# Measuring CKM Parameters with $C P$ Asymmetry and Isospin Analysis in $B \rightarrow \pi K^{\star}$ 

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#### Abstract

Isospin relations are used to eliminate hadronic uncertainties in various $C P$ asymmetries in $B^{0}$ decays via $b \rightarrow u \bar{u} s$, e.g. $B^{0} \rightarrow \pi^{0} K_{S}$. A clean measurement of the angle $\alpha$ of the unitarity triangle is thus made possible. The magnitudes of the tree and the penguin amplitudes can be measured.


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[^0]$C P$-asymmetries in $B^{0}$ decays into a final $C P$ eigenstate are free of hadronic uncertainties if a single CKM combination dominates the decay process. Within the standard model, most processes get contributions from both tree and penguin amplitudes [1]. In $b \rightarrow c \bar{c} s$ processes (e.g. $B^{0} \rightarrow \psi K_{S}$ ) both amplitudes canry the came CKM phase (to a very good approximation); extracting $\sin 2 \beta$ from this asymmetry is free of hadronic uncertainties [2]. In $b \rightarrow u \bar{u} d$ processes (e.g. $B^{0} \rightarrow \pi^{+} \pi^{-}$) the two amplitudes carry different CKM phases. It is expected that the contribution from the penguin amplitude is small (a few percent), but it could be larger than the naive expectation if the matrix element for the penguin operator is enhanced; extracting $\sin 2 \alpha$ from this asymmetry may suffer from hadronic uncertainties if this is indeed the case. In $b \rightarrow u \bar{u} s$ processes (e.g. $B^{0} \rightarrow \pi^{0} K_{S}$ ) the two amplitudes carry different CKM phases and are expected to be of the same order of magnitude; it is usually stated that one cannot cleanly extract values of CKM parameters from this asymmetry.

Recently, Gronau and London [3] have shown how to separate the CKM phase of the tree-level $B \rightarrow \pi \pi$ process from any penguin contamination. This is done by means of isospin analysis of various (charged and neutral) $B$ decays into $\pi \pi$. The three relevant amplitudes fulfill a triangle relation: once their magnitudes are known, the relative phases among them can be calculated. This will allow a determination of $\alpha$ completely free of hadronic uncertainties, independent of how large the penguin amplitude is.

In this work, we study the $C P$ asymmetry in $B^{0} \rightarrow \pi^{0} K_{S}$. As mentioned above, without isospin analysis this mode does not provide a clean theoretical determination of CKM parameters. Moreover, the analysis of ref. [3] applies to a case where isospin relates three amplitudes and cannot be applied in a straightforward way to the $\pi K$ mode, where isospin gives a relation among four amplitudes. However, we show that there is still a way to use isospin relations in order to cleanly measure CKM parameters (specifically, the angle $\alpha$ of the unitarity triangle) from $C P$ asymmetries in various $b \rightarrow u \bar{u} s$ modes. Finally, we explain how to measure the magnitude of the tree and penguin amplitudes.
$B^{+}$and $B^{0}$ decay into final $\pi K$ states via the quark subprocess $\bar{b} \rightarrow \bar{u} u \bar{s}$. The Hamiltonian acting on the $B$ can be written

$$
\begin{equation*}
\mathcal{H}|B\rangle=A_{0}|0,0\rangle+A_{1}|1,0\rangle . \tag{1}
\end{equation*}
$$

Let us define a decay amplitudes $A_{i j}$ by

$$
\begin{equation*}
A_{i j} \equiv\left\langle\pi^{i} K^{j}\right| \mathcal{H}|B\rangle \tag{2}
\end{equation*}
$$

Then the four amplitudes for $B^{+}$and $B^{0}$ decays into final $\pi K$ states can be written as

$$
\begin{align*}
A_{0+}=U-W, & \sqrt{\frac{1}{2}} A_{+0}=V+W \\
A_{00}=U+W, & \sqrt{\frac{1}{2}} A_{-+}=V-W \tag{3}
\end{align*}
$$

The amplitudes $U, V$ and $W$ absorb Clebsh-Jordan coefficients:

$$
\begin{equation*}
W \equiv \sqrt{\frac{1}{3}} A_{0}^{\prime}, \quad U \equiv \frac{1}{3}\left(2 A_{1}^{\prime \prime}+A_{1}^{\prime}\right), \quad V \equiv \frac{1}{3}\left(A_{1}^{\prime \prime}-A_{1}^{\prime}\right) \tag{4}
\end{equation*}
$$

where $A_{i}^{\prime}$ and $A_{i}^{\prime \prime}$ incorporate the change in magnitude as well as the strong phase shift corrections to $A_{i}$ due to hadronization and rescattering effects for final $I=1 / 2$ and $I=3 / 2$ states respectively.

