. 9 and Appendix A in Ref. 10 and can be derived from the ${ }^{-}$light-cone Hamiltonian $H_{L C}$ given in Eqs. (2.2)- (2.7).

The diagrams that must be considered in LCPTh are given in Figure 9 with light-cone time $x^{+}$flowing from left to right and momenta assigned as shown. Using $P^{+}$and $\vec{P}_{\perp}$ momentum conservation, $q$ and $q^{\prime}$ are

$$
\begin{align*}
& q^{+}=l_{i}^{+}-l_{f}^{+}=k_{f}^{+}-k_{i}^{+} \\
& \vec{q}_{\perp}=\vec{l}_{\perp i}-\vec{l}_{\perp f}=\vec{k}_{\perp f}-\vec{k}_{\perp i} \\
& q^{-}=\frac{q_{\perp}^{2}+\lambda^{2}}{q^{+}}  \tag{A.1}\\
& q^{\mu^{\prime}}=-q^{\mu}
\end{align*}
$$

Note that the photon's 4 -momentum, $q$, is on mass shell. Remember that $P^{-}$is not necessarily conserved, so

$$
\begin{equation*}
q^{-} \neq l_{i}^{-}-l_{f}^{-} \neq k_{f}^{-}-k_{i}^{-} . \tag{A.2}
\end{equation*}
$$

Using the LCPTh rules found in Ref. 9 or 10 and performing the sum over photon
polarizations gives the following for the three LCPTh graphs,

$$
\begin{align*}
T_{f i}^{(1)}= & e^{2} \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \frac{\eta^{\mu} \eta^{\nu}}{\left(q^{+}\right)^{2}} \\
T_{f i}^{(2)}= & e^{2} \theta\left(q^{+}\right) \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \\
& \times\left[-g^{\mu \nu}+\frac{\eta^{\mu} q^{\nu}+\eta^{\nu} q^{\mu}}{q^{+}}\right] \frac{1}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}  \tag{A.3}\\
T_{f i}^{(3)}= & e^{2} \theta\left(-q^{+}\right) \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \\
& \times\left[-g^{\mu \nu}+\frac{\eta^{\mu} q^{\nu}+\eta^{\nu} q^{\mu}}{q^{+}}\right] \frac{1}{-q^{+}\left(k_{i}^{-}-k_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}
\end{align*}
$$

where $\eta^{\mu}=\left(0,2, \overrightarrow{0}_{\perp}\right)$. Note that $T_{f i}^{(1)}$ diverges like $1 /\left(q^{+}\right)^{2}$ for small $q^{+}$. The sum of these three amplitudes is

$$
\begin{align*}
& T_{f i}= e^{2} A_{\mu} B_{\nu}\left\{\frac{\eta^{\mu} \eta^{\nu}}{\left(q^{+}\right)^{2}}+\left[-g^{\mu \nu}+\frac{\eta^{\mu} q^{\nu}+\eta^{\nu} q^{\mu}}{q^{+}}\right]\right. \\
&\left.\times\left[\frac{\theta\left(q^{+}\right)}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}+\frac{\theta\left(-q^{+}\right)}{-q^{+}\left(k_{i}^{-}-k_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}\right]\right\},  \tag{A.4}\\
& A_{\mu}=\bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right), \quad B_{\nu}=\bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) .
\end{align*}
$$

Writing out the components $\mu, \nu=+,-, 1,2$ explicitly, one finds after some algebra,

$$
\begin{align*}
A_{\mu} B_{\nu} & \theta\left(q^{+}\right)\left[\frac{\eta^{\mu} \eta^{\nu}}{\left(q^{+}\right)^{2}}+\frac{\eta^{\mu} q^{\nu}+\eta^{\nu} q^{\mu}}{q^{+}} \frac{1}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}\right] \\
& =A_{\mu} B_{\nu} \frac{\theta\left(q^{+}\right)}{q^{+}} \frac{1}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q^{+} q^{-}+i \epsilon}\left[\eta^{\mu}\left(l_{i}-l_{f}\right)^{\nu}+\eta^{\nu}\left(l_{i}-l_{f}\right)^{\mu}\right] \tag{A.5}
\end{align*}
$$

This expression can be summed with a similar expression for the $\theta\left(-q^{+}\right)$term to

-     - 

give

$$
\begin{align*}
T_{f i}= & e^{2} \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \\
\times\{ & -g^{\mu \nu}\left[\frac{\theta\left(q^{+}\right)}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q_{\perp}^{2}-\lambda^{2}+i \epsilon}+\frac{\theta\left(-q^{+}\right)}{q^{+}\left(k_{f}^{-}-k_{i}^{-}\right)-q_{\perp}^{2}-\lambda^{2}+i \epsilon}\right] \\
& +\theta\left(q^{+}\right) \frac{\eta^{\mu}\left(l_{i}-l_{f}\right)^{\nu}+\eta^{\nu}\left(l_{i}-l_{f}\right)^{\mu}}{q^{+}} \frac{1}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q_{\perp}^{2}-\lambda^{2}+i \epsilon} \\
& \left.+\theta\left(-q^{+}\right) \frac{\eta^{\mu}\left(k_{f}-k_{i}\right)^{\nu}+\eta^{\nu}\left(k_{f}-k_{i}\right)^{\mu}}{q^{+}} \frac{1}{q^{+}\left(k_{f}^{-}-k_{i}^{-}\right)-q_{\perp}^{2}-\lambda^{2}+i \epsilon}\right\} \tag{A.6}
\end{align*}
$$

This result is valid for on- or off-shell electrons and does not assume $P^{-}$momentum conservation. Note that this final expression for $T_{f i}$ diverges only like $1 / q^{+}$for small $q^{+}$. The leading $1 /\left(q^{+}\right)^{2}$ behavior from $T_{f i}^{(1)}$ is apparently cancelled by a similar singularity from $T_{f i}^{(2)}$ and $T_{f i}^{(3)}$.

The Feynman rules answer can be obtained by first enforcing four-momentum conservation (i.e. $k_{i}^{-}+l_{i}^{-}=k_{f}^{-}+l_{f}^{-}$),

$$
\begin{align*}
T_{f i}= & e^{2} \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \\
& \times \frac{1}{q^{+}\left(l_{i}^{-}-l_{f}^{-}\right)-q_{\perp}^{2}-\lambda^{2}+i \epsilon}\left[-g^{\mu \nu}+\frac{\eta^{\mu}\left(l_{i}-l_{f}\right)^{\nu}+\eta^{\nu}\left(l_{i}-l_{f}\right)^{\mu}}{q^{+}}\right] \tag{A.7}
\end{align*}
$$

and then requiring the electrons to be on-shell (i.e. $\bar{u}\left(l_{f}\right)\left(\mu_{i}-\mu_{f}\right) u\left(l_{i}\right)=\bar{u}\left(k_{f}\right)$ $\left.\left(\lambda_{i}-l_{f}\right) u\left(k_{i}\right)=\ldots=0\right)$,

$$
\begin{equation*}
T_{f i}=-e^{2} \bar{u}\left(l_{f}\right) \gamma_{\mu} u\left(l_{i}\right) \bar{u}\left(k_{f}\right) \gamma_{\nu} u\left(k_{i}\right) \frac{g^{\mu \nu}}{q_{F R}^{2}-\lambda^{2}+i \epsilon} \tag{A.8}
\end{equation*}
$$

$q_{F R}^{\mu}$ is defined to be $l_{i}^{\mu}-l_{f}^{\mu}=k_{f}^{\mu}-k_{i}^{\mu}$. This last answer is recognized as the familiar answer for Møller scattering using Feynman rules. Note that the above
analysis only holds for small $\lambda$. The conclusions should not be carried over to finite $\lambda$ theories.

