# INVARIANT TORI OF THE POINCARÉ RETURN MAP AS SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS* 

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#### Abstract

Functional difference equations characterize the invariant surfaces of the Poincaré return map of a general Hamiltonian system. Two different functional equations are derived. The first is analogous to the Hamilton-Jacobi equation and the second is a generalization of Moser's equation. Some properties of the equations, and schemes for solving them numerically, are discussed.


## INTRODUCTION

Jürgen Moser introduced a functional difference equation to characterize invariant curves of area-preserving mappings of an annulus. ${ }^{1}$ His proof of the Kolmogorov-Arnol'd-Moser (KAM) theorem for such maps, assumed to be very smooth but not analytic, was based on a close analysis of this equation. Proofs of refined versions of this "twist theorem" and its relatives also are based on the difference equation. ${ }^{2}$

Some time ago I became interested in applying difference equations to study invariant tori of general Hamiltonian systems. By describing the Poincaré map $\mathcal{M}$ of a surface of section in action-angle variables, one easily finds the generalization of Moser's equation, formulated in terms of $\mathcal{M}$. It seemed that this equation could lead to an efficient method for numerical approximation of invariant tori. I intended to project the equation onto a finite Fourier basis in the angle variables, then solve the projected equation by an iterative method. This procedure had been used to good effect in solving the Hamilton-Jacobi equation for invariant tori. ${ }^{3}$ By working with the difference equation, and thereby restricting attention to the Poincare section, one could effectively reduce the dimension of the problem by one unit, thus eliminating a costly integration that is required in the HamiltonJacobi framework.

Unfortunately, I could see no obvious way to satisfy the component of the projected equation arising from the constant element of the Fourier basis; i.e., the average of the original equation over angles. The corresponding average of the Hamilton-Jacobi equation had presented no problem. This issue appears as well in Moser's proof, but his method of handling it did not seem right for my purposes.

Faced with this difficulty, I was pleased to find a new type of functional difference equation ${ }^{4,5}$ (at least new to me) that corresponds closely to the IIamiltonJacobi equation. In fact, the solution of this equation is exactly the HamiltonJacobi generating function, restricted to the Poincaré section. In this formulation one could ignore the averaged equation, since the other components of the Fourier projection were sufficient to determine the generating function. Satisfaction of the averaged equation turned out to be automatic, although not obviously so.

[^0]I programmed a numerical solution by the projection-iteration algorithm, for a general four-dimensional symplectic map $\mathcal{M}$, and applied the program to a strongly nonlinear example from accelerator theory, a system with $21 / 2$ degrees of freedom. I found, as expected, a great saving of computation time in comparison to the Hamilton-Jacobi treatment. I used a Newton iteration, which proved to have a large region of convergence, but not quite as large as that of the analogous Hamilton-Jacobi calculation. Shortly after this bit of progress, I found an even better method to compute close approximations to invariant tori. ${ }^{6}$ The method is extremely direct, avoiding both partial differential equations and functional equations. It depends on fitting a toroidal surface, represented as a truncated Fourier series, to an appropriate set of points on a single orbit of $\mathcal{M}$. The idea of fitting surfaces to orbit points is not new, but I use a novel method of accomplishing the fit that has strong advantages. The result is an accurate, robust, and relatively inexpensive method of computing invariant tori. It provides all quantities of interest in the canonical formalism, with better numerical efficiency than seems possible in methods of semi-analytical character.

In spite of the success of this algorithm, I still think that functional difference equations for general Hamiltonian systems hold considerable interest. They provide an elegant and concise mathematical characterization of invariant tori that is lacking in the surface-fitting approach. A study of singularities of the equations, aided by numerical analysis and ideas of functional analysis, might help to clarify questions regarding breakup of invariant tori of high-dimensional systems. Moreover, it is not excluded that difference equations could rival the surface-fitting method in numerical power, after further development of solution algorithms. The generalized Moser equation has not yet been investigated numerically. As I will indicate presently, it no longer seems inferior in interest to the functional equation of Hamilton-Jacobi type.

I will show that the two equations arise as conjugation equations for two different time-evolution maps that arise naturally in the scheme of canonical transformations. These maps appear in a hierarchy, with the map for the equation of Hamilton-Jacobi type appearing first, and that for the generalized Moser equation second. I therefore refer to each equation as a Functional Difference Equation of the First (Second) Kind, accordingly.