Similarly, $\bar{B}^{0}$ and $B^{-}$decay into final $\pi K$ states via $b \rightarrow u \bar{u} s$. Here, the various amplitudes are given by:

$$
\begin{array}{ll}
\bar{A}_{0+}=\bar{U}-\bar{W}, & \sqrt{\frac{1}{2}} \bar{A}_{+0}=\bar{V}+\bar{W} \\
\bar{A}_{00}=\bar{U}+\bar{W}, & \sqrt{\frac{1}{2}} \bar{A}_{-+}=\bar{V}-\bar{W} \tag{5}
\end{array}
$$

where $\bar{A}_{i j}$ is the amplitude for the $C P$-conjugated process of $A_{i j}$, e.g. $\bar{A}_{+0}$ corresponds to $B^{-} \rightarrow \pi^{-} \bar{K}^{0}$. The amplitudes $\bar{U}, \bar{V}$ and $\bar{W}$ carry weak phases opposite to (but strong phases identical to) those of $U, V$ and $W$, respectively.

Measuring the eight decay rates gives the various $\left|A_{i j}\right|$ and $\left|\bar{A}_{i j}\right|$. The $C P$ asymmetry in the $\pi^{0} K_{S}$ mode gives

$$
\begin{equation*}
\operatorname{Asym}\left(B \rightarrow \pi^{0} K_{S}\right)=\operatorname{Im}\left[e^{-2 i\left(\phi_{M}+\phi_{K}+\phi_{T}\right)} \frac{e^{2 i \phi_{T}} \bar{A}_{00}}{A_{00}}\right] \tag{6}
\end{equation*}
$$

The phases $\phi_{M}$ and $\phi_{K}$ are the CKM phases in the mixing amplitudes for neutral $B$ and neutral $K$, respectively $\left[\phi_{M}=\arg \left(V_{t d}^{*} V_{t b}\right), \phi_{K}=\arg \left(V_{c s}^{*} V_{c d}\right)\right]$. The phase $\phi_{T}$ is the CKM phase in the tree diagram $\left[\phi_{T}=\arg \left(V_{u b}^{*} V_{u s}\right)\right]$.

If $b \rightarrow u \bar{u} s$ processes were dominated by tree diagrams (or if $\phi_{P}$, the CKM phase in the penguin diagram, equalled $\left.\phi_{T}\right), A_{i j}$ would be $e^{2 i \phi_{T}} \bar{A}_{i j}$, and the asymmetry in eq. (6) would reduce to $\sin 2\left(\phi_{M}+\phi_{K}+\phi_{T}\right)=\sin 2 \alpha$. This type of situation holds in $b \rightarrow c \bar{c} s$ and, probably, $b \rightarrow u \bar{u} d$ processes, which is the reason for the cleanliness in their theoretical interpretation. However, this is not the actual case for $b \rightarrow u \bar{u} s$ processes since (i) the penguin diagram depends on $\phi_{P}=\arg \left(V_{t b}^{*} V_{t s}\right)$, so that $\phi_{P} \neq \phi_{T}$, and (ii) while the penguin diagram is higher order in couplings (an $\left(\alpha_{S} / 12 \pi\right) \ln \left(m_{t}^{2} / m_{b}^{2}\right) \sim 0.02$ suppression), the tree diagram is CKM suppressed $\left(\mathrm{a}\left(\sin \theta_{C}\right)\left(V_{u b} / V_{c b}\right) \sim 0.02\right.$ suppression). Thus the two amplitudes are expected to be of the same order of magnitude. In general, then, $e^{2 i \phi_{T}} \bar{A}_{00} / A_{00} \neq 1$ and needs to be determined before $\alpha$ can be calculated from (6). This is done through isospin analysis, as we show below.