## APPENDIX B

The calculation of various self-mass diagrams is given in this appendix. The first to be considered is the familiar one-loop fermion self-mass diagram shown in Figure 10. The various momenta are

$$
\begin{align*}
p & =\left(x P, \frac{x^{2} p_{\perp}^{2}+m_{e}^{2}}{x P}, x \vec{p}_{\perp}\right) \\
k_{1} & =\left(y P, \frac{\left(\vec{k}_{\perp}+y \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y P}, \vec{k}_{\perp}+y \vec{p}_{\perp}\right)  \tag{B.1}\\
k_{2} & =\left((x-y) P, \frac{\left(-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{(x-y) P},-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right) .
\end{align*}
$$

The light-cone perturbation theory (LCPTh) amplitude for this process is

$$
\begin{align*}
T_{f i} & =\frac{g^{2}}{16 \pi^{3}} \frac{1}{P} \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{1}{y(x-y)} \frac{N}{D+i \epsilon} \\
N & =\bar{u}(p) \notin u\left(k_{2}\right) \bar{u}\left(k_{2}\right) \not 申^{*} u(p)  \tag{B.2}\\
D & =\frac{x^{2} p_{\perp}^{2}+m_{e}^{2}}{x P}-\frac{\left(\vec{k}_{\perp}+y \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y P}-\frac{\left(-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{(x-y) P}
\end{align*}
$$

The rules for LCPTh QED are derived in Appendix B in Ref. 9 and Appendix A
in Ref. 10. The photon spin sum can be done by using the relation

$$
\begin{equation*}
\sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu *}=-g^{\mu \nu}+\frac{\eta^{\mu} q^{\nu}+\eta^{\nu} q^{\mu}}{q^{+}} \tag{B.3}
\end{equation*}
$$

which holds for the spinors given in Eq. (2.10) with $\eta^{\mu}=\left(0,2, \overrightarrow{0}_{\perp}\right)$. Doing the numerator algebra and simplifying the denominator produces the desired answer

$$
\begin{equation*}
T_{f i}=-\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} x \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{\frac{1}{x(x-y)}\left[x^{2} k_{\perp}^{2}+y^{2} m_{e}^{2}\right]+\frac{2}{y^{2}}\left[x^{2} k_{\perp}^{2}+x(x-y) \lambda^{2}\right]}{x^{2} k_{\perp}^{2}+y^{2} m_{e}^{2}+\lambda^{2} x(x-y)-i \epsilon} \tag{B.4}
\end{equation*}
$$

$\delta_{s s^{\prime}}$ is a delta function between the incoming and outgoing fermion spins. Note that as expected from Lorentz invariance, this answer is independent of $\vec{p}_{\perp}$. If one changes variables to $z=y / x$, one also finds that the answer is independent of $P$ and $x$. Since $T_{f i}$ evidently does not depend on any of the quantum numbers of the incoming fermion, $T_{f i}$ can be considered to be a pure mass renormalization.

The quantities actually discretized are $x, y, \vec{p}_{\perp}=x \vec{p}_{\perp}$, and $\vec{k}_{\perp}^{\prime}=\vec{k}_{\perp}+y \vec{p}_{\perp}$ or $-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}$. The choice between these last two is irrelevant. Rewriting $T_{f i}$ in terms of these quantities gives
$T_{f i}=-\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} x \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{\frac{1}{x(x-y)}\left[x^{2}\left(\vec{k}_{\perp}^{\prime}-\frac{y}{x} \vec{p}_{\perp}\right)^{2}+y^{2} m_{e}^{2}\right]+\frac{2}{y^{2}}\left[x^{2}\left(\vec{k}_{\perp}^{\prime}-\frac{y}{x} \vec{p}_{\perp}\right)^{2}+x(x-y) \lambda^{2}\right]}{x^{2}\left(\vec{k}_{\perp}^{\prime}-\frac{y}{x} \vec{p}_{\perp}^{\prime}\right)^{2}+y^{2} m_{e}^{2}+\lambda^{2} x(x-y)-i \epsilon}$

This answer is discretized by replacing

$$
\begin{gather*}
x=\frac{n}{K}, \quad y=\frac{q}{K}, \quad \vec{p}_{\perp}=\frac{\pi \vec{n}_{\perp}}{L_{\perp}}, \quad \vec{k}_{\perp}^{\prime}=\frac{\pi \vec{q}_{\perp}}{L_{\perp}}, \\
\int d y=\frac{2}{K} \sum_{q}, \quad \int d^{2} \vec{k}_{\perp}=\left(\frac{\pi}{L_{\perp}}\right)^{2} \sum_{\vec{q}_{\perp}} \tag{B.6}
\end{gather*}
$$

where $\pi n / L$ and $\pi \vec{n}_{\perp} / L_{\perp}$ are the $P^{+}$and $\vec{P}_{\perp}$ of the incoming fermion, respectively, and $q=2,4,6, \ldots$. A factor of $1 / x$ is also necessary because in the continuum,

-     -         - 

factors of $1 / \sqrt{x}$ from external wavefunctions are conventionally associated with the wavefunctions themselves; whereas in the discretized case, the factors of $1 / \sqrt{x}$ are absorbed into $P^{-}$. These steps give the result

$$
\begin{equation*}
T_{f i}=-\delta_{s s^{\prime}} K \frac{2 \alpha}{L_{\perp}^{2}} \sum_{q, \vec{q}_{\perp}} \frac{\frac{1}{2 n(n-q)}\left[n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}\right]+\frac{n^{2}}{q^{2}}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}} \tag{B.7}
\end{equation*}
$$

where $\beta_{f}=\left(m L_{\perp} / \pi\right)^{2}$ and $\beta_{\gamma}=\left(\lambda L_{\perp} / \pi\right)^{2}$. The photon mass, $\lambda$, has been set equal to zero in the numerator in this last expression.