## CANONICAL MAPS

To establish notation and a point of view, we first review canonical transformations. It is appropriate to take the action-angle variables ( $\mathbf{I}, \boldsymbol{\Phi}$ ) of an underlying integrable system as the primary phase-space coordinates. The Hamiltonian will have the form

$$
\begin{equation*}
H(\mathbf{I}, \boldsymbol{\Phi}, \theta)=H_{o}(\mathbf{I})+V(\mathbf{I}, \mathbf{\Phi}, \theta), \tag{1}
\end{equation*}
$$

where $H_{o}$ represents an integrable system. The perturbation $V$ is $2 \pi$-periodic in $\boldsymbol{\Phi}$ and $\theta$. The independent variable of Hamilton's equations is $\theta$; it can be the time, or a monotonically increasing function of time. Bold-faced letters represent $d$ dimensional vectors. Owing to the $\theta$ dependence of the Hamiltonian, the effective dimension of phase space is $2 d+1$; the a system is said to have $d+\frac{1}{2}$ degrees of freedom.

A canonical transformation to new action-angle variables ( $\mathbf{J}, \boldsymbol{\Psi}$ ) is induced by a generating function $S(\mathbf{J}, \boldsymbol{\Phi}, \theta)=\mathbf{J} \cdot \boldsymbol{\Phi}+G(\mathbf{J}, \boldsymbol{\Phi}, \theta)$, where $G$ is $2 \pi$-periodic in $\Phi$ and $\theta$. The equations relating old and new variables are

$$
\begin{equation*}
\mathbf{I}=\mathbf{J}+G_{\boldsymbol{\Phi}}(\mathbf{J}, \boldsymbol{\Phi}, \theta), \quad \mathbf{\Psi}=\boldsymbol{\Phi}+G_{\mathbf{J}}(\mathbf{J}, \boldsymbol{\Phi}, \theta) \tag{2}
\end{equation*}
$$

where subscripts denote partial derivatives. If the transformation is ideal, so that $\mathbf{J}$ is invariant, then the first equation of (2) constitutes an explicit representation of a $(d+1)$-dimensional invariant torus, giving $I$ as a function of $\boldsymbol{\Phi}$ and $\theta$. The invariant parameter $\mathbf{J}$ serves to distinguish different tori. When $\mathbf{J}$ is invariant, the generator satisfies the Hamilton-Jacobi equation, which is the requirement that the new Hamiltonian be independent of $\boldsymbol{\Phi}$ :

$$
\begin{equation*}
H\left(\mathbf{J}+G_{\boldsymbol{\Phi}}(\mathbf{J}, \boldsymbol{\Phi}, \theta), \boldsymbol{\Phi}, \theta\right)+G_{\theta}(\mathbf{J}, \boldsymbol{\Phi})=H_{1}(\mathbf{J}, \theta) \tag{3}
\end{equation*}
$$

For a Poincare surface of section $\mathcal{S}$ we choose the surface in phase space defined by $\theta=0(\bmod 2 \pi)$. The Poincaré return $\operatorname{map} \mathcal{M}: \mathcal{S} \rightarrow \mathcal{S}$ gives the result of Hamiltonian flow during a change of $\theta$ by $2 \pi$ :

$$
\begin{equation*}
\mathcal{M}:\left.\left.(\mathbf{I}, \mathbf{\Phi})\right|_{\theta=2 \pi n} \mapsto(\mathbf{I}, \mathbf{\Phi})\right|_{2 \pi(n+1)} \tag{4}
\end{equation*}
$$

The following discussion is based entirely on this map. The object of interest will be a torus of dimension $d$, invariant under $\mathcal{M}$; namely, the intersection of the full invariant torus with the surface of section $\mathcal{S}$. Accordingly we define

$$
\begin{equation*}
\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi})=G_{\boldsymbol{\Phi}}(\mathbf{J}, \boldsymbol{\Phi}, 0) \tag{5}
\end{equation*}
$$

so that a representation of this $d$-dimensional torus is

$$
\begin{equation*}
\mathbf{I}=\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi}) \tag{6}
\end{equation*}
$$

For almost all purposes it is sufficient to know this $d$ - dimensional section of the full torus. Should the full torus be needed, it can be obtained by integrating the Hamilton-Jacobi equation as an initial-value problem in $\theta$, with initial data obtained from (6) and the integral of (6) with respect to $\boldsymbol{\Phi}$. The hard part of an $a b$ initio solution of the Hamilton-Jacobi equation, meeting the condition that $G$ be periodic in $\theta$, is avoided.