For this analysis it is convenient to define new quantities $\tilde{A}_{i j}$ :

$$
\begin{equation*}
\tilde{A}_{i j} \equiv e^{2 i \phi_{T}} \bar{A}_{i j} \tag{7}
\end{equation*}
$$

Note that the ratios $\tilde{A}_{i j} / A_{i j}$ are independent of phase conventions. What we need in order to extract $\alpha$ from the asymmetry in (6) is then ( $\tilde{A}_{00} / A_{00}$ ). Similarly, we define

$$
\begin{equation*}
\tilde{U} \equiv e^{2 i \phi_{T}} \bar{U}, \quad \tilde{V} \equiv e^{2 i \phi_{T}} \bar{V}, \quad \tilde{W} \equiv e^{2 i \phi_{T}} \bar{W} \tag{8}
\end{equation*}
$$

Let us examine the amplitudes in eqs. (3) and (5). They fulfill quadrilateral
relations:

$$
\begin{align*}
& A_{0+}+\sqrt{\frac{1}{2}} A_{+0}=A_{00}+\sqrt{\frac{1}{2}} A_{-+}  \tag{9}\\
& \tilde{A}_{0+}+\sqrt{\frac{1}{2}} \tilde{A}_{+0}=\tilde{A}_{00}+\sqrt{\frac{1}{2}} \tilde{A}_{-+}
\end{align*}
$$

This means that the four $A_{i}$ 's form a quadrilateral in the complex plane, and similarly the four $\tilde{A}_{i j}$ 's. The various decay rates give all eight sides of these two quadrilaterals. However, knowing the lengths of the sides of a quadrilateral, does not determine its angles. In $B \rightarrow \pi \pi$ decays, there are three (instead of four) amplitudes. The six decay rates give the six sides of two triangles and all angles are consequently determined. It is obvious that the same method cannot be extended in a straightforward way to the present case.

However, there is an additional important piece of information. The penguin operator is purely $I=0$ and, consequently, only tree diagrams contribute to $I=1$ transitions. This gives the following relations among the $U$ and $V$ amplitudes (which are pure $I=1$, as can be seen from their definition in eq. (4)):

$$
\begin{equation*}
\tilde{U}=U, \quad \tilde{V}=V \tag{10}
\end{equation*}
$$

Instead of eq. (5) we can now use

$$
\begin{array}{cl}
\tilde{A}_{0+}=U-\tilde{W}, & \sqrt{\frac{1}{2}} \tilde{A}_{+0}=V+\tilde{W} \\
\tilde{A}_{00}=U+\tilde{W}, & \sqrt{\frac{1}{2}} \tilde{A}_{-+}=V-\tilde{W} \tag{11}
\end{array}
$$

This implies the following two relations between the two quadrilaterals of eq. (9) (see fig. (1)):
(i) One of the two diagonals is common to the two quadrilaterals:

$$
\begin{equation*}
A_{00}+\sqrt{\frac{1}{2}} A_{-+}=\tilde{A}_{00}+\sqrt{\frac{1}{2}} \tilde{A}_{-+}=U+V \tag{12}
\end{equation*}
$$

(ii) The other (non-common) diagonals bisect each other:

$$
\begin{equation*}
A_{00}+A_{0+}=\tilde{A}_{00}+\tilde{A}_{0+}=2 U \tag{13}
\end{equation*}
$$

The crucial point is that knowing the eight sides of two quadrilaterals that fulfill conditions (i) and (ii) does determine (up to a twofold discrete ambiguity) all the angles. To demonstrate that, we write the equation for the length of the common diagonal, $|U+V|$ :

$$
\begin{align*}
& h\left(\left|A_{00}\right|,\left|A_{-+} / \sqrt{2}\right|,|U+V|\right)+h\left(\left|A_{0+}\right|,\left|A_{+0} / \sqrt{2}\right|,|U+V|\right)= \\
& h\left(\left|\tilde{A}_{00}\right|,\left|\tilde{A}_{-+} / \sqrt{2}\right|,|U+V|\right)+h\left(\left|\tilde{A}_{0+}\right|,\left|\tilde{A}_{+0} / \sqrt{2}\right|,|U+V|\right), \tag{14}
\end{align*}
$$

where $h(a, b, c) /(2 c)$ is (up to sign ambiguity) the height of a triangle of basis $c$ and sides $a$ and $b$ :

$$
\begin{equation*}
[h(a, b, c)]^{2}=2\left(a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}\right)-a^{4}-b^{4}-c^{4} \tag{15}
\end{equation*}
$$

Obviously, once $|U+V|$ is known, the angles within each quadrilateral are determined. In particular, $\tilde{A}_{00} / A_{00}$ is determined, allowing us to derive $\alpha=\phi_{M}+\phi_{K}+$ $\phi_{T}$ from the $C P$ asymmetry in $B^{0} \rightarrow \pi^{0} K_{S}$ (eq. (6)).