Ultraviolet and infrared regulators are implemented by requiring that the intermediate state in Fig. 10 satisfy

$$
\begin{equation*}
\sum_{i} \frac{k_{\perp i}^{2}+m_{i}^{2}}{x_{i}} \leq \Lambda^{2}, \quad \frac{\left(\vec{k}_{\perp}+y \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y} \geq \epsilon \tag{B.8}
\end{equation*}
$$

which in terms of the discrete variables given above reads

$$
\begin{align*}
\frac{q_{\perp}^{2}+\beta_{\gamma}}{q}+\frac{\left(\vec{n}_{\perp}-\vec{q}_{\perp}\right)^{2}+\beta_{f}}{n-q} & \leq \frac{1}{K}\left(\frac{\Lambda L_{\perp}}{\pi}\right)^{2}-\sum_{\mathrm{spec}} \frac{m_{\perp}^{2}+\beta}{m}  \tag{B.9}\\
\frac{q_{\perp}^{2}+\beta_{\gamma}}{q} & \geq \frac{1}{K}\left(\frac{L_{\perp}}{\pi}\right)^{2} \epsilon
\end{align*}
$$

Here, $\beta_{i}$ is equal to $\left(m_{i} L_{\perp} / \pi\right)^{2}$. The sum is over any spectator particles that might occur during the process. The correct mass counterterm that should be inserted in $H_{L C}$ to ensure that the fermion's bare mass is equal to its physical mass is the negative of Eq. (B.7) where the sum is over $q^{i}=0, \pm 1, \pm 2, \ldots$ and $q=2,4,6, \ldots, n-1$ that satisfy Eq. (B.9).

The next self-mass diagram to consider is shown in Figure 11. The momenta
are assigned to be

$$
\begin{align*}
p & =\left(x P, \frac{x^{2} p_{\perp}^{2}+m_{e}^{2}}{x P}, x \vec{p}_{\perp}\right) \\
k_{1} & =\left(y P, \frac{\left(\vec{k}_{\perp}+y \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y P}, \vec{k}_{\perp}+y \vec{p}_{\perp}\right), \\
k_{2} & =\left((x-y) P, \frac{\left(-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{(x-y) P},-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right),  \tag{B.10}\\
l_{1} & =\left(z P, \frac{\left(\vec{l}_{\perp}+z \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{z P}, \vec{l}_{\perp}+z \vec{p}_{\perp}\right), \\
l_{2} & =\left((x-z) P, \frac{\left(-\vec{l}_{\perp}+(x-z) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{(x-z) P},-\vec{l}_{\perp}+(x-z) \vec{p}_{\perp}\right)
\end{align*}
$$

and the answer in LCPTh is

$$
\begin{align*}
T_{f i} & =\frac{1}{2 x P}\left(\frac{g^{2}}{16 \pi^{3}}\right)^{2} \int_{0}^{x} d y d z \int d^{2} \vec{k}_{\perp} d^{2} l_{\perp} \frac{1}{y(x-y) z(x-z)} \frac{N}{D} \\
N & =\bar{u}(p) \not \subset\left(l_{1}\right) u\left(l_{2}\right) \bar{u}\left(l_{2}\right) \not \subset\left(l_{1}\right)^{*} \gamma^{+} \not \ell\left(k_{1}\right) u\left(k_{2}\right) \bar{u}\left(k_{2}\right) \not \subset\left(k_{1}\right)^{*} u(p) \\
D & =\left[\frac{x^{2} p_{\perp}^{2}+m_{e}^{2}}{x}-\frac{\left(\vec{k}_{\perp}+y \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y}-\frac{\left(-\vec{k}_{\perp}+(x-y) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{x-y}+i \epsilon\right] \\
& \times\left[\frac{x^{2} p_{\perp}^{2}+m_{e}^{2}}{x}-\frac{\left(\vec{l}_{\perp}+z \vec{p}_{\perp}\right)^{2}+\lambda^{2}}{y}-\frac{\left(-\vec{l}_{\perp}+(x-z) \vec{p}_{\perp}\right)^{2}+m_{e}^{2}}{x-z}+i \epsilon\right] \tag{B.11}
\end{align*}
$$

The numerator algebra is done by using the photon spin sum relation Eq. (B.3), applying symmetric integration to eliminate various terms proportional to $k^{i}$ and $l^{i}$ (upon simplification, the denominator turns out to only involve $k_{\perp}^{2}$ and $l_{\perp}^{2}$ ), and making use of the spinor properties shown in Appendix F. The answer for the numerator,

$$
\begin{equation*}
N=8 P m_{e}^{2} \frac{z y}{x} \delta_{s s^{\prime}}, \tag{B.12}
\end{equation*}
$$

turns out to only have a contribution from the spin-flip interaction of $H_{L C}$. The

-     -         - 

complete answer is then

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}}\left[\frac{m_{e} g^{2}}{8 \pi^{3}} \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{y}{x^{2} k_{\perp}^{2}+y^{2} m m_{e}^{2}+\lambda^{2} x(x-y)-i \epsilon}\right]^{2} \tag{B.13}
\end{equation*}
$$

Again, changing variables to $z=y / x$ demonstrates that this result is independent of $x, P$, and $\vec{p}_{\perp}$ and is therefore a pure mass renormalization.

Next, consider the case of $N$ one-loop fermion self-mass pieces all connected by instantaneous fermions shown in Figure 12. As above, momenta are assigned and the LCPTh answer is written down for $T_{f i}$. The numerator and denominator are both factorizable, giving an answer of

$$
\begin{equation*}
-T_{f i}^{(N)}=\left[\frac{g^{2}}{8 \pi^{3} x} \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{1}{y} \frac{1}{\frac{m_{e}^{2}}{x}-\frac{k_{\perp}^{2}+\lambda^{2}}{y}-\frac{k_{\perp}^{2}+m_{e}^{2}}{x-y}+i \epsilon}\right]^{N-2} T_{f i}^{(2)} \tag{B.14}
\end{equation*}
$$

where $T_{f i}^{(2)}$ is the answer for the diagram in Fig. 11. Using

$$
\begin{equation*}
\sum_{N=2}^{\infty}=\frac{1}{1-x} \tag{B.15}
\end{equation*}
$$

and substituting in Eq. (B.13) for $T_{f i}^{(2)}$ yields

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} \frac{m_{e}^{2}\left[\frac{g^{2}}{8 \pi^{3}} \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{y}{x^{2} k_{\perp}^{2}+y^{2} m_{e}^{2}+\lambda^{2} x(x-y)-i \epsilon}\right]^{2}}{1+\frac{g^{2}}{8 \pi^{3}} \int_{0}^{x} d y \int d^{2} \vec{k}_{\perp} \frac{x}{x^{2} k_{\perp}^{2}+y^{2} m_{e}^{2}+\lambda^{2} x(x-y)-i \epsilon}} \tag{B.16}
\end{equation*}
$$

as the amplitude for the process shown in Figure 13. Similarly to above, this result is discretized by re-writing in terms of $x, y, \vec{p}_{\perp}=x \vec{p}_{\perp}$, and $\vec{k}_{\perp}^{\prime}=\vec{k}_{\perp}+y \vec{p}_{\perp}$ and

-     -         - 

making the substitutions in Eq. (B.6) to give

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} K \frac{\beta_{f} \pi^{2}}{n L_{\perp}^{2}} \frac{\left[\frac{\alpha}{\pi^{2}} \sum_{q, \vec{q}_{\perp}} \frac{q}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}}\right]^{2}}{1+\frac{\alpha}{\pi^{2}} \sum_{q, \vec{q}_{\perp}} \frac{n-q}{n^{2}\left(\vec{q}_{\perp}-\frac{q}{n} \vec{n}_{\perp}\right)^{2}+q^{2} \beta_{f}+n(n-q) \beta_{\gamma}}} \tag{B.17}
\end{equation*}
$$

This answer is subject to the same regularization conditions as above, Eq. (B.9). The mass counterterm necessary in $H_{L C}$ is the negative of Eq. (B.17) subject to the conditions, Eq. (B.9). A combination of the mass counterterms, Eq. (B.7) and Eq. (B.17), provides the full mass renormalization needed in the truncated Fock space $\left(e^{-}, e^{-} \gamma\right)$ or $\left(e^{+} e^{-}, e^{+} e^{-} \gamma\right)$.