If the system happens to be autonomous, so that $V$ is independent of $\theta$, then the following discussion requires a small modification. The generator $G$ becomes independent of $\theta$, the surface of section is defined by fixing one component of $\boldsymbol{\Phi}$, and the torus of interest has dimension $d-1$.

It is useful to view the canonical transformation as being composed of two steps: $(\mathbf{I}, \boldsymbol{\Phi}) \mapsto(\mathbf{J}, \boldsymbol{\Phi})$ followed by $(\mathbf{J}, \boldsymbol{\Phi}) \mapsto(\mathbf{J}, \boldsymbol{\Psi})$. We define $\mathcal{U}$ to be the inverse of the first step, and $\mathcal{V}$ to be the second step:

$$
\begin{align*}
& \mathcal{U}(\mathbf{J}, \boldsymbol{\Phi})=(\mathbf{I}, \boldsymbol{\Phi})=(\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi}), \boldsymbol{\Phi})  \tag{7}\\
& \mathcal{V}(\mathbf{J}, \boldsymbol{\Phi})=(\mathbf{J}, \boldsymbol{\Psi})=(\mathbf{J}, \boldsymbol{\Phi}+\mathbf{v}(\mathbf{J}, \boldsymbol{\Phi})) \tag{8}
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{v}(\mathbf{J}, \boldsymbol{\Phi})=G_{\mathbf{J}}(\mathbf{J}, \boldsymbol{\Phi}, 0) \tag{9}
\end{equation*}
$$

The full canonical transformation is

$$
\begin{equation*}
\mathcal{V} \circ \mathcal{U}^{-1}(\mathbf{I}, \boldsymbol{\Phi})=(\mathbf{J}, \mathbf{\Psi}) \tag{10}
\end{equation*}
$$

A notation for its inverse is useful,

$$
\begin{equation*}
\mathcal{C}(\mathbf{J}, \mathbf{\Psi})=\mathcal{U} \circ \mathcal{V}^{-1}(\mathbf{J}, \mathbf{\Psi})=(\mathbf{I}, \boldsymbol{\Phi}) \tag{11}
\end{equation*}
$$

The map $\mathcal{M}$ induces two other maps of interest, via the transformations $\mathcal{U}$ and $\mathcal{C}$. Namely,

$$
\mathcal{N}(\mathbf{J}, \boldsymbol{\Phi})=\mathcal{U}^{-1} \circ \mathcal{M} \circ \mathcal{U}(\mathbf{J}, \boldsymbol{\Phi})=\left(\mathbf{J}^{\prime}, \boldsymbol{\Phi}^{\prime}\right)
$$

and

$$
\mathcal{O}(\mathbf{J}, \mathbf{\Psi})=\mathcal{C}^{-1} \circ \mathcal{M} \circ \mathcal{C}(\mathbf{J}, \mathbf{\Psi})=\left(\mathbf{J}^{\prime}, \mathbf{\Psi}^{\prime}\right)
$$



Fig. 1. A commutative diagram showing how the maps $\mathcal{N}$ and $\mathcal{O}$ are induced by the original map $\mathcal{M}$ and changes of variable. The transformations $\mathcal{U}$ and $\mathcal{V}$ are given in explicit form through derivatives of the generator; see Eqs. (7) and (8).

These relations are illustrated in the pleasantly symmetric commutative diagram of Fig. 1. Next we shall see that a difference equation of the first or second kind is obtained by putting $\mathbf{J}=\mathbf{J}^{\prime}$ in (12) or (13), respectively. For derivation of the equations it is convenient to have separate notations for the radial and angular components of $\mathcal{M}$. With $\mathcal{M}(\mathbf{I}, \boldsymbol{\Phi})=\left(\mathbf{I}^{\prime}, \boldsymbol{\Phi}^{\prime}\right)$ we write

$$
\begin{align*}
\mathbf{I}^{\prime} & =\mathbf{I}+\mathbf{R}(\mathbf{I}, \boldsymbol{\Phi}) \\
\mathbf{\Phi}^{\prime} & =\boldsymbol{\Phi}+\Theta(\mathbf{I}, \boldsymbol{\Phi}) \tag{12}
\end{align*}
$$

