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# ALCHEMY IN 1 + 1 DIMENSION: FROM BOSONS TO FERMIONS

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## ABSTRACT

Canonical massless fermion field is constructed from canonical boson field in $1+1$ dimensional space. Single-fermion states are expressed in terms of eigenstates of boson operators.

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# 1. INTRODUCTION

Matter is made of fermions and bosons. Spin and statistics is what makes a difference between the two. Fermions are characterized by half-integer values of spin, and by the Pauli exclusion principle. They are described by sets of quantum operators with simple *anti-commuting* rules. Bosons, on the other hand, have integer values of spin, a state can contain an arbitrary number of bosons with the same quantum numbers, and *commutators* rather than anticommutators characterize the corresponding operators. It is well known from the statistical, nuclear, and particle physics that an even number of fermions can form a boson. For example, quark-antiquark pairs (*i.e.,* pairs of fermions), are believed to compose mesons (which are bosons). This is really not a surprise: if some binding force keeps an even number of objects with a half-integer spin together, it must be possible to combine them into an object with an integer spin. Along these lines one might even tend to believe that fermions are perhaps more fundamental objects than bosons, and consider the latter merely as composed states.

However, such a view neglects among other things the fact that the opposite direction is also possible: fermions may be made out of suitably arranged bosons, at least in a lower dimensional space! The present article deals just with such a surprising relationship between fermions and bosons in the one-dimensional space. During the past thirty years this subject was thoroughly studied by many distinguished physicists, and with a good reason. The transformation of bosons to fermions and *vice versa* (or the "bosonization" of fermions, as it is sometimes called) might prove to be a very useful tool in getting a valuable insight into the long-standing problem of confinement in the quantum chromodynamics. The concept was also used to solve complicated, interacting models in $1 + 1$ dimension, by replacing them with simpler and/or non-interacting theories. Furthermore, the mere notion that fermions and bosons are deeply inter-related, has a beauty on its own. Yet, students are most often exposed to the subject only in highly specialized graduate courses. *E.g.,* the popular introductory-level textbooks on quantum field theory very rarely mention the bosonization. Similarly, in the last decade there was not even a single article on the topic in this Journal.

This paper is meant to be an elementary introduction to the fermion-boson duality. It considers the simplest possible situation: the world is reduced to one-dimensional segment of a finite length, and we study the possibility of forming free fermions in the segment, by using only the free, massless bosons. Truly, in one-space, one-time dimension $(1+1)$ the angular momentum is not defined, and we do not have to worry about the spin, but fermions and bosons are still distinguishable by their statistics. Our task is therefore to find the transformation from a set of commuting operators characterizing bosons, into another set of anticommuting operators corresponding to fermions. But why would anyone want to know anything about such a simplified world in which some non-interacting particles are kept in a segment? First, the finite length of the interval is really not a serious restriction. This length is an infra-red cutoff which can be set to infinity at the end of the analysis. Furthermore, the free and massless theory in $1+1$ dimension is simple enough to be easily absorbed by beginners and non-experts, and yet it contains almost all important elements needed in a more advanced study of massive and interacting systems. Once the interactions are introduced, the whole new world opens, and not only of the pure academic interest. For example, a better understanding of interacting one-dimensional systems might prove crucial for the development of synthetic metals, new types of transistors, or light-weight, rechargeable, high-energy-density batteries. More on these possibilities in the concluding section.

This article is primarily aimed at the first and second year students of graduate schools, but an undergraduate with some knowledge of relativistic quantum mechanics, and inclined to quantum field theory or condensed matter physics, could also benefit from it. Those readers not interested in the relativistic quantum fields, can neglect all the dynamics and consider this paper an exercise in transforming commuting into anticommuting variables within the framework of ordinary quantum mechanics. We begin by finding a general solution $\Phi(t,x)$ of the Klein-Gordon equation $\partial^\mu \partial_\mu \Phi = 0$ in a segment of a finite length $L$. The field $\Phi$ is described in terms of various time-independent operators, and the operators satisfy simple commutation rules (Section 2). The time evolution of "physical" states is determined by hamiltonian and the momentum operator, which are constructed in Section 3. The one-dimensional Dirac equation $i\gamma^\mu \partial_\mu \Psi = 0$, and general properties of a fermion field operator $\Psi(t,x)$ in a segment of length $L$, are studied in Section 4. Section 5 is the heart of this article: we use boson operators to construct the fermion field, and show that field operators in the resulting set satisfy the correct anticommutation rules. We then express fermionic annihilation and creation operators in terms of the bosonic counterparts, and discuss single particle states for fermions (Section 6). Delta functions relevant to finite intervals are described in Appendix A, and a brief review of the Klein's factor can be found in Appendix B.

In preparing this paper I benefited most from the two articles by Wolf and Zittartz,[1] in which one can also find a good list of references to the earlier works as well as the discussion on the relevance of the subject for the solid state and statistical physics. The articles by Boyanovsky,[2] Kogut and Susskind,[3] and Klaiber,[4] were also very useful in my study. For the lattice version of the problem see *e.g.,* the article by Shultz, Mattis and Lieb[5]. I truly enjoyed following this miraculous transformation of bosons to fermions, and hope that the readers will also find it exciting.

## 2. KLEIN-GORDON EQUATION

To begin, we consider the equation

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \Phi(t,x) = 0 \tag{1}$$

in the segment $x \in [-L/2, +L/2]$, for a real function $\Phi(t,x)$. The form of the equation (1) allows us to introduce the "charge density" $\rho(t,x) = \Phi'(t,x)/\sqrt{\pi}$, and the "current" $J(t,x) = -\dot{\phi}(t,x)/\sqrt{\pi}$, where "prime" and "dot" denote space and time derivatives. With these definitions, the "continuity equation", $\partial\rho/\partial t + \partial J/\partial x = 0$ is clearly satisfied. We are now in a position to construct the "total charge" $Q$, and the "mean current" $\widetilde{Q}$,

$$
\begin{aligned}
Q &= \int_{-L/2}^{+L/2} dx \rho(t,x) = \frac{1}{\sqrt{\pi}} \int_{-L/2}^{+L/2} dx \left[\frac{\partial\Phi}{\partial x}\right] \qquad , \\
\widetilde{Q} &= \int_{-L/2}^{+L/2} dx J(t,x) = -\frac{1}{\sqrt{\pi}} \int_{-L/2}^{+L/2} dx \left[\frac{\partial\Phi}{\partial t}\right] \qquad .
\end{aligned}
\tag{2}
$$

Note that at this stage the names "charge" and "current" are misleading. After all, we are dealing with the real function $\Phi(t,x)$ , which is expected to describe *chargeless* field. However, as we continue, the naming scheme will become more justified.

Our first goal is to find the solutions of eq. (1) for which $Q$ and $\widetilde{Q}$ are conserved (*i.e.*, time independent). In order to find such solutions, we assume that $\Phi(t,x) = T(t)F(x)$ . Eq. (1) leads to $F''(x)/F(x) = \ddot{T}(t)/T(t) = -\omega^2$ , where $\omega^2$ is an arbitrary constant. The conservation of charge and mean current then restricts the values of $\omega$ to a discrete set of values, $\omega \longrightarrow \omega_n = 2\pi n/L$ , with $n \geq 0$ . By superposing partial solutions for all allowed values of $\omega_n$, we can write the general solution of eq. (1) as

$$
\Phi(t,x) = R(t,x) + \varphi(t,x) \qquad , \tag{3}
$$

where

$$
R(t,x) = \frac{1}{\sqrt{\pi}}\widetilde{P} + \frac{\sqrt{\pi}}{L}(xQ - t\widetilde{Q}) \qquad , \tag{4a}
$$

$$
\varphi(t,x) = \sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(a_n e^{-i\frac{2\pi n}{L}(t-x)} + a_{-n} e^{-i\frac{2\pi n}{L}(t+x)}\right) + \text{c.c.} \qquad . \tag{4b}
$$

In eq. (3), the first term $R(t,x)$ corresponds to the zero-frequency mode $\omega = 0$ . It contains the constant operator $\widetilde{P}$, ( see eq. ($4a$) ), which is related to $\Phi$ by

$$\widetilde{P} = \frac{\sqrt{\pi}}{L} \int_{-L/2}^{+L/2} dx\, \Phi(0,x) \qquad . \tag{5}$$

$a_n$ and $a_{-n}$ in ($4b$) are conveniently normalized coefficients in the Fourier expansion for $\varphi$ , and $c.c.$ denotes the complex conjugate values. Note that

$$a_n = \frac{i}{\sqrt{4\pi|n|}} \int_{-L/2}^{+L/2} dx[\dot{\Phi}(0,x) - \mathrm{sgn}(n)\, \Phi'(0,x)]\, e^{-i\frac{2\pi n}{L}x} \qquad , \tag{6}$$

where the function $\mathrm{sgn}(n) \equiv n/|n|$ returns the sign of $n$.