## APPENDIX C

The equivalence of answers derived using Feynman's S-matrix analysis and using infinite momentum frame time-ordered perturbation theory ( $\mathrm{TOPTh}_{\infty}$ ) is demonstrated in this appendix for the one-loop fermion self-energy diagram in Feynman gauge. Since it is believed that light-cone perturbation theory (LCPTh) and $\mathrm{TOPTh}_{\infty}$ are mathematically equivalent, this demonstration makes the equivalence of LCPTh and Feynman rules results for one-loop radiative corrections plausible. The analysis for the fermion self-energy is done in Feynman gauge for convenience, though the analysis should be similar in light-cone gauge in the limit $\lambda \rightarrow 0$.

First, the Feynman rules answer for the fermion self-energy graph shown in

Fig. 14 is described briefly. We start with the familiar result

$$
\begin{equation*}
T_{f i}=-\frac{i g^{2}}{(2 \pi)^{4}} \int d^{4} k \frac{\bar{u}(p) \gamma^{\mu}\left(\not p-\nmid+m_{e}\right) \gamma_{\mu} u(p)}{\left[(p-k)^{2}-m_{e}^{2}+i \epsilon\right]\left(k^{2}-\lambda^{2}+i \epsilon\right)} \tag{C.1}
\end{equation*}
$$

A factor of $-i$ has been included to facilitate comparison with $\mathrm{TOPTh}_{\infty}$. Doing the numerator algebra, combining denominators, changing variables to $q^{\mu}=k^{\mu}-x p^{\mu}$, and eliminating terms proportional to $q^{\mu}$ by symmetric integration gives

$$
\begin{align*}
T_{f i} & =-\delta_{s s^{\prime}} \frac{i g^{2}}{(2 \pi)^{4}} \int d^{4} q \int_{0}^{1} d x \frac{4 m_{e}^{2}(1+x)}{\left[q^{2}-a^{2}+i \epsilon\right]^{2}}  \tag{C.2}\\
a^{2} & =m_{e}^{2} x^{2}+\lambda^{2}(1-x)
\end{align*}
$$

The delta function is between the spin of the incoming and outgoing fermion. Doing the $q^{0}$ integral by contour integration and then the $q^{3}$ integral by standard methods results in

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1} d x \int d^{2} \vec{q}_{\perp} \frac{2 m_{e}^{2}(1+x)}{q_{\perp}^{2}+x^{2} m_{e}^{2}+\lambda^{2}(1-x)-i \epsilon} \tag{C.3}
\end{equation*}
$$

This answer diverges like $\log q_{\perp}^{2}$ for large $\vec{q}_{\perp}$; it is therefore necessary to introduce a regulator such as subtracting a Pauli-Villars contribution.

Now consider the same process in $\mathrm{TOPTh}_{\infty}$. The $\mathrm{TOPTh}_{\infty}$ rules for QED in Feynman gauge are given in Ref. 25. Two graphs need to be considered, the usual time-ordering and the Z-graph. These are pictured in Fig. 15. Momenta are assigned to the various legs of the usual time-ordering contribution,

$$
\begin{array}{ll}
p=\left(E, \overrightarrow{0}_{\perp}, P\right), & k_{1}=\left(E_{1}, \vec{k}_{\perp}, x P\right), \\
E=\sqrt{P^{2}+m_{e}^{2}}, & E_{1}=\sqrt{x^{2} P^{2}+\lambda_{\perp}^{2}}, \\
\left.E E_{2},-\vec{k}_{\perp},(1-x) P\right) \\
& E_{2}=\sqrt{(1-x)^{2} P^{2}+m_{\perp}^{2}}
\end{array}
$$

$$
\begin{equation*}
\lambda_{\perp}^{2}=k_{\perp}^{2}+\lambda^{2}, \quad m_{\perp}^{2}=k_{\perp}^{2}+m_{e}^{2} \tag{C.4}
\end{equation*}
$$

The time-ordered perturbation theory answer for this graph is

$$
\begin{align*}
T_{f i} & =\frac{g^{2}}{4(2 \pi)^{3}} P \int_{-\infty}^{\infty} d x \int d^{2} \vec{k}_{\perp} \frac{1}{E_{1} E_{2}} \frac{N}{D+i \epsilon}-(\lambda \rightarrow \Lambda), \\
N & =\bar{u}(p) \phi u\left(k_{2}\right) \bar{u}\left(k_{2}\right) \not^{*} u(p),  \tag{C.5}\\
D & =E-E_{1}-E_{2} .
\end{align*}
$$

A Pauli-Villars contribution has been subtracted for ultraviolet regularization. The TOPTh $_{\infty}$ answer is gotten by letting $P$ approach infinity, and the numerator is evaluated with the help of the relation

$$
\begin{equation*}
\sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu *}=-g^{\mu \nu} \tag{C.6}
\end{equation*}
$$

which holds in Feynman gauge. This gives the result

$$
\begin{gather*}
T_{f i}=\lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{-\infty}^{\infty} d x \int d^{2} \vec{k}_{\perp}[I(\lambda, P)-I(\Lambda, P)] \\
I(\lambda, P)=\frac{1}{\sqrt{x^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}} \sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}} \frac{\sqrt{1+\left(\frac{m_{e}}{P}\right)^{2}} \sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}-(1-x)-2\left(\frac{m_{e}}{P}\right)^{2}}{\sqrt{1+\left(\frac{m_{e}}{P}\right)^{2}}-\sqrt{x^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}}-\sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}+i \epsilon} \tag{C.7}
\end{gather*}
$$

for the usual time-ordering in $\mathrm{TOPTh}_{\infty}$. Note that all the square roots are assumed to be positive.

The usual procedure is then to take the limit $P \rightarrow \infty$ inside the integral to simplify $I(\lambda, P)$. This is valid as long as one is not near the points $x=0,1$, which are singular for $P=\infty$. It is necessary to do a more detailed analysis near these two points. The integral is split into three regions: $x<0,0<x<1, x>1$.

