5377
February 1992
T

$D = 0$ Matrix Model as Conjugate Field Theory*

Shahar Ben-Menahem

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

The $D = 0$ matrix model is reformulated as a nonlocal quantum field theory in two dimensions, in which the interactions occur on the one-dimensional line of hermitian matrix eigenvalues. The field can be thought of as a fluctuation in the potential V, and is conjugate to the density of matrix eigenvalues which appears in the Jevicki collective field theory. The classical solution of the field equation is either unique or labeled by a discrete index. Such a solution corresponds to the Dyson sea modified by an entropy term. The modification smoothes the sea edges, and interpolates between different eigenvalue bands for multiple-well potentials. Our classical eigenvalue density contains nonplanar effects, and satisfies a local nonlinear Schrödinger equation with similarities to the Marinari-Parisi $D = 1$ reformulation. The quantum fluctuations about a classical solution are computable, and the IR and UV divergences are manifestly removed to all orders. The quantum corrections greatly simplify in the double scaling limit, and include both string-perturbative and nonperturbative effects. The latter are unambiguous for V bounded from below, and can be compared with the various nonperturbative definitions of these theories proposed in the literature.

Submitted to *Nuclear Physics* **B**

⋆ Work supported by the Department of Energy, contract DE-AC03-76SF00515.

1. Introduction.

The study of discretized two-dimensional quantum gravity (matrix models) has received much attention in recent years, thanks to the uncovering of the solvability of a range of such models in the double scaling limit [1−8]. A matrix model describes the statistical mechanics of random surfaces in the large-N limit, where N is the dimension of the matrix (we restrict our attention to hermitian matrix models). The procedure for solving these models consists of three conceptual steps. In the first step, integrating out the angular modes of the random matrix leaves us with its N real eigenvalues as the dynamical variables. Then, the large-N limit is performed via a WKB-like procedure†. And finally, as N is taken to infinity, the matrix-model potential must approach one of its critical values in accordance with the double-scaling limit. The averages of various quantities over random surfaces can then be found as functions of the string coupling, which is held fixed in this limit. To any given order in the string-perturbative expansion, the matrix model results may be compared with results from the corresponding continuum models, which consist of the quantum Liouville theory coupled to various matter fields.[9,10[11−14] . Such comparisons have been carried out for low genus [15−18].

In addition to perturbative series in the string coupling, one finds in these models also nonperturbative effects [19]. In the $D = 1$ model, the leading such effect comes from the tunneling of a single matrix eigenvalue (=fermion) out of the potential well, or between different potential wells. The contribution to, say, the free energy will behave as $\exp(-const/g_{string})$ for low string couplings. In ref.19 it was pointed out that for the $D = 0$ models, as well, there are saddle-point configurations where a single eigenvalue leaves the potential well‡, and that these configurations give rise to nonperturbative effects of the same form. However, for even-k multicritical $D = 0$ models, including pure gravity itself, there are ambigu-

† For the $D = 1$ model, this is a true quantum-mechanical WKB procedure for N independent fermions in a potential well.

‡ In the $D = 0$ case, this is the effective potential, namely the matrix-model potential plus the mean potential due to the Coulomb repulsion of the other N-1 particles.

ities in defining the theory nonperturbatively, which can be traced to the fact that the critical quartic potential is unbounded from below. Various nonperturbative definitions have been proposed in the literature[20−31] . For the even-k multicritical models, these definitions can and do disagree with one another, since the perturbative series is not Borel-summable. But even for the well-defined models, a systematic derivation of the nonperturbative physics directly in $D = 0$ has so far been lacking.

A further line of development has been the reformulation of the $D = 1$ matrix model as a string field theory. In matrix language, the correlators that one calculates in this model are of any number of boundary operators, each such boundary having an arbitrary length. Each boundary is in a single slice of the embedding dimension[§]. The same information is contained in the n-point functions of the density of matrix eigenvalues; it is essentially this density which has been suggested as the string field[32] . It is a function of λ, the matrix eigenvalue, and the embedding dimension; hence the field theory is two-dimensional. In ref. [32], Das and Jevicki used collective coordinate techniques to transform the quantum mechanics of N particles (in a bosonic formulation) into two-dimensional quantum field theory. The kinetic term in the action is that of a massless field, which corresponds to the massless tachyon in the continuum effective field theory [12]. Some aspects of this correspondence remain unclear— for instance, the identification of matrix eigenvalue with (a function of the) Liouville zero-mode. Nevertheless, much progress has been made in understanding the Das-Jevicki collective field theory, making string-perturbative calculations with it, and comparing the results with those obtained via other methods[33,34] . Nonperturbative effects, appearing in the form of solitons and instantons of the $D = 1$ field theory, have also been investigated[35,36] .