In the classical field theory, (3) and (4) describe a special solution of the Klein-Gordon equation called "plasmon". Upon quantization, $\Phi(t,x)$ becomes a Hermitian field operator satisfying equal-time commutation relations for boson fields,

$$\begin{aligned} &[\Phi(t,x), \dot{\Phi}(t,y)] = i\Delta_L(x-y) \qquad , \\ &[\Phi(t,x), \Phi(t,y)] = [\dot{\Phi}(t,x), \dot{\Phi}(t,y)] = 0 \qquad . \end{aligned} \tag{7}$$

Here, $\Delta_L$ is an equivalent of the Dirac delta function, relevant for the finite interval $[-L/2, +L/2]$ (see Appendix A). The constants $\widetilde{P}, Q, \widetilde{Q}$ and $a_n$ from eqs. (4) now also become operators, the first three being Hermitian. The complex conjugate part in ($4b$) is replaced by a Hermitian conjugate part. From relations (7), and with definitions (2) to (6), we find

$$[a_n, a_m^\dagger] = \delta_{nm} \qquad , \qquad [\widetilde{Q}, \widetilde{P}] = i \qquad , \tag{8}$$

while all the other commutators vanish.

As we might have expected, the boson field is described by an infinite set of harmonic oscillators with frequencies $\omega_n = 2\pi|n|/L$, and characterized by annihilation and creation operators $a_n\,(a_n^\dagger)$, acting in the Hilbert space $\mathcal{S}_a$. In addition to these local degrees of freedom, there are other, global operators in the expansion of the field. These are $\widetilde{Q}$ with its conjugate pair $\widetilde{P}$, and the operator $Q$. We usually neglect those operators when the value of $x$ is unrestricted, but in the final interval they do play a central role. Since the global operators $Q$ and $\widetilde{Q}$ commute mutually as well as with all $a_n(a_n^\dagger)$ operators, they generate two new Hilbert spaces, $\mathcal{S}_Q$ and $\mathcal{S}_{\widetilde{Q}}$. Consequently, the total space of states $\mathcal{S}_B$ in our problem is the tensor product

$$\mathcal{S}_B = \mathcal{S}_Q \otimes \mathcal{S}_{\widetilde{Q}} \otimes \mathcal{S}_a \qquad . \tag{9}$$

Here $B$ stands for bosons. We shall see later that the Hilbert space $\mathcal{S}_F$, corresponding to fermions, is a subspace in $\mathcal{S}_B$. In other words, not all the states in $\mathcal{S}_B$ will be used to build fermions.

To round out the discussion of massless, non-interacting bosons, we decompose the field $\Phi$ into the right-moving ($\Phi_+$), and left-moving ($\Phi_-$) components,

$$\Phi(t,x) = \frac{1}{\sqrt{2}}\Phi_+(t,x) + \frac{1}{\sqrt{2}}\Phi_-(t,x) \qquad . \tag{10}$$

Our interest in these right- and left-moving sub-systems will become justified later, when we observe that the fermion field can also be decomposed into the right- and left-moving pieces. The two new field operators, $\Phi_+$ and $\Phi_-$, depend respectively on $t - x$ and $t + x$ combinations of variables. Unfortunately, due to the presence of constant terms in eq. (4), the partition (10) is not unique. It is convenient to introduce another new operator, $P$, which by assumption is conjugate to the charge operator $Q$ (*i.e.*, $[Q,P] = i$). $P$ is an operator in $\mathcal{S}_Q$ space, and therefore it commutes with both $\widetilde{Q}$ and $\widetilde{P}$, and with all $a_n(a_n^\dagger)$ operators. With the aid of $P$, we can achieve a highly symmetric partition,[6] where $\Phi_\pm$ are defined as

$$\Phi_\pm(t,x) = R_\pm(t,x) + \varphi_\pm(t,x) \qquad , \tag{11}$$

and

$$R_\pm(t,x) = \frac{1}{\sqrt{2\pi}}(\widetilde{P} \pm P) - \frac{\sqrt{\pi}}{L\sqrt{2}}\,(\widetilde{Q} \pm Q)(t \mp x)$$

$$\begin{aligned}\varphi_\pm(t,x) &= \sum_{n>0} \frac{1}{\sqrt{2\pi n}}\,(a_{\pm n} e^{-i\frac{2\pi n}{L}(t\mp x)} + a^\dagger_{\pm n} e^{+i\frac{2\pi n}{L}(t\mp x)}\,) \\ &= \varphi_\pm^{(+)}(t,x) \;+\; \varphi_\pm^{(-)}(t,x) \qquad .\end{aligned} \tag{12}$$

In expression (12), $\varphi_\pm^{(+)}$ is the positive-frequency component, and $\varphi_\pm^{(-)}$ describes the negative-frequency part. Note that functions $\varphi_\pm$ are periodic, $\varphi_\pm(t,x+L) = \varphi_\pm(t,x)$, while this is not true for the complete solutions $\Phi_\pm$ .

In this section we found the general form of the Klein-Gordon field, and decomposed the field into right- and left-moving parts, keeping global and local degrees of freedom separated. The dynamics of the boson (plasmon) field is determined by the hamiltonian of the system, and in the next section we shall construct this operator and define the vacuum.

## 3. HAMILTONIAN FOR BOSONS

The hamiltonian, and the momentum operator, determine the time evolution of states in a system. In analogy with the procedure applied to the three-dimensional Klein-Gordon theory,[7] we define

$$H_B = \int\limits_{-L/2}^{+L/2} dx \frac{1}{2}(\dot{\Phi}^2 + {\Phi'}^2) \qquad , \qquad K_B = \int\limits_{-L/2}^{+L/2} dx(-\dot{\Phi}\Phi') \qquad . \tag{13}$$

According to eq. (3), the field $\Phi$ is the sum of $R$ and $\varphi$ , and from $(4b)$ it follows that $\int \dot{\varphi} = \int \varphi' = 0$ . The hamiltonian therefore reduces to

$$H_B = \frac{\pi}{2L^2} \int\limits_{-L/2}^{+L/2} dx(Q^2 + \widetilde{Q}^2) + \frac{1}{2} \int\limits_{-L/2}^{+L/2} dx(\dot{\varphi}^2 + {\varphi'}^2) \qquad . \tag{14}$$

This shows that, except when $L \to \infty$ , the "charge" and the "mean current" of a state also contribute to the total energy. Since the second integral in eq. (14) represents the standard harmonic contribution, we can immediately write

$$H_B = \frac{\pi}{2L}(Q^2 + \widetilde{Q}^2) + \sum_{n\neq 0} \omega_n (a_n^\dagger a_n + \frac{1}{2}) \qquad . \tag{15}$$

In eq. (15), $\omega_n = 2\pi|n|/L$ is the frequency (and, at the same time the energy) of the massless harmonic excitations. The expression still contains an infinite energy of the vacuum, $\frac{1}{2}\sum \omega_n$, which should be subtracted. As a convenient shorthand for this subtraction we introduce the normal ordering with respect to the $\mathcal{S}_B$ vacuum state. For an operator $X$, let $\vdots X \vdots$ denotes

$$\vdots X \vdots = X - \langle\, \emptyset \,|\, X \,|\, \emptyset \,\rangle \qquad , \tag{16}$$

where $|\, \emptyset \,\rangle$ is the state with zero charge and mean current, $Q\,|\, \emptyset \,\rangle = \widetilde{Q}\,|\, \emptyset \,\rangle = 0$, and no oscillators, $a_n \,|\, \emptyset \,\rangle = 0$. Using this definition, we may now express the normalized hamiltonian for bosons as

$$H_B = \vdots \int_{-L/2}^{+L/2} dx\, \frac{1}{2}(\dot{\Phi}^2 + \Phi'^2) \,\vdots \;=\; \frac{\pi}{2L}(Q^2 + \widetilde{Q}^2) + \sum_{n\neq 0} \omega_n a_n^\dagger a_n \quad . \tag{17}$$

In the tensor product notation we can describe the vacuum state as

$$|\, \emptyset \,\rangle = |0\rangle_Q \otimes |0\rangle_{\widetilde{Q}} \otimes |0\rangle_a \qquad . \tag{18}$$

Here, $|0\rangle_Q$ denotes an eigenstate of $Q$ with the eigenvalue zero, $Q\,|0\rangle_Q = 0$. Similarly, $\widetilde{Q}\,|0\rangle_{\widetilde{Q}} = 0$, and $a_n\,|0\rangle_a = 0$ for all allowed values of $n$. By construction, $|\, \emptyset \,\rangle$ is the state with the lowest energy, $H_B\,|\, \emptyset \,\rangle = 0$.