1. In the first region, $E \rightarrow P\left[1+\frac{1}{2}\left(\frac{m}{P}\right)^{2}\right], E_{1} \rightarrow-x P\left[1+\frac{1}{2}\left(\frac{\lambda_{1}}{x P}\right)^{2}\right]$, and $E_{2} \rightarrow$ $(1-x) P\left[1+\frac{1}{2}\left(\frac{m_{\perp}}{(1-x) P}\right)^{2}\right]$ as $P \rightarrow \infty . I(\lambda, P)$ approaches

$$
\begin{equation*}
-\frac{1}{x(1-x)} \frac{\frac{1}{2}(1-x) \frac{m_{e}^{2}}{P^{2}}+\frac{1}{2} \frac{m_{\perp}^{2}}{(1-x) P^{2}}-2 \frac{m_{e}^{2}}{P^{2}}}{2 x} \underset{P \rightarrow \infty}{\longrightarrow} 0 \tag{C.8}
\end{equation*}
$$

which is non-singular. Therefore, taking the limit before doing the $x$ integral is allowed, giving the result

$$
\begin{equation*}
T_{f i}^{(1)}=0 . \tag{C.9}
\end{equation*}
$$

-2 . In this region, $E \rightarrow P\left[1+\frac{1}{2}\left(\frac{m}{P}\right)^{2}\right], E_{1} \rightarrow x P\left[1+\frac{1}{2}\left(\frac{\lambda_{1}}{x P}\right)^{2}\right], E_{2} \rightarrow$ $(1-x) P\left[1+\frac{1}{2}\left(\frac{m_{\perp}}{(1-x) P}\right)^{2}\right]$ and

$$
\begin{align*}
I(\lambda, P) & \rightarrow \frac{1}{x(1-x)} \frac{(1-x) m_{e}^{2}+\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}-4 m_{e}^{2}}{m_{e}^{2}-\frac{k_{\perp}^{2}+\lambda^{2}}{x}-\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}+i \epsilon} \\
& =\frac{1}{1-x} \frac{(1-x)^{2} m_{e}^{2}+k_{\perp}^{2}+m_{e}^{2}-4 m_{e}^{2}(1-x)}{x(1-x) m_{e}^{2}-(1-x)\left(k_{\perp}^{2}+\lambda^{2}\right)-x\left(k_{\perp}^{2}+m_{e}^{2}\right)+i \epsilon} \tag{C.10}
\end{align*}
$$

as $P \rightarrow \infty . I(\lambda, P)$ has a singularity near $x=1$. The integral for region 2 is split again into two parts

$$
\begin{equation*}
T_{f i}^{(2)}=\lim _{\epsilon \rightarrow 0} \lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}}\left[\int_{0}^{1-\epsilon} d x+\int_{1-\epsilon}^{1} d x\right] \int d^{2} \vec{k}_{\perp}[I(\lambda, P)-I(\Lambda, P)] \tag{C.11}
\end{equation*}
$$

(a) In the region $0<x<1-\epsilon$, we are away from the singularity so the
limit $P \rightarrow \infty$ can be taken inside the integral to produce the answer

$$
\begin{equation*}
T_{f i}^{(2 a)}=\lim _{\epsilon \rightarrow 0} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1-\epsilon} d x \int d^{2} \vec{k}_{\perp} \frac{1}{x(1-x)} \frac{(1-x) m_{e}^{2}+\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}-4 m_{e}^{2}}{m_{e}^{2}-\frac{k_{\perp}^{2}+\lambda^{2}}{x}-\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}+i \epsilon}-(\lambda \rightarrow \Lambda) \tag{C.12}
\end{equation*}
$$

(b) The non-singular part of $I(\lambda, P)$ is expanded in powers of $(1-x)$ to give the form

$$
\begin{equation*}
I(\lambda, P)=\frac{1}{\sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}} \sum_{n=0}^{\infty} A_{n}(\lambda, P)(1-x)^{n} \tag{C.13}
\end{equation*}
$$

for $I(\lambda, P)$. The contribution to $T_{f i}$ is then

$$
\begin{align*}
T_{f i}^{(2 b)}= & \lim _{\epsilon \rightarrow 0} \lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{1-\epsilon}^{1} d x \int d^{2} \vec{k}_{\perp}  \tag{C.14}\\
& \frac{1}{\sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}} \sum_{n=0}^{\infty}\left[A_{n}(\lambda, P)-A_{n}(\Lambda, P)\right](1-x)^{n}
\end{align*}
$$

Since $\lambda$ and $\Lambda$ appear in $I$ only as $\lambda / P$ and $\Lambda / P$, it must be that $A_{n}(\lambda, P)-A_{n}(\Lambda, P)$ approaches zero at least like $1 / P$ as $P \rightarrow \infty$. One can expand $A_{n}$ in powers of $1 / P$ to see this. As $P \rightarrow \infty$, the most divergent $x$ integral is

$$
\begin{equation*}
\int_{1-\epsilon}^{\epsilon} d x \frac{(1-x)^{0}}{\sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}}=\log \left(\frac{\epsilon+\sqrt{\epsilon^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}}{\frac{\left|m_{\perp}\right|}{P}}\right) \underset{P \rightarrow \infty}{\longrightarrow} \log \frac{2 \epsilon P}{\left|m_{\perp}\right|} . \tag{C.15}
\end{equation*}
$$

The final answer as $P \rightarrow \infty$ is then

$$
\begin{equation*}
T_{f i}^{(2 b)} \rightarrow \frac{1}{P} \log P \rightarrow 0 \tag{C.16}
\end{equation*}
$$

3. Finally, in the third region, $x>1$,

$$
\begin{equation*}
I(\lambda, P) \underset{P \rightarrow \infty}{\longrightarrow} \frac{1}{x(1-x)} \tag{C.17}
\end{equation*}
$$

which is singular near $x=1$. As above, the integral is split into two pieces, one for $1<x<1+\epsilon$ and one for $1+\epsilon<x<\infty$. In the first region, the non-singular part of $I(\lambda, P)$ is expanded in powers of $(x-1)$, similarly to Eq. (C.13). Again, we find that $A_{n}(\lambda, P)-A_{n}(\Lambda, P) \rightarrow 1 / P$ as $P \rightarrow \infty$ and that the $x$ integrals diverge at most like $\log P$. Thus, this region gives a. zero contribution to $T_{f i}$. The limit $P \rightarrow \infty$ can be taken inside the $x$ integral for $1+\epsilon<x<\infty$ since we are away from the singularity to give

$$
\begin{equation*}
T_{f i}^{(3)}=\lim _{\epsilon \rightarrow 0} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{1+\epsilon}^{\infty} d x \int d^{2} \vec{k}_{\perp}\left[\frac{1}{x(1-x)}-\frac{1}{x(1-x)}\right]=0 \tag{C.18}
\end{equation*}
$$

The contributions from the three $x$ regions are now summed to give the final answer for the usual time-ordering, one-loop fermion self-energy diagram,

$$
\begin{align*}
T_{f i} & =\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1} d x \int d^{2} \vec{k}_{\perp} \frac{1}{x(1-x)} \frac{(1-x) m_{e}^{2}+\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}-4 m_{e}^{2}}{m_{e}^{2}-\frac{k_{\perp}^{2}+\lambda^{2}}{x}-\frac{k_{\perp}^{2}+m_{e}^{2}}{1-x}+i \epsilon}-(\lambda \rightarrow \Lambda) \\
& =\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1} d x \int d^{2} \vec{k}_{\perp} \frac{1}{1-x} \frac{\left(2-2 x-2 x^{2}\right) m_{e}^{2}-k_{\perp}^{2}}{k_{\perp}^{2}+x^{2} m_{e}^{2}+(1-x) \lambda^{2}-i \epsilon}-(\lambda \rightarrow \Lambda) . \tag{C.19}
\end{align*}
$$