FUNCTIONAL DIFFERENCE EQUATION OF THE FIRST KIND: AN ANALOG OF THE HAMILTON-JACOBI EQUATION

Let us write Eq. 12 as

$$
\begin{equation*}
\mathcal{M} \circ \mathcal{U}(\mathbf{J}, \boldsymbol{\Phi})=\mathcal{U} \circ \mathcal{N}(\mathbf{J}, \boldsymbol{\Phi}) \tag{13}
\end{equation*}
$$

In accord with Fig. 1, this means that passing from ( $\mathbf{J}, \boldsymbol{\Phi}$ ) to ( $\mathbf{I}^{\prime}, \boldsymbol{\Phi}^{\prime}$ ) by either of two routes in the diagram gives the same result. We put $\mathbf{J}=\mathbf{J}^{\prime}$ to characterize an
invariant torus of $\mathcal{M}$. Introducing the representation (7) of $\mathcal{U}$, and the notation (12), we see that (13) takes the form

$$
\begin{gather*}
(\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi})+\mathbf{R}(\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi}), \boldsymbol{\Phi}), \quad \mathbf{\Phi}+\Theta(\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi}), \boldsymbol{\Phi}))  \tag{14}\\
=\left(\mathbf{J}+\mathbf{u}\left(\mathbf{J}, \boldsymbol{\Phi}^{\prime}\right), \quad \boldsymbol{\Phi}^{\prime}\right)
\end{gather*}
$$

Combining the two components of Eq. (14), we obtain the Functional Difference Equation of the First Kind,

$$
\begin{equation*}
\mathbf{u}(\boldsymbol{\Phi}+\Theta(\mathbf{J}+\mathbf{u}(\boldsymbol{\Phi}), \boldsymbol{\Phi}))-\mathbf{u}(\boldsymbol{\Phi})=\mathbf{R}(\mathbf{J}+\mathbf{u}(\boldsymbol{\Phi}), \boldsymbol{\Phi}) \tag{15}
\end{equation*}
$$

The first argument of $\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi})$ is not written, since a $\mathbf{J}$ dependence is obviously induced by the occurrence of $\mathbf{J}$ as a parameter in Eq. (15). With less formality, one deduces (15) by requiring that $\mathbf{I}^{\prime}=\mathbf{J}+\mathbf{u}\left(\mathbf{J}, \boldsymbol{\Phi}^{\prime}\right)$ follow from $\mathbf{I}=\mathbf{J}+\mathbf{u}(\mathbf{J}, \boldsymbol{\Phi})$.

FUNCTIONAL DIFFERENCE EQUATION OF THE SECOND KIND: THE GENERALIZATION OF MOSER'S EQUATION

To apply a similar argument to Eq. (13), we define functions to represent the radial and angular components of the map $\mathcal{C}(\mathbf{J}, \mathbf{\Psi})=(\mathbf{I}, \boldsymbol{\Phi})$, the inverse of the canonical transform; namely,

$$
\begin{equation*}
\mathbf{I}=\boldsymbol{\rho}_{o}(\mathbf{J})+\boldsymbol{\rho}(\mathbf{J}, \boldsymbol{\Psi}), \quad \mathbf{\Phi}=\mathbf{\Psi}+\boldsymbol{\alpha}_{o}(\mathbf{J})+\boldsymbol{\alpha}(\mathbf{J}, \mathbf{\Psi}) \tag{16}
\end{equation*}
$$

The functions $\rho$ and $\boldsymbol{\alpha}$ are $2 \pi$-periodic in $\boldsymbol{\Psi}$, and have zero mean with respect to $\boldsymbol{\Psi}$. Thus $\boldsymbol{\rho}_{o}$ and $\boldsymbol{\alpha}_{o}$ are the constant terms in the Fourier series for $\mathbf{I}$ and $\boldsymbol{\Phi}-\boldsymbol{\Psi}$ as functions of $\boldsymbol{\Psi}$. When restricted to an invariant torus of $\mathcal{M}$, the map $\mathcal{O}$ has the form

$$
\begin{equation*}
\mathcal{O}(\mathbf{J}, \mathbf{\Psi})=(\mathbf{J}, \mathbf{\Psi}+2 \pi \boldsymbol{\nu}(\mathbf{J})) \tag{17}
\end{equation*}
$$