The above analysis can be applied in the same way to additional hadronic final states for $b \rightarrow u \bar{u} s$ processes where the non-strange meson is an isovector: $\rho K_{S}$, $\pi K^{*}$ and others. Note that, unlike the case of a very small penguin contribution, the asymmetries in these various modes are not expected to be all equal. The reason is that $\tilde{A}_{00} / A_{00}$ depends on strong interaction effects and is, therefore, different for different modes. However, a single value of $\alpha$ should, of course, arise from all asymmetries.

We note that there is a discrete twofold ambiguity in the determination of $\tilde{A}_{00} / A_{00}$, which corresponds to reflecting both quadrilaterals along the $U+V$ axis. This will give two solutions, one corresponds to the correct $\tilde{A}_{00} / A_{00}$ and the other to its conjugate. Measuring the asymmetries in various hadronic modes will resolve the ambiguity: only the true values of $\tilde{A}_{00} / A_{00}$ will give a consistent solution for $\alpha$. (This ambiguity is not to be confused with another discrete ambiguity which persists even in the case of $\tilde{A}_{00} / A_{00}=1[4]$.)

An isospin analysis carries useful information besides the $C P$-violating CKM phases. Assuming the standard model, we can actually extract measures of the tree and the penguin contribution to $W$. To show this, we use

$$
\begin{align*}
& W=P_{0} e^{i\left(\delta_{P_{1 / 2}}+\phi_{P}\right)}+T_{0} e^{i\left(\delta_{T_{1 / 2}}+\phi_{T}\right)}, \\
& \bar{W}=P_{0} e^{i\left(\delta_{P_{1 / 2}}-\phi_{P}\right)}+T_{0} e^{i\left(\delta_{T_{1 / 2}}-\phi_{T}\right)}, \tag{16}
\end{align*}
$$

where $P_{0}$ and $T_{0}$ denote penguin and tree diagrams, respectively and $\delta$ and $\phi$ denote strong and weak phases, respectively. We obtain the following relations:

$$
\begin{align*}
& T_{0}=\frac{\left|W-\tilde{W} e^{2 i\left(\phi_{P}-\phi_{T}\right)}\right|}{\sqrt{2\left[1-\cos 2\left(\phi_{T}-\phi_{P}\right)\right]}} \\
& P_{0}=\frac{|W-\tilde{W}|}{\sqrt{2\left[1-\cos 2\left(\phi_{T}-\phi_{P}\right)\right]}} \tag{17}
\end{align*}
$$

The quantities $W$ and $\tilde{W}$ can be determined from the isospin analysis. Within the standard model the penguin amplitude depends on the CKM combination $V_{t b}^{*} V_{t s}$. Consequently, $\phi_{T}-\phi_{P}$ is the angle $\gamma$ of the unitarity triangle, which can be directly measured in $C P$ asymmetries in $B_{s}$ decays or calculated from $C P$ asymmetries in $B_{d}$ decays. We conclude that the full isospin analysis allows a determination of $P_{0}$ and $T_{0}$, and is, therefore, useful not only for our understanding of $C P$ violation but also of hadronic physics. (In the case of $B \rightarrow \pi \pi$, comparing the penguin and tree contributions to $I=1 / 2$ transitions is even simpler. There, $\phi_{T}-\phi_{P}=\phi_{T}+\phi_{M}=\alpha$, measured by the $C P$ asymmetry in the $\pi \pi$ mode itself.)

In conclusion: $C P$ asymmetries in $B \rightarrow \pi^{0} K_{S}$ suffer from hadronic uncertainties because they get comparable contributions from penguin and tree diagrams. The simple isospin analysis of ref. [3] does not help because here the amplitudes fulfill quadrilateral relations. However, due to the isospin properties of the penguin and tree operators, there is still a way to use isospin relations in order to eliminate the hadronic uncertainties and cleanly measure the angle $\alpha$ of the unitarity triangle. The same analysis can be applied to other hadronic modes of the $b \rightarrow u \bar{u} s$ process. The list of $C P$ asymmetries in $B^{0}$ decays which yield to a clean theoretical interpretation is thus significantly expanded.

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## FIGURE CAPTIONS

Fig. 1: The two quadrilaterals of eq. (9). Note that $U+V$ is a common diagonal, while the non-common diagonals bisect each other.


Fig. 1


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