Note that this result diverges like $\Lambda^{2}$ for large $\Lambda . \Lambda$ term

$$
\begin{equation*}
1=\frac{k_{\perp}^{2}+x^{2} m_{e}^{2}+(1-x) \lambda^{2}}{k_{\perp}^{2}+x^{2} m_{e}^{2}+(1-x) \lambda^{2}} \tag{C.20}
\end{equation*}
$$

can be added to the first term in the integrand and an alogous term with $\lambda$
replaced by $\Lambda$ subtracted from the second term to give

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1} d x \int d^{2} \vec{k}_{\perp} \frac{2 m_{e}^{2}+\lambda^{2}}{k_{\perp}^{2}+x^{2} m_{e}^{2}+(1-x) \lambda^{2}-i \epsilon}-(\lambda \rightarrow \Lambda) \tag{C.21}
\end{equation*}
$$

Now turn to the Z-graph contribution. A procedure similar to the above for the usual time-ordering is applied. The momenta arc assigned to be

$$
\begin{gather*}
p=\left(E, \overrightarrow{0}_{\perp}, P\right), \quad k_{1}=\left(E_{1}, \vec{k}_{\perp},-x P\right), \quad k_{2}=\left(E_{2},-\vec{k}_{\perp},-(1-x) P\right), \\
E=\sqrt{P^{2}+m_{e}^{2}}, \quad E_{1}=\sqrt{x^{2} P^{2}+\lambda_{\perp}^{2}}, \quad E_{2}=\sqrt{(1-x)^{2} P^{2}+m_{\perp}^{2}} \\
\lambda_{\perp}^{2}=k_{\perp}^{2}+\lambda^{2}, \quad m_{\perp}^{2}=k_{\perp}^{2}+m_{e}^{2} \tag{C.22}
\end{gather*}
$$

The $\mathrm{TOPTh}_{\infty}$ result for the Z-graph including Pauli-Villars regularization is

$$
T_{f i}=\lim _{P \rightarrow \infty} \frac{g^{2}}{4(2 \pi)^{3}} P \int_{-\infty}^{\infty} d x \int d^{2} \vec{k}_{\perp} \frac{1}{E_{1} E_{2}} \frac{N}{D+i \epsilon}-(\lambda \rightarrow \Lambda)
$$

$$
\begin{equation*}
N=-\bar{u}(p) \not \phi^{*} v\left(k_{2}\right) \bar{v}\left(k_{2}\right) \notin u(p) \tag{C.23}
\end{equation*}
$$

$$
D=-E-E_{1}-E_{2}
$$

Doing the numerator algebra gives

$$
\begin{gather*}
T_{f i}=\lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{-\infty}^{\infty} d x \int d^{2} \vec{k}_{\perp}[I(\lambda, P)-I(\Lambda, P)], \\
I(\lambda, P)=\frac{1}{\sqrt{x^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}} \sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}} \frac{\sqrt{1+\left(\frac{m_{e}}{P}\right)^{2}} \sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}+(1-x)+2\left(\frac{m_{e}}{P}\right)^{2}}{\sqrt{1+\left(\frac{m_{e}}{P}\right)^{2}}+\sqrt{x^{2}+\left(\frac{\lambda_{1}}{P}\right)^{2}}+\sqrt{(1-x)^{2}+\left(\frac{m_{\perp}}{P}\right)^{2}}-i \epsilon} \tag{C.24}
\end{gather*} .
$$

Again, we find potential singularities in $I(\lambda, P)$ near $x=0,1$. The integral is again split into three regions: $x>1,0<x<1, x<0$.

1. For $x>1, E \rightarrow P\left[1+\frac{1}{2}\left(\frac{m}{P}\right)^{2}\right], E_{1} \rightarrow x P\left[1+\frac{1}{2}\left(\frac{\lambda_{1}}{x P}\right)^{2}\right]$, and $E_{2} \rightarrow$ $(x-1) P\left[1+\frac{1}{2}\left(\frac{m_{\perp}}{(1-x) P}\right)^{2}\right]$ as $P \rightarrow \infty$ and

$$
\begin{equation*}
I(\lambda, P) \underset{P \rightarrow \infty}{\longrightarrow} \frac{1}{x(x-1)} \frac{\frac{1}{2}(x-1) \frac{m_{e}^{2}}{P^{2}}+\frac{1}{2} \frac{m_{\perp}^{2}}{(x-1) P^{2}}-2 \frac{m_{e}^{2}}{P^{2}}}{2 x} \rightarrow 0 \tag{C.25}
\end{equation*}
$$

which is non-singular. The limit $P \rightarrow \infty$ can be taken inside to give

$$
\begin{equation*}
T_{f i}^{(1)}=0 \tag{C.26}
\end{equation*}
$$

-2. In the second region,

$$
\begin{equation*}
I(\lambda, P) \underset{P \rightarrow \infty}{ } \frac{1}{x} \tag{C.27}
\end{equation*}
$$

which is singular near $x=0$. The integral is split into two pieces,

$$
\begin{equation*}
T_{f i}^{(2)}=\lim _{\epsilon \rightarrow 0} \lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}}\left[\int_{0}^{\epsilon} d x+\int_{\epsilon}^{1} d x\right] \int d^{2} \vec{k}_{\perp}[I(\lambda, P)-I(\Lambda, P)] \tag{C.28}
\end{equation*}
$$

(a) The non-singular part of $I(\lambda, P)$ is expanded in powers of $x$ for the region $0<x<\epsilon$ to give

$$
\begin{equation*}
I(\lambda, P)=\frac{1}{\sqrt{x^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}}} \sum_{n=0}^{\infty} A_{n}(\lambda, P) x^{n} \tag{C.29}
\end{equation*}
$$

Focus specifically on the contribution of the term $A_{0}$ to $T_{f i}$,

$$
\begin{align*}
& T_{f i}^{\left(2 a_{0}\right)}=\lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int d^{2} \vec{k}_{\perp}\left[A_{0}(\lambda, P) \int_{0}^{\epsilon} \frac{d x}{\sqrt{x^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}}}-(\lambda \rightarrow \Lambda)\right] \\
& \quad=\lim _{P \rightarrow \infty} \delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int d^{2} \vec{k}_{\perp}\left[A_{0}(\lambda, P) \log \left(\frac{\epsilon+\sqrt{\epsilon^{2}+\left(\frac{\lambda_{\perp}}{P}\right)^{2}}}{\frac{\frac{\left|\lambda_{\perp}\right|}{P}}{}}\right)-(\lambda \rightarrow \Lambda)\right] . \tag{C.30}
\end{align*}
$$

As $P \rightarrow \infty, A_{0}(\lambda, P)$ and $A_{0}(\Lambda, P)$ both approach one and the log approaches $\log \frac{2 \epsilon P}{\left|\lambda_{\perp}\right|}$. Using these relations, we find

$$
\begin{equation*}
T_{f i}^{\left(2 a_{0}\right)}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int d^{2} \vec{k}_{\perp} \log \frac{\left|\Lambda_{\perp}\right|}{\left|\lambda_{\perp}\right|}=\delta_{s s^{\prime}} \frac{g^{2}}{16 \pi^{3}} \int d^{2} \vec{k}_{\perp} \log \frac{k_{\perp}^{2}+\Lambda^{2}}{k_{\perp}^{2}+\lambda^{2}} \tag{C.31}
\end{equation*}
$$