(Allowing a priori a function $\boldsymbol{\nu}(\mathbf{J}, \mathbf{\Psi})$ in the angular part, one finds that the symplectic condition forbids any $\boldsymbol{\Psi}$-dependence of $\boldsymbol{\nu}$.) Here $\boldsymbol{\nu}(\mathbf{J})$ is the winding number of $\mathcal{M}$ on the torus (the "tune" of accelerator physics). Now write Eq. (13) with $\mathbf{J}=\mathbf{J}^{\prime}$ in the form

$$
\begin{equation*}
\mathcal{M} \circ \mathcal{C}(\mathbf{J}, \Psi)=\mathcal{C} \circ \mathcal{O}(\mathbf{J}, \boldsymbol{\Psi}) \tag{18}
\end{equation*}
$$

and introduce the representation (16) of $\mathcal{C}$. The radial and angular components of the resulting Functional Difference Equation of the Second Kind are

$$
\begin{align*}
\rho(2 \pi \boldsymbol{\nu}+\boldsymbol{\Psi})-\rho(\mathbf{\Psi}) & =\mathbf{R}\left(\rho_{o}+\boldsymbol{\rho}(\mathbf{\Psi}), \mathbf{\Psi}+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\mathbf{\Psi})\right)  \tag{19}\\
\boldsymbol{\alpha}(2 \pi \boldsymbol{\nu}+\mathbf{\Psi})-\boldsymbol{\alpha}(\mathbf{\Psi})+2 \pi \boldsymbol{\nu} & =\Theta\left(\rho_{o}+\rho(\mathbf{\Psi}), \mathbf{\Psi}+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\mathbf{\Psi})\right)
\end{align*}
$$

Reference to the $\mathbf{J}$ dependence of the various functions has been suppressed.
A solution of this equation defines an invariant torus parametrized as in Eq. (16), with $\Psi$ as "curve parameter." In the earlier representation (6), there was no auxiliary curve parameter, the natural toroidal coordinate $\boldsymbol{\Phi}$ itself sufficing to specify a point on the torus. To see a rough correspondence of the two
representations, we note the lowest order iterative solution of the second equation in (2), for $\boldsymbol{\Phi}$ as a function of $\boldsymbol{\Psi}$ :

$$
\begin{equation*}
\boldsymbol{\Phi}=\mathbf{\Psi}-G_{\mathbf{J}}(\mathbf{J}, \mathbf{\Psi}, 0)+O\left(V^{2}\right) \tag{20}
\end{equation*}
$$

This follows, because $G$ is of order $V$, the perturbation strength. Substituting in the first equation of (2), we find

$$
\begin{equation*}
\mathbf{I}=\mathbf{J}+G_{\mathbf{\Psi}}(\mathbf{J}, \mathbf{\Psi}, 0)+O\left(V^{2}\right)=\mathbf{J}+\mathbf{u}(\mathbf{J}, \mathbf{\Psi})+\mathcal{O}\left(V^{2}\right) \tag{21}
\end{equation*}
$$

Since $G_{\boldsymbol{\Psi}}$ has zero mean with respect to $\boldsymbol{\Psi}$, a comparison to (16) gives

$$
\begin{equation*}
\boldsymbol{\rho}_{o}(\mathbf{J}) \approx \mathbf{J}, \quad \boldsymbol{\alpha}_{o}(\mathbf{J}) \approx-\left\langle G_{\mathbf{J}}(\mathbf{J}, \cdot)\right\rangle, \tag{22}
\end{equation*}
$$

where equality holds with corrections of order $V^{2}$, and the angular brackets imply averaging with respect to $\boldsymbol{\Psi}$ over $[0,2 \pi]^{d}$. One may think of the value of $\boldsymbol{\rho}_{o}(\mathbf{J})$ as a surrogate for $\mathbf{J}$, and as a parameter to distinguish different invariant tori.

## PROPERTIES AND NUMERICAL SOLUTION OF THE EQUATION OF THE FIRST KIND

To enforce periodicity in angle variables, we look for solutions represented as Fourier series,

$$
\begin{equation*}
\mathbf{u}(\Phi)=\sum_{\mathbf{m} \in \mathcal{T}} \mathbf{u}_{\mathbf{m}} e^{i \mathbf{m} \cdot \boldsymbol{\Phi}} \tag{23}
\end{equation*}
$$

For a numerical solution the set $\mathcal{T}$ of mode numbers will be finite, but does not necessarily consist of all modes up to some cutoff; for numerical efficiency it is useful to drop insignificant modes that lie below the highest significant mode. From Eq. (5) it follows that $\mathbf{u}_{\mathbf{m}}=i \mathbf{m} G_{\mathbf{m}}$, where $G_{\mathbf{m}}$ is the Fourier coefficient of $G$. Thus $\mathcal{T}$ does not include vector $\mathbf{m}=\mathbf{0}$, and that is an important feature in the solution of Eq. (15).