Another approach to $D = 1$ field theory has been to retain its formulation as N-fermion quantum mechanics, but second-quantize the fermions and then bosonize them[37−39] . This approach is equivalent to the Das-Jevicki formulation.

§ Otherwise, the angular matrix variables are excited and the calculations cannot be done using current methods.

Although the collective-field method has also been applied to the $D = 0$ matrix models[40−43], it seems to us less compelling there than for $D = 1$, because of its unusual kinetic term, and although a perturbative scheme has been outlined[44,45], this program has not, to our knowledge, been carried out as far as the corresponding $D = 1$ field theory¶.

In this paper, we develop an alternative field theory formalism for the $D = 0$ matrix models. Our field theory is two-dimensional, but the extra dimension is an auxiliary one and drops out of the formalism at some stage. What is left is a nonlocal quantum mechanics, with a function of λ playing the role of time. This is interesting in view of the idea [32] that the matrix eigenvalue is related to the Liouville zero-mode. The Jevicki-Sakita $D = 0$ collective field theory is also a nonlocal quantum mechanics. We have found, however, that our formalism has the following desirable properties, which make it worth pursuing:

A. The entropy term* in the action, which in the collective field approach appears through the Jacobian, is for the first time incorporated into the classical solution. This results in smoothing of the Dyson sea edges, interpolation between different eigenvalue bands for multiple-well potentials (determining their relative population), and an unambiguous treatment of nonperturbative (instanton) effects, directly in a $D = 0$ field-theory framework, for multicritical models that can occur for a bounded-from-below potential.

B. Our classical eigenvalue distribution, $\rho(\lambda)$, satisfies a local nonlinear Schrödinger equation, with $1/N$ playing the role of $\hbar$. The Schrödinger potential, $V_1(\lambda)$, appearing in this equation is similar to, but distinct from, the Marinari-Parisi $D = 1$ reformulation. On the other hand, in the planar limit it has an exact corresposondence with the Dyson sea effective potential, mentioned above.

¶ In ref. [45] , Lechtenfeld uses our equations to set up a perturbative scheme. But since he performs the genus expansion at an early stage, one is left with manifest divergences, and in addition nonperturbative information is lost. We avoid these problems, as will be seen below.

∗ This term is $N \int \rho \ln \rho \, d\lambda$, with $\rho(\lambda)$ the eigenvalue density.

C. Not only are quantum corrections computable, but most of them vanish in the double scaling limit. The only quantum corrections which survive this limit, apart from the semiclassical functional determinant, are a set of Feynamn graphs which can be exactly summed.

The rest of this paper is organized as follows. In section 2, the matrix-model partition sum is recast as that of a massless two-dimensional field, $A(r)$, with (non-local) interactions confined to an infinite line (the 'eigenvalue axis'). The equation of motion is written, and its classical solution is expressed as an eigenvalue distribution, $\rho(\lambda)$, and shown to be either unique or labeled by a discrete index. The two-dimensional equation of motion becomes a one-dimensional integro-differential equation on the eigenvalue axis; this in turn leads to a weaker** local nonlinear Schrödinger equation. The interpretation of this equation is discussed, as well as the nature of its solution and how it is determined uniquely (up to the possible discrete index).

In section 3, the quantum fluctuations about a classical solution are studied, and an expression derived for the partition sum in terms of the propagator of the A field (the 'conjugate field'). The IR and UV singularities are seen to cancel in a trivial way, to all orders. The one-dimensional Green's equation for the propagator is presented, as well as a partial solution. Section 4 is devoted to a discussion of the double scaling limit, including nonperturbative effects***. We see there that the quartic $k = 2$ classical solution is unique at the level of string perturbation

** 'Weaker' because the Schrödinger equation has more solutions than the integro-differential equation.

*** Most of the discussion in section 4 is limited to the case of the quartic potential as it approaches its $k = 2$ critical point. Since neither the theory nor our formalism are well defined for this potential, only the string-perturbative series discussed in that section is meaningful. Therefore, our description of an instanton ansatz for this model, described in sec.4, is not necessarily more reliable than any of the previous nonperturbative definitions of pure gravity referenced above. Nevertheless, it is reassuring that our exponentially-suppressed tunneling factor, eq.(39a), agrees with that appearing in other approaches (it is the prefactor that is ambiguous). The main point of our instanton calculation is, however, that the conjugate-field formalism applies to any well-defined realization of a critical model, and a similar instanton calculation for such a realization would be rigorous.

theory. In section 5 we restate our conclusions. Some mathematical details are reserved for the appendix. Many results, presented or stated without proof in this paper, will be exposed more fully in a follow-up publication, to appear soon.