Analysis of the other terms $A_{n}, n=1,2,3, \ldots$ reveals that their contribution to $T_{f i}$ all approach zero as $P \rightarrow \infty$. So, the complete answer for the region $0<x<\epsilon$ is

$$
\begin{equation*}
T_{f i}^{(2 a)}=\delta_{s s^{\prime}} \frac{g^{2}}{16 \pi^{3}} \int d^{2} \vec{k}_{\perp} \log \frac{k_{\perp}^{2}+\Lambda^{2}}{k_{\perp}^{2}+\lambda^{2}} \tag{C.32}
\end{equation*}
$$

(b) For $\epsilon<x<1$ the integrand is non-singular, so the limit can be taken inside the integral to give

$$
\begin{equation*}
T_{f i}^{(2 b)}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{\epsilon}^{1} d x \int d^{2} \vec{k}_{\perp}\left[\frac{1}{x}-\frac{1}{x}\right]=0 \tag{C.33}
\end{equation*}
$$

3. For $x<0$, the results are similar to $0<x<1$. There is a singularity in $I(\lambda, P)$ near $x=0$. Expanding $I$ in powers of $-x$ for $-\epsilon<x<0$ reveals a contribution identical to Eq. (C.32) from the term $A_{0}$. All other contributions vanish as $P \rightarrow \infty$.

Summing contributions from $x>1,0<x<1$ and $x<0$ gives the total result

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int d^{2} \vec{k}_{\perp} \log \frac{k_{\perp}^{2}+\Lambda^{2}}{k_{\perp}^{2}+\lambda^{2}} \tag{C.34}
\end{equation*}
$$

for the Z-graph contribution to the one-loop fermion self-energy diagram. This answer can be re-written as

$$
\begin{equation*}
T_{f i}=\delta_{s s^{\prime}} \frac{g^{2}}{8 \pi^{3}} \int_{0}^{1} d x \int d^{2} \vec{k}_{\perp} \frac{-\lambda^{2}+2 m_{e}^{2} x}{k_{\perp}^{2}+x^{2} m_{e}^{2}+(1-x) \lambda^{2}-i \epsilon}-(\lambda \rightarrow \Lambda) \tag{C.35}
\end{equation*}
$$

Note that this answer disagrees with the Z-graph answer using a naive application of the tree graph rule for including backward moving particles given in Refs. 25 and 8 . Of course, their rule continues to remain valid for tree graphs.

- Summing this result with that for the usual time-ordering Eq. (C.21) yields an answer identical to the Feynman rules answer Eq. (C.3), demonstrating the equivalence of using $\mathrm{TOPTh}_{\infty}$ and Feynman rules for the one-loop fermion selfenergy. The final answer in $\mathrm{TOPTh}_{\infty}$ is just the Feynman rules answer.

Summarizing, the usual time-ordering graph gives an answer in $\mathrm{TOPTh}_{\infty}$ that diverges like $\Lambda^{2}$ and is equal to the usual LCPTh answer for the fermion self-energy. There are no contributions to this graph from the regions near $x=0$ or 1 . The Z-graph contribution in $\mathrm{TOPTh}_{\infty}$ only has a contribution near $x=0$ and sums with the usual time-ordering graph to give the familiar Feynman rules answer. This final answer diverges like $\ln \Lambda$ because the leading $\Lambda^{2}$ divergence cancels. In order to reconcile the LCPTh and Feynman rules answers for the one-loop fermion self-energy, an extra piece equal to the $\mathrm{TOPTh}_{\infty}$ Z-graph must be added to the light-cone Hamiltonian and the LCPTh rules.

One final note: it should be noted that the method of implementing a PauliVillars ultraviolet regulator in Feynman gauge used above is not appropriate in light-cone gauge unless a modification is made. The problem in light-cone gauge is that the transverse degrees of freedom are mass dependent, but the longitudinal degree (i.e. the instantaneous interaction) is not. Consequently, the Pauli-Villars counterterm has (up to a sign) exactly the same instantaneous piece as the true pho-- ton, and (at least at tree-level) a suppressed transverse piece for large $\Lambda$. Therefore, the counterterm cancels the instantaneous piece and leaves the transverse piece unmodified as $\Lambda \rightarrow \infty$. The full answer at tree-level would be just the transverse interaction, which is incorrect and not gauge invariant.

This problem can be remedied by introducing a dynamical longitudinal photon with derivative coupling proportional to the photon mass squared. However, since the photon mass is used here only as an infrared regulator and is ultimately sent to zero, no consequences of significance arise from the improper treatment of the photon mass term in this work. Implementing a Pauli-Villars regulator in lightcone gauge would, however, require the addition of heavy longitudinal photons. ${ }^{42}$

## APPENDIX D

A set of useful spinor properties is given in this appendix.

| $\bar{u}(k, s) \ldots u\left(k, s^{\prime}\right)$ | $\uparrow \longrightarrow \uparrow$ | $\uparrow \longrightarrow \downarrow$ |
| :---: | :---: | :---: |
| $\bar{u} u$ | $\downarrow \longrightarrow \downarrow$ | $\left(s^{\prime} \rightarrow s\right)$ |
| $\bar{u} \gamma^{\mu} u$ | $\downarrow m_{e}$ | 0 |
| $\bar{u} \gamma^{+} \gamma^{-} u$ | $2 k^{\mu}$ | 0 |
| $\bar{u} \gamma^{-} \gamma^{+} u$ | $4 m_{e}$ | $4\left[ \pm k^{1}+i k^{2}\right]$ |
| $\bar{u} \gamma^{+} \gamma^{i} u$ | $4 m_{e}$ | $4\left[\mp k^{1}-i k^{2}\right]$ |
| $\bar{u} \gamma^{i} \gamma^{+} u$ | 0 | $2 k^{+}\left[ \pm \delta^{i 1}+i \delta^{i 2}\right]$ |
| $\bar{u} \gamma^{-} \gamma^{+} \gamma^{-} u$ | 0 | $2 k^{+}\left[\mp \delta^{i 1}-i \delta^{i 2}\right]$ |
| $\bar{u} \gamma^{-} \gamma^{+} \gamma^{i} u$ | $8\left[\frac{k_{1}^{2}+m_{c}^{2}}{k^{+}}\right]$ | 0 |
| $\bar{u} \gamma^{i} \gamma^{+} \gamma^{-} u$ | $4\left[k^{i} \mp i \epsilon^{i j} k^{j}\right]$ | $4 m_{e}\left[ \pm \delta^{i 1}+i \delta^{i 2}\right]$ |
| $\bar{u} \gamma^{i} \gamma^{+} \gamma^{j} u$ | $2 k^{+}\left[\delta^{i j} \pm i \epsilon^{i j} k^{j}\right]$ | $4 m_{e}\left[\mp \delta^{i 1}-i \delta^{i 2}\right]$ |