Now substitute (23) in Eq. (15), and take the Fourier transform of the equation to obtain
$\mathbf{u}_{\mathbf{m}}=\int_{0}^{2 \pi}\left[\frac{d \boldsymbol{\Phi}}{2 \pi}\right] e^{-i \mathbf{m} \cdot \boldsymbol{\Phi}}\left[\sum_{\mathbf{n} \in \mathcal{T}} \mathbf{u}_{\mathbf{n}} \exp (\mathbf{i n} \cdot(\boldsymbol{\Phi}+\boldsymbol{\Theta}(\mathbf{J}+\mathbf{u}(\boldsymbol{\Phi}), \boldsymbol{\Phi})))-\mathbf{R}(\mathbf{J}+\mathbf{u}(\boldsymbol{\Phi}), \boldsymbol{\Phi})\right]$.
With m restricted to $\mathcal{T}$, Eq. (24) constitutes a finite set of nonlinear equations for the amplitudes $\mathbf{u}_{\mathbf{m}}$. As in the case of the corresponding Fourier method applied to the Hamilton-Jacobi equation, the $\mathbf{m}=\mathbf{0}$ mode does not appear, and the parameter $\mathbf{J}$, destined to approximate the invariant action, is a fixed input.

To put the equations (24) in a form suitable for a simple iterative solution, add and subtract $\mathbf{u}\left(\boldsymbol{\Phi}+\Theta_{o}(\mathbf{J})\right)$ in Eq. (15), where $\Theta_{o}(\mathbf{J})=\langle\Theta(\mathbf{J}, \cdot)\rangle$. Then take the Fourier transform and rearrange to obtain

$$
\begin{equation*}
\mathbf{u}_{\mathbf{m}}=\frac{1}{e^{i \mathbf{m} \cdot \Theta_{o}(\mathbf{J})}--1}\left[\mathbf{R}(\mathbf{J}+\mathbf{u}(\cdot), \cdot)-\mathbf{u}(\cdot+\Theta(\mathbf{J}+\mathbf{u}(\cdot), \cdot))+\mathbf{u}\left(\cdot+\Theta_{o}(\mathbf{J})\right)\right]_{\mathbf{m}} \tag{25}
\end{equation*}
$$

where $f(\cdot)_{\mathbf{m}}$ denotes the Fourier coefficient of $f(\boldsymbol{\Phi})$. Now the right-hand side consists of terms that are $O(V)$ and $O(d V / d \mathrm{I})$, as one can determine by looking at the expressions for $\mathbf{R}$ and $\Theta$ in lowest order perturbation theory. By an application of the contraction mapping principle under appropriate restrictions
on $\mathbf{R}$ and $\Theta$, one can show that simple iteration of (25), beginning with $\mathbf{u}=\mathbf{0}$ as the lowest iterate, will give a solution. This requires that $\mathbf{J}$ be chosen so that the potential "small divisor," $\exp (i \mathbf{m} \cdot \Theta(\mathbf{J}))-1$, not be too small for any $\mathbf{m}$ in the finite set $\mathcal{T}$. In practice it has been easy to find such a $\mathbf{J}$, for mode sets $\mathcal{T}$ large enough to give quite accurate solutions. Again, the situation with respect to iterative solutions is quite analogous to that for the Hamilton-Jacobi system, except that the problem has been simplified through elimination of the $\theta$ coordinate.

If $\mathcal{T}$ included all integer vectors, the iterative solution would fail, due to small divisors. For solutions of arbitrary accuracy one can carry out a sequence of canonical transformations, so that there is an equation like (15), with a different $\operatorname{map}(\mathbf{R}, \Theta)$, at each step of the sequence. The relevant formalism is in close correspondence to the "super-convergent" form of perturbation theory in the Hamilton-Jacobi framework.

The requirements for convergence of the simple iterative solution with finite $\mathcal{T}$ are fairly restrictive. Newton's method applied to Eq. (24) has a much bigger range of convergence. As was mentioned above, a Newton iteration was programmed for an arbitrary system with $d=2$. The Jacobian matrix required for Newton's method involves $\partial \mathbf{R} / \partial \mathbf{I}$ and $\partial \Theta / \partial \mathbf{I}$. These were approximated by simple divided differences. In an application to an accelerator problem (betatron motion in a sextupole lattice) the map ( $\mathbf{R}, \Theta$ ) was evaluated by symplectic numerical integration of Hamilton's equations. The results are reported briefly in Ref. 5.