We can excite the vacuum $|0\rangle_a$ in $\mathcal{S}_a$, by applying the creation operators in the standard way. *E.g.,* for an arbitrary integer value of $n$, we can form single particle states of the frequency $\omega_n$, by constructing the vectors $a_n^\dagger\,|0\rangle_a \equiv |a_n\rangle_a$ or

$a^\dagger_{-n}\,|0\rangle_a \equiv |a_{-n}\rangle_a$ . The situation is slightly different in $\mathcal{S}_Q$ and $\mathcal{S}_{\widetilde{Q}}$ subspaces. While in $\mathcal{S}_a$ the frequencies of oscillators are quantized, here we find no restrictions to the values of $Q$ and $\widetilde{Q}$. Consequently, both may assume arbitrary real eigenvalues from $-\infty$ to $+\infty$ . In order to create various non-zero eigenstates of $Q$ and $\widetilde{Q}$ , it is convenient to use the conjugate operators $P$ and $\widetilde{P}$ . *E.g.,* consider the state $|q\rangle_Q \equiv \exp(-iqP)\,|0\rangle_Q$ . It is easy to see that $Q\,|q\rangle_Q = q\,|q\rangle_Q$ , and thus, $|q\rangle_Q$ is indeed an eigenstate of $Q$ with the eigenvalue $q$ . The proof is left as an exercise for the reader. In a similar way, starting with $|0\rangle_{\widetilde{Q}}$ , and by making use of the conjugate operator $\widetilde{P}$ , we construct all the eigenvectors of $\widetilde{Q}$ .

In analogy with the procedure applied to hamiltonian, we also subtract the momentum of the vacuum from the momentum operator in (13), and write

$$K_B =\vdots \int_{-L/2}^{+L/2} dx(-\dot{\Phi}\Phi') \vdots \quad = \quad \frac{\pi}{L}\,Q\widetilde{Q} + \sum_{n\neq 0} k_n a^\dagger_n a_n, \tag{19}$$

where $k_n = 2\pi n/L$ is the momentum corresponding to the excitation $a^\dagger_n\,|0\rangle_a$ . The second term in (19) is the standard harmonic contribution. From the first term we find again that for any finite length $L$, the charge and current modify values of the operator.

In this section we defined the vacuum $|\,\emptyset\,\rangle$ as a tensor product of states with zero charge, zero mean current, and with no harmonic excitations. Having constructed the vacuum, we renormalized the hamiltonian and momentum operator by subtracting corresponding vacuum expectation values. Next, we turn our attention to fermions.

## 4. HOW SHOULD FERMIONS LOOK?

In $1+1$ dimension, the Dirac equation for massless fermions is

$$i(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x})\Psi(t,x) = 0 \qquad , \tag{20}$$

where in the "chiral" representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \Psi = \begin{pmatrix} \Psi_+(t,x) \\ \Psi_-(t,x) \end{pmatrix} \qquad . \tag{21}$$

For the components $\Psi_\pm$ , we find that

$$(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x})\Psi_\pm(t,x) = 0 \qquad . \tag{22}$$

Consequently, it becomes clear that $\Psi_+$ is a function of $t-x$ combination, and describes a propagation in the positive (right) direction along the $x$ axis, while the $\Psi_-$ is a function of $t+x$ , thus corresponding to propagation in the negative (left) direction.

We are interested in a particular set of solutions of eq. (22), which satisfy anti-periodic boundary conditions, $\Psi_\pm(\frac{L}{2},t) = -\Psi_\pm(-\frac{L}{2},t)$ . Namely, it turns out that for such anti-periodic solutions we can most easily accomplish the intended transformation of bosons into fermions. Given the boundary conditions, we can write the general solutions of eq. (22) as

$$\Psi_\pm(t,x) = \frac{1}{\sqrt{L}} \sum_{-\infty}^{+\infty} (D_\pm)_n \; e^{i\frac{2\pi}{L}(n+\frac{1}{2})(x\mp t)} \qquad , \tag{23}$$

where $(D_+)_n$ and $(D_-)_n$ are appropriately normalized constants defined by

$$(D_\pm)_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} d\eta \; e^{-i\frac{2\pi}{L}(n+\frac{1}{2})\eta} \; \Psi_\pm(0,\eta) \qquad . \tag{24}$$

We can now quantize the theory. The main difference from the procedure in the Klein-Gordon theory is our use here of anticommutators instead of commutators.

We require that

$$\{\Psi_r(t,x), \Psi_{r'}{}^{\dagger}(t,y)\} = \delta_{rr'}\, \widetilde{\Delta}_L(x,y) \qquad , \qquad (r,r'=\pm) \tag{25}$$

with all the other anticommutators vanishing. In (25), $\widetilde{\Delta}_L$ is an antisymmetric delta function which is described more thoroughly in the Appendix A. The difference between $\Delta_L$ and $\widetilde{\Delta}_L$ lies in the fact that $\Delta_L(x) \to \infty$ when $x \to \pm L$, while in the same limit $\widetilde{\Delta}_L(x) \to -\infty$. In the limit $L \to \infty$, both $\Delta_L$ and $\widetilde{\Delta}_L$ are replaced by the ordinary Dirac delta function $\delta(x)$. We point out that the antisymmetric fields $\Psi_\pm$ require an antisymmetric delta function in (25).

Upon quantization, the constants (24) turn into operators whose algebra is assigned by the anticommutators (25). It is convenient to replace operators in (24) by a set of new operators

$$\begin{aligned}
b_{n\geq 0} &= (D_+)_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx e^{-i\frac{2\pi}{L}(n+\frac{1}{2})x}\ \Psi_+(0,x) \\
b_{n<0} &= (D_-)_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx e^{-i\frac{2\pi}{L}(n+\frac{1}{2})x}\ \Psi_-(0,x) \\
d_{n\geq 0} &= (D_+)^{\dagger}_{-n-1} = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx e^{-i\frac{2\pi}{L}(n+\frac{1}{2})x}\ \Psi^{\dagger}_+(0,x) \\
d_{n<0} &= (D_-)^{\dagger}_{-n-1} = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx e^{-i\frac{2\pi}{L}(n+\frac{1}{2})x}\ \Psi^{\dagger}_-(0,x) \qquad .
\end{aligned} \tag{26}$$

The new operators have simple anticommutators,

$$\{b_n, b^{\dagger}_m\} = \{d_n, d^{\dagger}_m\} = \delta_{nm} \qquad , \tag{27}$$

as we verify by direct calculation. All the other anticommutators of operators in (26) vanish. We recognize the operators (26) as the annihilation operators for fermions. Likewise, the hermitian conjugates of (26) are fermionic creation operators. From (23) we obtain

$$\begin{aligned}
\Psi_+(t,x) &= \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} \left( b_n e^{-i\frac{2\pi}{L}(n+\frac{1}{2})(t-x)} + d_n^\dagger e^{i\frac{2\pi}{L}(n+\frac{1}{2})(t-x)} \right) \quad , \\
\Psi_-(t,x) &= \frac{1}{\sqrt{L}} \sum_{n=1}^{\infty} \left( b_{-n} e^{-i\frac{2\pi}{L}(n-\frac{1}{2})(t+x)} + d_{-n}^\dagger e^{i\frac{2\pi}{L}(n-\frac{1}{2})(t+x)} \right) \quad .
\end{aligned} \tag{28}$$

Consequently, for non-negative values of $n$, $b_n$ and $b_n^\dagger$ are annihilation and creation operators for the right-moving massless fermions with energy $\epsilon_n = 2\pi(n+\frac{1}{2})/L$. Similarly, $d_n$ and $d_n^\dagger$ correspond to the right-moving massless antifermions. For $n<0$, the corresponding operators describe left-moving fermions and left-moving antifermions. We are now in the position to define the fermion vacuum, $|\phi_0\rangle$. This is the state for which $b_n|\phi_0\rangle = 0$ and $d_n|\phi_0\rangle = 0$, for all values of $n$.

We can also introduce the hamiltonian and the momentum operator for the system of massless Dirac particles. They are readily constructed in parallel with the three-dimensional theory[7],

$$\begin{aligned}
H_F &=: \int_{-L/2}^{+L/2} dx[i\Psi^\dagger(t,x)\dot{\Psi}(t,x)]: \\
K_F &=: \int_{-L/2}^{+L/2} dx[-i\Psi^\dagger(t,x)\Psi'(t,x)]: \quad .
\end{aligned} \tag{29}$$

Columns in (29) denote the normal ordering with respect to the fermion vacuum $|\phi_0\rangle$. With the aid of (28), $H_F$ and $K_F$ may be rewritten as

$$\begin{aligned}
H_F &= \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} |n+\frac{1}{2}| \ :[b_n^\dagger b_n - d_n d_n^\dagger]: = \sum_{n=-\infty}^{+\infty} \epsilon_n (b_n^\dagger b_n + d_n^\dagger d_n) \quad , \\
K_F &= \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} (n+\frac{1}{2}) \ :[b_n^\dagger b_n - d_n d_n^\dagger]: = \sum_{n=-\infty}^{+\infty} \ell_n (b_n^\dagger b_n + d_n^\dagger d_n) \quad ,
\end{aligned} \tag{30}$$

where $\ell_n = \frac{2\pi}{L}(n+\frac{1}{2})$ is the momentum of the n-th excitation, and $\epsilon_n = |\ell_n|$ its energy.