$$
\begin{aligned}
& \bar{v}(k, s) v\left(k, s^{\prime}\right)=-2 m_{e} \delta_{s s^{\prime}} \\
& \bar{v}(k, s) \gamma^{\mu} v\left(k, s^{\prime}\right)=2 k^{\mu} \delta_{s s^{\prime}} \\
& \bar{v}(k, s) u\left(k, s^{\prime}\right)=\bar{u}(k, s) v\left(k, s^{\prime}\right)=0 \\
& \bar{u}(k, s)\left[\gamma^{\mu} \gamma^{\nu} \gamma^{\sigma}+\gamma^{\sigma} \gamma^{\nu} \gamma^{\mu}\right] u\left(k, s^{\prime}\right)=\bar{v}(k, s)\left[\gamma^{\mu} \gamma^{\nu} \gamma^{\sigma}+\gamma^{\sigma} \gamma^{\nu} \gamma^{\mu}\right] v\left(k, s^{\prime}\right) \\
& \quad=\left[4 g^{\mu \nu} k^{\sigma}-4 g^{\mu \sigma} k^{\nu}+4 g^{\nu \sigma} k^{\mu}\right] \delta_{s s^{\prime}} \\
& \bar{v}(k, s) \gamma^{\mu} \gamma^{\nu} \gamma^{\sigma} v\left(k^{\prime}, s^{\prime}\right)=\bar{u}\left(k^{\prime}, s^{\prime}\right) \gamma^{\sigma} \gamma^{\nu} \gamma^{\mu} u(k, s) \\
& i=j=1,2 \quad \mu, \nu, \sigma=0,1,2,3 \text { or }+,-, 1,2
\end{aligned}
$$

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## TABLE CAPTIONS

1: Definitions in light-cone quantization.
2: A comparison of light-cone and equal-time quantization.

## FIGURE CAPTIONS

1) Light-cone diagrams for QED interactions.
2) Decomposition of positronium into Fock states.
3) Light-cone perturbation theory graphs contributing to Møller scattering. $k_{1}^{+}$ is assumed to be larger than $k_{3}^{+}$.
4) Comparison of ground state energy with (Y) and without (N) infrared cut-off.
5) One-loop LCPTh radiative corrections to fermion line.
6) Fermion structure function from diagonalization. $\alpha=0.3, K=17, L_{\perp}=$

- $10 \frac{1}{m_{e}}, \Lambda=2.3 m_{e}$.

7) Some fermion mass counterterms needed to include the ( $\left.e^{+} e^{-} \gamma \gamma\right)$ Fock state.
8) Photon mass counterterms needed to include the ( $\left.e^{+} e^{-} e^{+} e^{-}\right)$Fock state.
9) Three graphs that occur in LCPTh for tree-level Møller scattering.
10) One-loop fermion self-mass.
11) One-loop fermion self-mass diagrams joined by instantaneous fermion.
12) $N$ one-loop fermion self-mass pieces chained by $N-1$ instantaneous fermions.
13) Sum of $N$ chained one-loop fermion self-mass diagrams.
14) One-loop fermion self-energy.
15) One-loop fermion self-energy contributions in time-ordered perturbation theory. The right graph is typically referred to as the Z-graph.

Table 1
Definitions in light-cone quantization.

| Variables | $\begin{aligned} \tau & =\text { light-cone time }=x^{+}=x^{0}+x^{3} \\ x^{-} & =\text {light-cone position }=x^{0}-x^{3} \\ \vec{x}_{\perp} & =\left(x^{1}, x^{2}\right) \end{aligned}$ |
| :---: | :---: |
| Covariant, notation | $A^{\mu}=\left(A^{+}, A^{-}, \vec{A}_{\perp}\right)$ |
| Metric | $\left(g^{\prime \prime \nu}\right)=\left(\begin{array}{rrrr}0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{array}\right)$ |
| Dot product | $x \cdot y=x^{\mu} g_{\mu \nu} y^{\nu}=\frac{1}{2}\left(x^{+} y^{-}+x^{-} y^{+}\right)-\vec{x}_{\perp} \cdot \vec{y}_{\perp}$ |
| Mass shell condition | $P^{+} P^{-}=\vec{P}_{\perp}^{2}+M^{2}$ |
| Derivative | $\begin{array}{lll} \partial_{+}=\frac{\partial}{\partial x^{+}}, & \partial_{-}=\frac{\partial}{\partial x^{-}}, & \partial_{i}=\frac{\partial}{\partial x^{i}} \\ \partial^{+}=2 \partial_{-}, & \partial^{-}=2 \partial_{+}, & \partial^{i}=-\partial_{i} \end{array}$ |
| Underscore notation | $\begin{gathered} \underline{x}=\left(x^{-}, \vec{x}_{\perp}\right), \quad \underline{k}=\left(k^{-}, \vec{k}_{\perp}\right) \\ \underline{k} \cdot \underline{x}=\frac{1}{2} k^{+} x^{-}-\vec{k}_{\perp} \cdot \vec{x}_{\perp} \end{gathered}$ |

Table 2
A comparison of light-cone and equal-time quantization.

|  | Instant Form | Front Form |
| :--- | :---: | :---: |
| Hamiltonian | $H=\sqrt{P^{2}+m^{2}}+V$ | $P^{-}=\frac{P_{\perp}^{2}+m^{2}}{P^{+}}+V$ |
| Conserved quantities | $E, \vec{P}$ | $P^{-}, \quad P^{+}, \overrightarrow{P_{\perp}}$ |
| Momenta | $P_{z}<>0$ | $P^{+}>0$ |
| Bound state equation | $H \psi=E \psi$ | $P^{+} P^{-} \psi=M^{2} \psi$ |
| Vacuum | Complicated | Trivial |


Diagram 13 1415


Fig. 1


Fig. 2

$$
\begin{array}{cc}
k_{1}-k_{3} & k_{1}-\frac{\zeta}{\zeta q} k_{3} \\
k_{2} \xrightarrow[q_{3}]{q_{2}} k_{4} & k_{2} \xlongequal{q^{+}=k_{1}^{+}-k_{3}^{+}=k_{4}^{+}-k_{2}^{+}} \\
-\vec{q}_{\perp}^{+} \equiv k_{1}^{+}-k_{3}^{+}=k_{4}^{+}-k_{2}^{+} \\
\vec{k}_{1 \perp}-\vec{k}_{3 \perp}=\vec{k}_{4 \perp}-\vec{k}_{2 \perp} & \vec{q}_{\perp} \equiv \vec{k}_{1 \perp}-\vec{k}_{3 \perp}=\vec{k}_{4 \perp}-\vec{k}_{2 \perp}
\end{array}
$$

Fig. 3


Fig. 4

Fig. 5


Fig. 6

$10-89$
$6482 A 26$

Fig. 7


Fig. 8

$$
\cdots
$$



Fig. 9


Fig. 10


Fig. 11

$10-89$
6482A33

Fig. 12


Fig. 13


Fig. 14



6482A14

Fig. 15


[^0]:    $\star$ Work supported by the Department of Energy, contract DE-AC03-76SF00515.