Although the $\mathbf{m}=\mathbf{0}$ projection of Eq. (15) is not involved in the computation just outlined, it still constitutes an equation that must be satisfied if the resultant torus is really to be invariant under $\mathcal{M}$. It was not too surprising to find that the $\mathbf{m}=\mathbf{0}$ equation was automatically satisfied to high accuracy in the numerical computation. The reason for this can be traced to existence of the lowest Poincaré invariant integral. Another requirement that is satisfied automatically and somewhat mysteriously in the numerical work is the relation between different components of the vector $\mathbf{u}_{\mathbf{m}}$. Since $\mathbf{u}=G_{\boldsymbol{\Phi}}$ is a gradient of the scalar $G$, there should be relations between Fourier coefficients: $\left(u_{\mathrm{m}}\right)_{j}=i m_{j} G_{\mathbf{m}}$, so that $\left(u_{\mathrm{m}}\right)_{j} /\left(u_{\mathrm{m}}\right)_{k}=m_{j} / m_{k}$ for $m_{k} \neq 0$. Not only are these relations satisfied, they also allow a sharp reduction in the number of equations to be solved; with $d=2$ the number of equations is almost cut in half.

## PROPERTIES AND POSSIBLE NUMERICAL SOLUTION OF THE EQUATION OF THE SECOND KIND

A numerical solution of the equation of the second kind, Eq. (19), can be pursued in the same way. We introduce finite Fourier expansions of $\boldsymbol{\rho}(\boldsymbol{\Psi})$ and $\boldsymbol{\alpha}(\boldsymbol{\Psi})$, then take a Fourier transform of the equation to obtain

$$
\begin{array}{r}
\rho_{\mathbf{m}}=\frac{1}{e^{2 \pi i \mathbf{m} \cdot \boldsymbol{\nu}}-1} \mathbf{R}_{\mathbf{m}}\left(\boldsymbol{\rho}_{o}+\boldsymbol{\rho}(\cdot) \cdot \cdot+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\cdot)\right), \\
\boldsymbol{\alpha}_{\mathbf{m}}=\frac{1}{e^{2 \pi i \mathbf{m} \cdot \boldsymbol{\nu}}-1} \Theta_{\mathbf{m}}\left(\boldsymbol{\rho}_{o}+\boldsymbol{\rho}(\cdot), \cdot+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\cdot)\right),  \tag{26}\\
\mathbf{m} \neq \mathbf{0} .
\end{array}
$$

In addition, the projection onto the $\mathbf{m}=\mathbf{0}$ mode gives

$$
\begin{align*}
\mathbf{0} & =\mathbf{R}_{o}\left(\boldsymbol{\rho}_{o}+\boldsymbol{\rho}(\cdot), \cdot+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\cdot)\right),  \tag{27}\\
2 \pi \boldsymbol{\nu} & =\Theta_{o}\left(\boldsymbol{\rho}_{o}+\boldsymbol{\rho}(\cdot), \cdot+\boldsymbol{\alpha}_{o}+\boldsymbol{\alpha}(\cdot)\right) . \tag{28}
\end{align*}
$$

To specify a solution of Eqs. (26)-(28), we must choose an input parameter to be held fixed in the course of solution. Following the viewpoint of the previous section, we could take $\boldsymbol{\rho}_{o}$, which is approximately equal to $\mathbf{J}$, as that parameter. Then $\boldsymbol{\nu}$ would be eliminated by substituting (28) in (26), and (26),(27) would be solved for $\alpha_{\mathrm{m}}, \rho_{\mathrm{m}}, \mathrm{m} \neq 0$, and $\boldsymbol{\alpha}_{o}$. A possible drawback of this option is that $\boldsymbol{\nu}$ will change in the course of iteration, possibly allowing a small divisor to develop so as to damage convergence. It may be more suitable to take $\boldsymbol{\nu}$ as a fixed input, in order to gain the best possible control of small divisors. Then all the Fourier coefficients, including those for $\mathbf{m}=\mathbf{0}$, would be found by solving (26), (27), (28). Without numerical experience, it is difficult to say which of these two choices should be preferred, even though fixed $\boldsymbol{\nu}$ is suggested by the KAM proof in which control of small divisors through a Diophantine condition is an essential element. Also, fixed $\boldsymbol{\nu}$ would be appropriate for studying the breakup of tori as the perturbation strength is increased.