The charge and axial charge of the fermions can also be defined in the standard way: $\mathbf{q} \; = :\int dx \Psi^\dagger \Psi :$ , and $\tilde{\mathbf{q}} \; = :\int dx \Psi^\dagger \gamma^0 \gamma^1 \Psi :$ . It turns out that the newly-defined operators satisfy

$$\begin{aligned} [\mathbf{q}\,, \Psi_\pm\,] &= -\Psi_\pm \qquad , \\ [\tilde{\mathbf{q}}\,, \Psi_\pm\,] &= \mp\Psi_\pm \qquad . \end{aligned} \tag{31}$$

Therefore, $\Psi_+$ operator decreases the charge and the axial charge by one unit. Similarly, $\Psi_-$ operator changes the charge for $-1$ unit, and axial charge for $+1$ unit.

In this section we analyzed general form of the solutions of Dirac equation for massless particles. We expressed the fields and the dynamic observables in terms of creation and annihilation operators. In the next section we demonstrate that there are many similarities between the components $\Psi_\pm(t,x)$ and the combinations of plasmon fields, $\exp[\pm i\sqrt{2\pi}\Phi_\pm(t,x)\,]$ .

## 5. FROM BOSONS TO FERMIONS

In Section 2, we constructed operators $Q$ and $\widetilde{Q}$ from the current-like structure $\partial^\mu\Phi$. We called them "charge" and "mean current", although the field $\Phi$ was real, and – consequently – the plasmons were chargeless. In this section we show that these global degrees of freedom in the expansion of the plasmon field really become the charge and mean current (or "axial charge") of the newly created fermions. Anticipating the result, and knowing that the charge and the mean current can assume only some discrete values, we restrict our analysis to a subspace of the total Hilbert space $\mathcal{S}_B$ .

The subspace, which we name $\mathcal{S}_F$ (F for fermions), consists of those states from $\mathcal{S}_B$ for which both $Q$ and $\widetilde{Q}$ have integer eigenvalues, and $Q - \widetilde{Q}$ is an even number. This condition may be rewritten as $Q - \widetilde{Q} \to 2n$ , $Q + \widetilde{Q} \to 2m$ , where $n$ and $m$ are arbitrary integers. This simply says that if fermion states are to have integer charges, then the resulting currents may differ from the charges only by an even number of units. We immediately observe that the vector $|\,\emptyset\,\rangle$ , which describes the plasmon vacuum, is one of the vectors in the subspace $\mathcal{S}_F$ .

Having defined the new Hilbert subspace $\mathcal{S}_F \subset \mathcal{S}_B$ , we made the first step in the construction of fermion fields from the boson counterparts. Next, we recall two useful operator relations,[8]

$$A\ e^B = e^B\,(A + [A,B]\,) \tag{32a}$$

$$e^A e^B = e^B e^A\ e^{[A,B]} = e^{A+B}\ e^{\frac{1}{2}[A,B]} \qquad , \tag{32b}$$

which are valid if the commutator $[A,B]$ is a number, and not another operator. With the aid of (32), it is straightforward to prove that

$$\left[Q, e^{\pm i\,\sqrt{2\pi}\Phi_\pm}\right] = -e^{\pm i\,\sqrt{2\pi}\Phi_\pm} \qquad , \tag{33}$$

$$\left[\widetilde{Q}, e^{\pm i\,\sqrt{2\pi}\Phi_\pm}\right] = \mp e^{\pm i\,\sqrt{2\pi}\Phi_\pm} \qquad , \tag{34}$$

and

$$i(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x})\begin{pmatrix} e^{+i\,\sqrt{2\pi}\Phi_+} \\ e^{-i\,\sqrt{2\pi}\Phi_-} \end{pmatrix} = 0 \qquad . \tag{35}$$

The new combinations, $\exp[\pm i\sqrt{2\pi}\Phi_\pm\,]$, transform eigenstates of $Q$ to eigenstates of $Q-1$ (relation (33) ), change the values of the mean current by $-1\,(+1)$ unit (relation (34) ), and satisfy the Dirac equation (35). Although these are exactly the properties we expect of the components $\Psi_\pm$ of fermion fields (compare to eqs. (31) and (20) ), the operators $\exp[\pm i\sqrt{2\pi}\Phi_\pm\,]$ are not yet the right combinations. Namely, they turn out to be incorrectly normalized. Instead, we introduce the operators

$$\chi_r(t,x) = \frac{1}{\sqrt{L}} e^{i\sqrt{2\pi}\,rR_r(t,x)}\ e^{i\sqrt{2\pi}\,r\varphi_r^{(-)}(t,x)}\ e^{i\sqrt{2\pi}\,r\varphi_r^{(+)}(t,x)} \qquad . \tag{36}$$

Here, $r$ denotes $+$ or $-$ signs, and $\varphi_r^{(-)}$ ( $\varphi_r^{(+)}$ ) are negative- (or positive-)

frequency components of the operators $\varphi_{\pm}(t,x)$ ( see eq. (12) and below ). The operators (36) differ only slightly from those used in expressions (33) to (35): the normalization constant $1/\sqrt{L}$ is added, and the exponent $i\sqrt{2\pi}r(R_r+\varphi_r^{(-)}+\varphi_r^{(+)})$ is factorized into three separate exponents. We observe that in the new arrangement (36) all creation operators $a_n^\dagger$ are to the left of annihilation operators $a_n$ . As a convenient shorthand for such an ordering , we may introduce the $\mathcal{N}_a$ symbol, and write $\chi_r(t,x) = \mathcal{N}_a\,(\,e^{i\,r\,\sqrt{2\pi}\,\Phi_r}\,)/\sqrt{L}$ . The changes introduced in (36), as we shall see, improve the normalization without affecting the Dirac equation and the commutators with $Q$ and $\widetilde{Q}$ in (33) and (34). Consequently, operators (36) become prime candidates for the description of fermions. It only remains to check the equal-time anticommutators of the operators $\chi_+$ and $\chi_-$ .

To do that, we first consider the product $\chi_r(t,x)\chi_s^\dagger(t,y)$ . With the aid of (32b), we write

$$\begin{aligned}\chi_r\chi_s^\dagger &= \frac{1}{L}e^{i\sqrt{2\pi}\,rR_r}\,e^{-i\sqrt{2\pi}\,sR_s}\,e^{i\sqrt{2\pi}\,r\varphi_r^{(-)}}\,e^{i\sqrt{2\pi}\,r\varphi_r^{(+)}}\,e^{-i\sqrt{2\pi}\,s\varphi_s^{(-)}}\,e^{-i\sqrt{2\pi}\,s\varphi_s^{(+)}}\\ &= \frac{1}{L}e^{i\sqrt{2\pi}(rR_r-sR_s)}\,e^{i\sqrt{2\pi}(r\varphi_r^{(-)}-s\varphi_s^{(-)})}\,e^{i\sqrt{2\pi}(r\varphi_r^{(+)}-s\varphi_s^{(+)})}\times\\ &\qquad e^{rs\pi[R_r,R_s]}\,e^{2rs\pi[\varphi_r^{(+)},\varphi_s^{(-)}]}\quad .\end{aligned} \tag{37}$$

In an analogous way we rewrite the product $\chi_s^\dagger(t,y)\chi_r(t,x)$ as

$$\begin{aligned}\chi_s^\dagger\chi_r &= \frac{1}{L}e^{i\sqrt{2\pi}(rR_r-sR_s)}\,e^{i\sqrt{2\pi}(r\varphi_r^{(-)}-s\varphi_s^{(-)})}\,e^{i\sqrt{2\pi}(r\varphi_r^{(+)}-s\varphi_s^{(+)})}\times\\ &\qquad e^{-rs\pi[R_r,R_s]}\,e^{2rs\pi[\varphi_s^{(+)},\varphi_r^{(-)}]}\quad ,\end{aligned} \tag{38}$$