Let us see how a computation at fixed $\boldsymbol{\nu}$ might go, of course with a finite mode set $\mathcal{T}$, and with weak perturbation $V$. First note that existence of solutions for arbitrary nonresonant (Diophantine) $\boldsymbol{\nu}$ cannot be expected. The domain of winding numbers is an intrinsic feature of the $\operatorname{map} \mathcal{M}$, and can be sharply restricted. In typical applications one knows the domain of $\mathbf{J}$ of interest, and has no choice but to deal with the corresponding range of $\boldsymbol{\nu}$. The first step is then to choose a $\boldsymbol{\nu}$ corresponding to the desired domain of $\mathbf{J}$. This can be done by applying the approximation $2 \pi \boldsymbol{\nu}(\mathbf{J}) \approx \Theta_{o}(\mathbf{J}, \cdot)$, or simply by following orbits of $\mathcal{M}$ and finding the average change of $\boldsymbol{\Phi}$ per iteration of $\mathcal{M}$, which defines $2 \pi \boldsymbol{\nu}$ in the limit of infinitely many iterations. Having chosen $\boldsymbol{\nu}$, let us define $\hat{\boldsymbol{\rho}}_{o}$ by $\nu=\Theta_{o}\left(\hat{\boldsymbol{\rho}}_{o}, \cdot\right)$.

To discuss the structure of the equations, we define a $2 d$-dimensional vector $x=\left(\boldsymbol{\rho}_{o}-\hat{\boldsymbol{\rho}}_{o}, \boldsymbol{\alpha}_{o}\right)$ and a big vector $y$ having the Fourier coefficients for $\mathbf{m} \neq \mathbf{0}$ as components: $y=\left(\boldsymbol{\rho}_{\mathbf{m}}, \boldsymbol{\alpha}_{\mathbf{m}} ; \mathbf{m} \neq \mathbf{0}\right)$. For fixed $x$, Eq. (26) can be solved for $y$ by simple iteration, since the system has the form $y=A(y, x)$ with $A$ being a contractive operator as far as $y$ is concerned. On the other hand, Eqs. (27) and (28) have a form $B(x, y)=0$ which does not lend itself to a simple iterative solution for $x$ at fixed $y$. Instead, an iteration of Newton type is needed. A hybrid iterative scheme like the following might be able to handle both equations together:

$$
\begin{gather*}
y^{(p+1)}=A\left(y^{(p)}, x^{(p)}\right)  \tag{29}\\
B\left(x^{(p)}, y^{(p+1)}\right)+B_{x}\left(x^{(p)}, y^{(p+1)}\right)\left(x^{(p+1)}-x^{(p)}\right)=0,  \tag{30}\\
x^{(0)}=0, \quad y^{(0)}=0, \quad B_{x}=\partial B / \partial x
\end{gather*}
$$

This algorithm has not been analyzed in detail, but it seems plausible, since we deal with small $(x, y)$ and $B(0,0)$ is small. The angular component of $B(0,0)$ is 0 by our definition of $\hat{\boldsymbol{\rho}}_{o}$, and the radial component is $O\left(V^{2}\right)$. The latter can be proved by a simple perturbative analysis of the Hamilton equation $\dot{\mathbf{I}}=-\partial V / \partial \Phi$ that defines the map $\mathbf{R}$.

All Fourier transforms in this scheme can be done by the Fast Fourier Transform, whereas slow transforms are required in the treatment of equations of the first kind. That should allow very fast execution of the scheme. One might conjecture that the convergence properties would be similar to those of normal-form perturbation theory, which has been rather successful in solving the conjugation equation (13) in Cartesian coordinates. ${ }^{7}$ On the other hand, one would expect the iterative method to be a great deal faster than perturbation theory, at least for the purpose of computing a single torus, whenever the perturbation series has to be carried to high orders for sufficient accuracy.

## CONCLUSION

Functional difference equations provide an elegant framework for the study and computation of invariant tori of an arbitrary Hamiltonian system. Being based on the Poincaré return map, they effectively reduce the dimension of the system to be analyzed by one unit. The generalized Moser equation seems particularly interesting for further work, since it has not yet been treated numerically, and has potential advantages over the equation of Hamilton-Jacobi type.

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