By (8), the commutators in eqs. (37) and (38) are

$$
\begin{aligned}
[R_r(t,x), R_s(t,y)\,] &= \delta_{rs}\, r\, \frac{i}{L}(x-y) \qquad , \\
[\,\varphi_r^{(+)}(t,x)\,,\,\varphi_s^{(-)}(t,y)\,] &= \frac{1}{2\pi}\delta_{rs} \sum_{n>0} \frac{1}{n}\, e^{i\frac{2\pi n}{L} r(x-y)} \qquad , \\
[\,\varphi_s^{(+)}(t,y), \varphi_r^{(-)}(t,x)] &= \frac{1}{2\pi}\delta_{rs} \sum_{n>0} \frac{1}{n}\, e^{-i\,\frac{2\pi n}{L} r(x-y)} \qquad ,
\end{aligned}
\tag{39}
$$

and they all have zero value when the signs of $r$ and $s$ are not equal. Consequently, for $r \neq s$ , expressions (37) and (38) coincide, and we find the troublesome result that a *commutator* instead of an *anticommutator* vanishes,

$$
[\,\chi_r(t,x), \chi_s^\dagger(t,y)] = 0 \qquad\qquad (\text{for } r \neq s) \qquad . \tag{40}
$$

The situation looks less unpleasant for $r = s$ , because we shall be able to combine (37) and (38) into an anticommutator (see below). However, the right hand side in the resulting expression gets much more complicated than we might have expected. Indeed, with the aid of (A.8), and by using (37) to (39), we can write for $r = s$,

$$
\begin{aligned}
&\{\chi_r(t,x),\ \chi_r^\dagger(t,y)\} = \\
&\quad e^{i\sqrt{2\pi}\, r[R_r(t,x)-R_r(t,y)]}\ \mathcal{N}_a\left(e^{i\sqrt{2\pi}\, r[\varphi_r(t,x)-\varphi_r(t,y)]}\right)\ \widetilde{\Delta}_L(x-y) \quad .
\end{aligned}
\tag{41}
$$

The alarm caused by this expression is lessened when we observe that the term $\mathcal{N}_a(\ldots)$ in fact reduces to one, due to the presence of $\widetilde{\Delta}_L(x-y)$ function. We verify this in a direct calculation, by noticing that within the range of interest $\widetilde{\Delta}_L(x-y)$ contains $\delta(x-y-L), \delta(x-y)$ and $\delta(x-y+L)$ terms (see Appendix A). On the other hand, the functions $\varphi_r(t,x)$ are periodic, and particularly, $\varphi_r(t,x\pm L)-\varphi_r(t,x) = 0$. Therefore, indeed $\mathcal{N}_a(\ldots) \;\to\; \mathbf{1}$ . There is another exponential factor on the right hand side of (41) which we would like to see eliminated. Now it becomes crucial that the analysis is carried out in the $\mathcal{S}_F$ subspace. Namely, by (12), $i\sqrt{2\pi}\, r\ [R_r(t,x) - R_r(t,x\pm L)\,] = \mp i\pi(\widetilde{Q} + r\ Q)$ . But, since $\widetilde{Q}+Q$ and $\widetilde{Q}-Q$

combinations have even eigenvalues in the subspace, the remaining factor in (41) also takes the value of one. The anticommutator (41) is now simply

$$\{\chi_r, \chi_r^\dagger\} = \Big( \delta(x-y) - e^{i\pi(\widetilde{Q}+rQ)}\delta(x-y-L) - e^{-i\pi(\widetilde{Q}+rQ)}\delta(x-y+L) + \ \ldots\Big)$$
$$\longrightarrow \widetilde{\Delta}_L(x-y) \qquad . \tag{42}$$

Therefore, for $r = s$, in contrast with the $r \neq s$ case (eq. (40) ), the anticommutator of $\chi_r$ and $\chi_s^\dagger$ operators looks fine, and we can claim at least a partial success.

We turn next to the product $\chi_r(t,x)\chi_s(t,y)$. There is a complete parallel here with the previous discussion, and we readily find that for $r \neq s$ ,

$$[\chi_r(t,x), \chi_s(t,y)] = 0 \qquad , \tag{43}$$

and for $r = s$ ,

$$\{\chi_r(t,x), \chi_r(t,y)\} = 0 \qquad . \tag{44}$$

Again, a commutator instead of an anticommutator appears for $r \neq s$ , while for $r = s$ , the expression has the form corresponding exactly to the Fermi statistics.

Our attempt to determine the equal-time anticommutators of $\chi_r$ operators apparently met some serious difficulties. In expressions (40) and (43), a wrong sign appeared between combinations of operators. However, the solution of this particular problem turns out to be very simple. To accomplish the sign change in (40) and (43), and at the same time preserve the relations (42) and (44), we only have to multiply $\chi_r$ by a suitably chosen "Klein's factor". The procedure is thoroughly explained in the Appendix B. According to the prescription, we change $\chi_r(t,x) \ \rightarrow \exp[i\pi Q(1+r)/2]\,\chi_r(t,x) \ = \ \Psi_r(t,x)$, and finally obtain a set of fields obeying the correct anticommutation rules for all values of $r$ and $s$ ,

$$\begin{aligned} \{\Psi_r(t,x), \Psi_s^\dagger(t,y)\} &= \delta_{rs}\widetilde{\Delta}_L(x-y) \quad , \\ \{\Psi_r(t,x), \Psi_s(t,y)\} &= 0 \qquad . \end{aligned} \tag{45}$$

The newly created operators $\Psi_\pm(t,x)$ therefore have all the properties required of the fermion fields. They satisfy the Dirac equation for components, change charge and current for one unit, and yet, are entirely expressed in terms of boson operators. The "miraculous" transformation of bosons into fermions is thus achieved: fermions become a kind of collective excitation modes of bosons.

In concluding this discussion of the transformation, we present the expression for the Fermi field with the main parts factorized in two different ways. The transition from one form to the other is readily achieved with the aid of (32).

$$\Psi(t,x) = \begin{pmatrix} \Psi_+(t,x) \\ \Psi_-(t,x) \end{pmatrix} = \frac{1}{\sqrt{L}} \begin{pmatrix} e^{i\pi Q}\, e^{+i\sqrt{2\pi}R_+}\, e^{+i\sqrt{2\pi}\varphi_+^{(-)}}\, e^{+i\sqrt{2\pi}\varphi_+^{(+)}} \\ e^{-i\sqrt{2\pi}R_-}\, e^{-i\sqrt{2\pi}\varphi_-^{(-)}}\, e^{-i\sqrt{2\pi}\varphi_-^{(+)}} \end{pmatrix} \tag{46}$$

$$= \frac{1}{\sqrt{L}} \begin{pmatrix} e^{i\pi Q}\, e^{i(P+\widetilde{P})}\, e^{-i\frac{\pi}{L}(Q+\widetilde{Q}-1)(t-x)}\, e^{+i\sqrt{2\pi}\varphi_+^{(-)}}\, e^{+i\sqrt{2\pi}\varphi_+^{(+)}} \\ e^{i(P-\widetilde{P})}\, e^{-i\frac{\pi}{L}(Q-\widetilde{Q}-1)(t+x)}\, e^{-i\sqrt{2\pi}\varphi_-^{(-)}}\, e^{-i\sqrt{2\pi}\varphi_-^{(+)}} \end{pmatrix} . \tag{47}$$

We shall use both of these forms in the following section.

## 6. SINGLE – PARTICLE STATES FOR FERMIONS

In the preceding section we constructed fermion field operators from the boson field operators. In this section we discuss vectors of states, and construct single-particle fermion states in terms of plasmon states. We first rewrite the annihilation operators $b$ and $d$. Following from (26) and (47),

$$b_{n\geq 0} = \frac{1}{L} \int\limits_{-L/2}^{+L/2} dx\ e^{-i\frac{2\pi}{L}(n+1)x} e^{i\pi Q} e^{i(P+\widetilde{P})} e^{i\frac{\pi}{L}(Q+\widetilde{Q})x} e^{i\sqrt{2\pi}\,\varphi_+^{(-)}} e^{i\sqrt{2\pi}\,\varphi_+^{(+)}}$$

$$b_{n<0} = \frac{1}{L} \int\limits_{-L/2}^{+L/2} dx\ e^{-i\frac{2\pi}{L}nx} e^{i(P-\widetilde{P})} e^{-i\frac{\pi}{L}(Q-\widetilde{Q})x} e^{-i\sqrt{2\pi}\,\varphi_-^{(-)}} e^{-i\sqrt{2\pi}\,\varphi_-^{(+)}}$$

$$d_{n\geq 0} = \frac{1}{L} \int\limits_{-L/2}^{+L/2} dx\ e^{-i\frac{2\pi}{L}nx} e^{-i\frac{\pi}{L}(Q+\widetilde{Q})x} e^{-i(P+\widetilde{P})} e^{-i\pi Q} e^{-i\sqrt{2\pi}\,\varphi_+^{(-)}} e^{-i\sqrt{2\pi}\,\varphi_+^{(+)}} \tag{48}$$

$$d_{n<0} = \frac{1}{L} \int\limits_{-L/2}^{+L/2} dx\ e^{-i\frac{2\pi}{L}(n+1)x} e^{i\frac{\pi}{L}(Q-\widetilde{Q})x} e^{-i(P-\widetilde{P})} e^{i\sqrt{2\pi}\,\varphi_-^{(-)}} e^{i\sqrt{2\pi}\,\varphi_-^{(+)}}$$

Here, $\varphi_\pm$ is a shorthand for operators $\varphi_\pm(t=0,x)$. The fermion vacuum state $|\phi_0\rangle$ should contain no fermions, and our first task is to find the state for which $b_n\,|\phi_0\rangle = d_n\,|\phi_0\rangle = 0$. Such a state indeed exists in the subspace $\mathcal{S}_F$, and – perhaps not surprisingly – turns out to be exactly the boson vacuum $|\,\emptyset\,\rangle$. In other words, the state with no fermions coincides with the chargeless, currentless state with no bosons, $|\phi_0\rangle = |\,\emptyset\,\rangle$. We shall demonstrate this in the next paragraph, by proving that $b_{n\geq 0}\,|\,\emptyset\,\rangle = 0$. In an analogous way it is possible to establish the similar relations for all $b_n$ and $d_n$ operators.

Let us apply $b_{n\geq 0}$ to the boson vacuum. The operator most to the right in $b_{n\geq 0}$ is $\exp[i\sqrt{2\pi}\varphi_+^{(+)}\,]$, and we first observe that $\varphi_+^{(+)}$ contains only the annihilation operators $a_n$. Therefore,

$$\exp[i\sqrt{2\pi}\,\varphi_+^{(+)}\,]\,|\,\emptyset\,\rangle = [\mathbf{1} + i\sqrt{2\pi}\varphi_+^{(+)} - \pi(\varphi_+^{(+)})^2 + \ldots]\,|\,\emptyset\,\rangle = |\,\emptyset\,\rangle\ , \tag{49}$$

because only the first term in the formal expansion is non-vanishing. Furthermore, $\exp[i\pi(Q+\widetilde{Q})x/L]\ |0\rangle_Q \otimes |0\rangle_{\widetilde{Q}} = |0\rangle_Q \otimes |0\rangle_{\widetilde{Q}}$ (recall that $Q\,|0\rangle_Q = \widetilde{Q}\,|0\rangle_{\widetilde{Q}} = 0$), and $\exp[i\pi Q]\ \exp[i(P+\widetilde{P})]\ |0\rangle_Q \otimes |0\rangle_{\widetilde{Q}} = \exp[i\pi Q]\,|-1\rangle_Q \otimes |-1\rangle_{\widetilde{Q}} = -\ |-1\rangle_Q \otimes$

$|-1\rangle_{\tilde{Q}}$ (recall that *e.g.,* $\exp[-iPq]\ |0\rangle_Q = |q\rangle_Q$ ). Consequently, when $b_{n\geq 0}$ is applied to the vacuum state $|\,\emptyset\,\rangle$ , only the factor which have $\varphi_+^{(-)}$ operator in the exponent can survive. We find

$$b_{n\geq 0}\ |\,\emptyset\,\rangle\ =-\ |-1\rangle_Q \otimes |-1\rangle_{\tilde{Q}} \otimes \frac{1}{L}\int_{-L/2}^{+L/2} dx\ e^{-i\frac{2\pi}{L}(n+1)x} e^{i\sqrt{2\pi}\varphi_+^{(-)}}\ |0\rangle_a \qquad . \tag{50}$$

At first we might think that the expansion for $\exp[i\sqrt{2\pi}\,\varphi_+^{(-)}]$ does create at least some non-zero states, but in fact this does not happen. Namely, none of the terms in the expansion (compare to the similar expression in eq. (49) ) matches correctly the factor $\exp[-i\,2\pi(n+1)x/L]$ in the integrand. (Recall that for an integer $p$ , the integral $\int \exp[-i\,2\pi(n+1)x/L]\ \exp[-i\,2\pi px/L]$ is different from zero only for $p=-n-1<0$ , but that never happens if $n\geq 0$ ). Therefore, from (50) it follows that $b_{n\geq 0}\,|\,\emptyset\,\rangle = 0$ . In a very similar way we can treat the other operators in (48), and conclude that for all of them, $b_n\,|\,\emptyset\,\rangle = 0$ , $d_n\,|\,\emptyset\,\rangle = 0$ . Consequently, $|\,\emptyset\,\rangle$ and $|\phi_0\rangle$ are really identical, and the state with the lowest plasmon energy at the same time is the state with no fermions or antifermions. Henceforth, we use only one symbol, $|\phi_0\rangle$ , for the vacuum state. In the same spirit, the two normal orderings coincide, $:A: = \vdots A \vdots$ , and from now on we use only the $:A:$ notation.

Turning next to the operators which determine the dynamics of systems, we demonstrate that the fermion hamiltonian and the momentum operators (29) are equivalent to the plasmon hamiltonian (17) and momentum (19). We first rewrite the original expressions (29) as

$$\begin{aligned} H_F &= \lim_{\epsilon\to 0} : \int_{-L/2}^{+L/2} dx\ [\ i\ \Psi^\dagger(t, y=x+\epsilon)\dot{\Psi}(t,x)\ ] : \\ K_F &= \lim_{\epsilon\to 0} : \int_{-L/2}^{+L/2} dx\ [\ -i\ \Psi^\dagger(t, y=x+\epsilon)\Psi'(t,x)\ ] : \qquad . \end{aligned} \tag{51}$$

There is a good reason for the introduction of the point splitting in eq. (51).

The product of two fields given by (46) or (47) is highly divergent if both fields are evaluated at the same point $(t,x)$, and we must define a limiting procedure in order to handle the resulting divergences and make sense of the product. According to the procedure, the limit should be taken only *after* the integration.

In (51) we need the space and time derivatives of the components $\Psi_\pm$ . From (47), we find

$$\begin{aligned}\dot{\Psi}_\pm(t,x) = \frac{1}{\sqrt{L}}\, & e^{i\frac{\pi}{2}(1\pm 1)Q}\, e^{i(P\pm\widetilde{P})}\, e^{-i\frac{\pi}{L}(Q\pm\widetilde{Q}-1)(t-x)} \\ & \{\, e^{\pm i\sqrt{2\pi}\varphi_\pm^{(-)}}\, e^{\pm i\sqrt{2\pi}\varphi_\pm^{(+)}}\, [\,\pm i\sqrt{2\pi}\,\dot{\varphi}_\pm^{(+)}\,] \\ & \quad + [-i\frac{\pi}{L}(Q\pm\widetilde{Q}-1) \pm i\sqrt{2\pi}\,\dot{\varphi}_\pm^{(-)}\,]\, e^{\pm i\sqrt{2\pi}\varphi_\pm^{(-)}}\, e^{\pm i\sqrt{2\pi}\varphi_\pm^{(+)}}\ \} \ . \end{aligned} \tag{52}$$

(The space derivatives are given by $\Psi_\pm{}' = \mp\dot{\Psi}_\pm$ ). In order to find the hamiltonian, we multiply eq. (52) by $i\Psi_\pm^\dagger(t,y)$, and with the aid of (32) bring all $\varphi_r^{(-)}$ operators to the left of all $\varphi_r^{(+)}$ operators. After some rearrangement, we obtain

$$\begin{aligned} i\,\Psi_\pm^\dagger(t,y)\dot{\Psi}_\pm(t,x) = \mp\frac{1}{2L}[\,\sin\frac{\pi(y-x)}{L}]^{-1}\ & e^{\mp i\frac{\pi}{L}(Q\pm\widetilde{Q})(y-x)} \\ & \{\ T_\pm\,[-i\frac{\pi}{L}(Q\pm\widetilde{Q}) \pm i\sqrt{2\pi}\,\dot{\varphi}_\pm^{(+)}(t,x)\ \mp\frac{\pi}{L}\cot\frac{\pi(y-x)}{L}\,] \\ & \quad + [\pm i\sqrt{2\pi}\,\dot{\varphi}_\pm^{(-)}(t,x)\,]\,T_\pm\ \}\ , \end{aligned} \tag{53}$$

where

$$T_\pm = e^{\mp i\sqrt{2\pi}[\,\varphi_\pm^{(-)}(t,y)-\varphi_\pm^{(-)}(t,x)\,]}\ e^{\mp i\sqrt{2\pi}[\,\varphi_\pm^{(+)}(t,y)-\varphi_\pm^{(+)}(t,x)\,]} \qquad . \tag{54}$$

Now it is not difficult to make an expansion to the second order in the "small" quantity $\epsilon = y - x$ , and we readily show that

$$i\Psi_\pm^\dagger(t,x+\epsilon)\dot{\Psi}_\pm(t,x) = \frac{1}{2\pi\epsilon^2} + \frac{1}{2}\Big[:(\,\Phi_\pm{}'(t,x)\,)^2: \mp\frac{i}{\sqrt{2\pi}}\varphi_\pm{}''(t,x) - \frac{\pi}{6L^2}\Big] + \mathcal{O}(\epsilon). \tag{55}$$

Recall that the notation $:A:$ means $A - \langle\phi_0|\,A\,|\phi_0\rangle$ , and therefore

$$:(\Phi_\pm{}')^2: = \frac{\pi}{2L^2}(Q\pm\widetilde{Q})^2 + \frac{\sqrt{2\pi}}{L}(Q\pm\widetilde{Q})\varphi_\pm{}' + (\varphi_\pm^{(+)'})^2 + 2\varphi_\pm^{(-)'}\varphi_\pm^{(+)'} + (\varphi_\pm^{(-)'})^2 . \tag{56}$$

The first, divergent term in (55), as well as the constant in the brackets, drop out when the vacuum expectation value is subtracted. The integral of the remaining $\epsilon$-independent term can be evaluated, and since $\int dx\ \varphi_\pm{}''(t,x) = 0$ , we finally obtain

$$\lim_{\epsilon\to 0} : \int_{-L/2}^{+L/2} dx\, i\,(\,\Psi_+^\dagger\dot{\Psi}_+ + \Psi_-^\dagger\dot{\Psi}_-\,):$$
$$= : \int_{-L/2}^{+L/2} dx\,\frac{1}{2}[\,(\Phi_+')^2 + (\Phi_-')^2\,]: \;=\; : \int_{-L/2}^{+L/2} dx\,\frac{1}{2}[\,(\dot{\Phi})^2 + (\Phi')^2\,]: \quad . \tag{57}$$

This proves that hamiltonians of free massless bosons and free massless fermions are equivalent, $H_F = H_B$ . In an analogous way we can verify that the momentum operators are equivalent, $K_F = K_B$ .

As anticipated before, the operators $Q$ and $\widetilde{Q}$ , which were generated as some local degrees of freedom in the plasmon theory, become true charge and mean current (or "axial-current") in the equivalent theory of fermions. We can prove that

$$\begin{aligned} \lim_{\epsilon\to 0} : \int_{-L/2}^{+L/2} dx\ \Psi^\dagger(t,x+\epsilon)\Psi(t,x): &= Q \quad , \\ \lim_{\epsilon\to 0} : \int_{-L/2}^{+L/2} dx\ \Psi^\dagger(t,x+\epsilon)\gamma^0\gamma^1\Psi(t,x): &= \widetilde{Q} \quad . \end{aligned} \tag{58}$$

Again, the split point limit must be used to handle the divergences. The proof of (58) with the use of the method applied in (51) to (56) is straightforward, and it is left to the reader as an exercise.

As discussed above, our fermions and bosons have the common vacuum state, and their dynamics is determined by the same operators. But ultimately, there must be a difference between bosons and fermions. Indeed, the equivalence goes up to the point where single-particle states are constructed. As an illustration, we consider the states generated by the creation operator $b^\dagger_{n\geq 0}$ . From (48),

$$b^\dagger_{n\geq 0} = \frac{1}{L}\int\limits_{-L/2}^{+L/2} dx\; e^{i\frac{2\pi}{L}(n+1)x}\; e^{-i\frac{\pi}{L}(Q+\widetilde{Q})x}\; e^{-i(P+\widetilde{P})} e^{-i\pi Q}\; e^{-i\sqrt{2\pi}\varphi_+^{(-)}} e^{-i\sqrt{2\pi}\varphi_+^{(+)}} \quad . \tag{59}$$

We want to construct a state with the momentum $\ell_n = (2n+1)\pi/L$ and energy $\epsilon_n = |\ell_n|$ , which describes a fermion moving in positive direction. With the aid of formalism used in eq. (49) and below, we write

$$|1(\epsilon_n, \ell_n, n\geq 0)\rangle = b^\dagger_{n\geq 0}\,|\phi_0\rangle = |1\rangle_Q \otimes |1\rangle_{\widetilde{Q}} \otimes \frac{1}{L}\int\limits_{-L/2}^{+L/2} dx\; e^{i\frac{2\pi}{L}nx} e^{-i\sqrt{2\pi}\varphi_+^{(-)}(0,x)}\; |0\rangle_a \;. \tag{60}$$

However,

$$e^{-i\sqrt{2\pi}\varphi_+^{(-)}(0,x)} = \mathbf{1} - \sum_{m>0}\frac{i}{\sqrt{m}} e^{-i\frac{2\pi}{L}mx}\, a^\dagger_m - \frac{1}{2}\sum_{m,s>0}\frac{1}{\sqrt{ms}} e^{-i\frac{2\pi}{L}(m+s)x}\, a^\dagger_m a^\dagger_s + \ldots \quad , \tag{61}$$

and (60) becomes

$$\begin{aligned}|1(\epsilon_n, \ell_n, n\geq 0)\,\rangle = \;& |1\rangle_Q \otimes |1\rangle_{\widetilde{Q}} \otimes \;\{\; \delta_{n,0}\,|0\rangle_a - \sum_{m>0}\frac{i}{\sqrt{m}}\delta_{n,m}\;|a_m\rangle_a \\ & -\frac{1}{2}\sum_{m,s>0}\frac{1}{\sqrt{ms}}\delta_{n,m+s}\;|a_m; a_s\rangle_a + \ldots\;\} \quad .\end{aligned} \tag{62}$$

From (62), the few lowest lying states (*e.g.,* for $n = 0, 1$, and 2) are

$$\begin{aligned}
|1\,(\frac{\pi}{L},\frac{\pi}{L},n=0)\,\rangle \;&=\; |1\rangle_Q \otimes |1\rangle_{\widetilde{Q}} \otimes |0\rangle_a \\
|1\,(\frac{3\pi}{L},\frac{3\pi}{L},n=1)\,\rangle \;&=-i\;\, |1\rangle_Q \otimes |1\rangle_{\widetilde{Q}} \otimes |a_1\rangle_a \\
|1\,(\frac{5\pi}{L},\frac{5\pi}{L},n=2)\,\rangle \;&=\; |1\rangle_Q \otimes |1\rangle_{\widetilde{Q}} \otimes \left[\; |2a_1\rangle_a + i\sqrt{2}\,|a_2\rangle_a \right] \quad .
\end{aligned} \tag{63}$$

The states (63) are genuine fermion states in the sense that there can be only one fermion per state. Addition of another identical fermion is not possible. *E.g.,* we can easily demonstrate that

$$b^\dagger_{n=1}\;\; |1\,(\frac{3\pi}{L},\frac{3\pi}{L},n=1)\,\rangle \;=\; 0 \qquad , \tag{64}$$

and therefore a state with two right-moving, $n = 1$ fermions, does not exist. Consequently, the Pauli principle is valid even for our composed fermions.

## 7. SUMMARY AND CONCLUSIONS

The one-dimensional space has a remarkable property: fermion systems can be completely described in terms of canonical one-dimensional boson fields. We illustrated this equivalence in the most simple example, by playing with the free, massless objects. While this example has mainly an academic value, it should be pointed out again that the same technique and methods could be extended to the more interesting cases of massive and interacting theories. This paper emphasizes the unusual fact that fermions can be constructed from bosons. In reality, most often we use the presented techniques in the other direction: to make a transition from relatively complicated fermion systems to much simpler and better understood boson models. For example, in a complete parallel with the earlier discussion, we can verify the equivalence between the Schwinger model (which is quantum electrodynamics of massless fermions in $1+1$ dimension), and the theory of massive, but free bosons. The Schwinger model with massive fermions is, on the other hand, equivalent to the massive Sine-Gordon theory of bosons, *etc.* The one-dimensional models which were originally used as theoretical laboratories to explore some features of quantum field theory, more recently were directly applied to real condensed matter systems. The best known example is the polyacetylene (see *e.g.,* Ref. 9 ). The material, which is an organic polymer, consists of parallel chains of $(CH)$ groups. Electrons are moving primarily along the chains, while hopping between chains is strongly suppressed. The system is therefore effectively one-dimensional. Some nonlinear topological excitations ('solitons'), which appear in all such quasi one-dimensional polymers, interact with fermions giving rise to all kind of remarkable electrical and optical properties of polyacetylene: fractionization of charge[10], semiconducting and metallic attributes ('synthetic metals') when suitably doped with donor or acceptor species[11], *etc.*

It is not known with certainty whether the equivalence between fermions and bosons will hold in higher dimensions, or whether it is only a peculiarity related to the topology of the one-dimensional space. The one-dimensional results can not be simply generalized to the three-dimensional space, because the spin degrees of freedom further complicate the picture. Still, the initial 'pure' field theoretical studies[12], and the analyses of the Skyrme effective model[13,14] seem to strongly support the equivalence. Speculations of any kind are usually risky, but if one day the fermion-boson duality becomes proven even in the $(3+1)$ dimension, perhaps we shall have to revise the standard classification of matter. Fermions and bosons might turn out to be just different faces of some even more fundamental entity.

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# APPENDIX A

## DELTA FUNCTIONS IN FINITE INTERVAL

In the one-dimensional space divided into cells of length $L$, it is useful to define the following combinations of $\delta$ functions:

$$\Delta_L(x) = \sum_{n=-\infty}^{\infty} \delta(x - nL) = \ \ldots + \delta(x-L) + \delta(x) + \delta(x+L) + \ \ldots \qquad , \quad (A.1)$$

$$\widetilde{\Delta}_L(x) = \sum_{n=-\infty}^{\infty} (-)^n \, \delta(x - nL) = \ \ldots - \delta(x-L) + \delta(x) - \delta(x+L) + \ \ldots \qquad , \quad (A.2)$$

Here, $\delta(x)$ denotes the usual Dirac delta function. The first combination, $\Delta_L(x)$, is a periodic function, $\Delta_L(x+L) = \Delta_L(x)$ , while the second one, $\widetilde{\Delta}_L(x)$ is an antiperiodic function, $\widetilde{\Delta}_L(x+L) = -\widetilde{\Delta}_L(x)$. However, both have the same limit when $L \to \infty$ : $\Delta_L(x)$ , $\widetilde{\Delta}_L(x) \to \delta(x)$.

For any $a$ inside the interval $[-L/2, +L/2]$, we can write

$$\begin{aligned} \int_{-L/2}^{+L/2} dx \; \Delta_L(x-a) = \int_{-L/2}^{+L/2} dx \; \widetilde{\Delta}_L(x-a) &= 1 \qquad , \\ \int_{-L/2}^{+L/2} dx \; \Delta_L(x-a) f(x) = \int_{-L/2}^{+L/2} dx \; \widetilde{\Delta}_L(x-a) f(x) &= f(a) \ . \end{aligned} \qquad (A.3)$$

These and other similar properties follow from the presence of the Dirac $\delta$ functions in (A.1) and (A.2). We point out that care should be taken at the boundaries, when $a$ approaches $\pm L/2$ . Then the values of integrals in (A.3) are different. For example,

$$\int_{-L/2}^{+L/2} dx\ \widetilde{\Delta}_L(x \mp \frac{L}{2}) = 0\ , \quad \int_{-L/2}^{+L/2} dx\ \Delta_L(x \mp \frac{L}{2}) f(x) = \frac{1}{2}\left[f(-\frac{L}{2}) + f(\frac{L}{2})\right]\ ,$$

$$\int_{-L/2}^{+L/2} dx\ \widetilde{\Delta}_L(x \mp \frac{L}{2}) f(x) = \mp\frac{1}{2}\left[f(-\frac{L}{2}) - f(\frac{L}{2})\right] \quad . \tag{A.4}$$

$\Delta_L(x)$ is a periodic function of period $L$, and we can expand it in a Fourier series. The coefficients turn out to be all equal, having the value $1/L$,

$$\Delta_L(x) = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{i\frac{2\pi}{L}mx} \quad . \tag{A.5}$$

Similarly, $\widetilde{\Delta}_L(x)$, although antiperiodic in the interval $L$, is a periodic function of period $2L$, and can also be expanded,

$$\widetilde{\Delta}_L(x) = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{i\frac{2\pi}{L}(m+\frac{1}{2})x} \quad . \tag{A.6}$$

Another useful form for $\widetilde{\Delta}_L(x)$ is based on the identity

$$\sum_{m=0}^{\infty} e^{im\alpha} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} e^{in\alpha}\right) \quad . \tag{A.7}$$

To verify (A.7) up to a constant, we first differentiate both sides. Then, by choosing a special value for $\alpha$, we show that the constant is zero. Consequently, the relation (A.7) is valid, and we can rewrite the expression (A.6) as

$$\widetilde{\Delta}_L(x) = \frac{1}{L} e^{\left( i\frac{\pi}{L}x + \sum_{n=1}^{\infty} \frac{1}{n} \exp i\frac{2\pi}{L}nx \right)} \ + \ \frac{1}{L} e^{\left( -i\frac{\pi}{L}x + \sum_{n=1}^{\infty} \frac{1}{n} \exp -i\frac{2\pi}{L}nx \right)} \ . \tag{A.8}$$

Note that the sum over negative values of $m$ in (A.6) was transformed into the sum over positive $m$ by changing $m \to -m-1$.

The Dirac delta function $\delta$ is a derivative of the (Heaviside's) step-function $\theta$. Likewise, we can define the step-function $\Theta_L(x)$, relevant for a finite interval $L$,

$$\begin{aligned}
\Theta_L(x) &= \frac{1}{2} + \frac{x}{L} - \frac{i}{2\pi} \sum_{n\neq 0} \frac{1}{n} \exp i \frac{2\pi}{L} nx \qquad , \\
\lim_{L\to\infty} \Theta_L(x) &= \theta(x) \qquad , \qquad {\Theta_L}'(x) = \Delta_L(x) \qquad .
\end{aligned} \tag{A.9}$$

The distribution $\Theta_L(x)$ is particularly useful in expressing the equal time commutators $[\Phi_\pm(t,x), \Phi_\pm(t,y)]$ between the components of $\Phi(t,x)$.

## APPENDIX B
## KLEIN'S FACTOR

From Section 5, we know that

$$\begin{aligned}
\chi_r\chi_{r'} - \chi_{r'}\chi_r &= 0 \qquad , \qquad \chi_r\chi^\dagger_{r'} - \chi^\dagger_{r'}\chi_r = 0 \qquad (r \neq r') \\
\chi_r\chi_{r'} + \chi_{r'}\chi_r &= 0 \qquad , \qquad \chi_r\chi^\dagger_{r'} + \chi^\dagger_{r'}\chi_r \sim \widetilde{\Delta}_L \qquad (r = r') \qquad .
\end{aligned} \tag{B.1}$$

We would like to change $\chi_r$ in such a way that the commutators for $r \neq r'$ become anticommutators. However, the results for $r = r'$, the equations of motion for the components $\chi_r$, and the relations (33) to (34) should not be affected by this change. One way to achieve such a transformation is to multiply $\chi_r$ by unitary, $r$-dependent operators $F_r$. We immediately observe that $F_r$ not only must be $(t,x)$ independent, but also should commute with all space-time dependent pieces in $\chi_r$. If this were not true, an additional $(t,x)$ dependence would have been introduced into the product $F_r\chi_r$, thus spoiling the Dirac equation. It is convenient to assume that $F_r$ have the form

$$F_r = e^{\frac{i\pi}{2}[(\alpha+\beta r)Q + (\sigma+\rho r)\widetilde{Q}]} \qquad , \tag{B.2}$$

and then determine the real constants $\alpha, \beta, \sigma$ and $\rho$, by requiring that (compare to (45) and (33)-(35) )

$$\{(F_r\chi_r),(F_{r'}\chi_{r'})\}_+ = 0 \qquad , \qquad \{(F_r\chi_r),(F_{r'}\chi_{r'})^\dagger\}_+ = \delta_{rr'}\,\widetilde{\Delta}_L \quad , \tag{B.3}$$

$$[(F_r\chi_r),Q]_- = (F_r\chi_r) \qquad , \qquad [(F_r\chi_r),\widetilde{Q}]_- = r\,(F_r\chi_r) \qquad , \tag{B.4}$$

and

$$i(\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x})\begin{pmatrix}F_+\chi_+\\ F_-\chi_-\end{pmatrix} = 0 \qquad . \tag{B.5}$$

With the form (B.2), relations (B.4) and (B.5) are automatically satisfied for any values of the parameters, as we easily verify by direct calculation. On the other hand, (B.3) can be satisfied only if

$$\sigma - \beta = \text{odd} \quad \text{integer} \qquad . \tag{B.6}$$

The proof that the condition (B.6) follows from (B.3) is straightforward with the use of (B.1) and (36), and it is left as an exercise for the reader. Since (B.6) is the only condition on the parameters in $F_r$ , the choice of the Klein's factor is not unique. Following Ref. 1, in this paper we chose $\alpha = \beta = 1$ , $\sigma = \rho = 0$ . This gives

$$F_r = \exp\frac{i\pi}{2}(1+r)Q \qquad . \tag{B.7}$$

However, any other choice (provided that (B.6) is satisfied), would be equally acceptable. Different Klein's factors would merely change the phases of spinors, without affecting physical quantities.

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