Finally, an explanatory note is in order. An early version of this preprint, containing most of the results (with the notable exception of point **C** above), has been circulating privately since November 1990. That earlier version has sometimes been confused in the literature with SLAC-PUB-5262, an altogether distinct work.

2. Conjugate Field Formalism and Classical Solutions.

The partition function of the $D = 0$ matrix model is,

$$Z\{V\} \equiv \int [d\lambda] \exp\Big(-N \sum_{i=1}^{N} V(\lambda_i) + \sum_{i \neq j} \ln |\lambda_i - \lambda_j|\Big) \tag{1}$$

Where V is the matrix-model potential, which we assume to be a polynomial.

We may formally rewrite (C a divergent number)

$$\exp\Big(\sum_{i \neq j} \ln |\lambda_i - \lambda_j|\Big) = C \int [dA] \exp\Big(\sqrt{4\pi} i \sum_{j=1}^{N} A(\lambda_j) - \frac{1}{2} \int (\partial A)^2 d\lambda dx\Big) \tag{2}$$

Which is the path integral over a massless field A in two dimensions, with N point charges on the $x = 0$ ('eigenvalue') axis. x is an auxiliary dimension, and $A(\lambda) \equiv A(\lambda 0)$. We shall use r to denote a general point (λ, x) in the plane.

This path integral is beset by infrared and ultraviolet divergences. The UV divergence is regulated by smearing the point charges; $A(\lambda_j)$ is replaced by

$$\frac{1}{\epsilon} \int_{\lambda_j}^{\lambda_j + \epsilon} d\lambda A(\lambda) \tag{2a}$$

in eq. (2). The infrared divergence is regulated by introducing a uniformly charged circle in the plane, centered about the origin and with a large radius L. The total charge of this circle is $-N$, which screens the N point-charges. All dependences on ϵ and L will cancel in a simple way. In what follows, we will mostly ignore the need to regulate the path integral; a more careful treatment reveals that this naive approach is the correct one. Introducing the normalized charge density,

$$\rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i) \tag{3}$$

We may instead think of Z as a path integral over the overcomplete variables $\{\rho(\lambda)\}$, with integrand

$$\exp\bigl(-N^2 \int d\lambda \rho(\lambda) V(\lambda) + N^2 \int\int d\lambda d\mu \rho(\lambda)\rho(\mu) \ln|\lambda - \mu|\bigr) \tag{4}$$

This is the collective-field approach of Jevicki and Sakita. But we prefer to use the conjugate field $A(r)$ as our dynamical field, since it has a standard kinetic term and trivial Jacobian. The other merits of the conjugate field theory have been listed in points **A, B** and **C** of the introduction.

Combining eqs. (1) and (2), and using (2a) to regulate the UV divergence in C, we obtain the following expression for the partition function*:

$$Z\{V\} = C_1(L,N)\epsilon^{-N} \int [dA] \bigl(I\{A\}\bigr)^N \exp\bigl(-\frac{i}{\sqrt{\pi}}\frac{N}{L}\oint ds A - \frac{1}{2}\int (\partial A)^2 d^2 r\bigr) \tag{5}$$

$$I\{A\} \equiv \int d\lambda \exp\bigl(-NV(\lambda) + i\sqrt{4\pi}A(\lambda)\bigr) \tag{6}$$

In equation (5), the linear term is an integral over the charged circle, and C_1 is IR divergent, UV finite, and V-independent. All integrals over λ, here and below, range over the whole real axis, unless the limits of integration are explicitly indicated.

$\star$ We define the measure $[dA]$ to include the infinite, but $\{L, \epsilon, N, V(\lambda)\}$ independent, factor of $(\det' \partial^2)^{1/2}$.

The euclidean action is, apart from the IR-regulator source term,

$$S\{A\} \equiv \frac{1}{2}\int (\partial A)^2 d^2 r - N \ln I\{A\}. \tag{6a}$$

This is a nonlocal action, with the interactions occuring on the eigenvalue axis. Away from the axis, A is a free massless field.

$i\sqrt{4\pi}A/N$ can be thought of as a quantum fluctuation in the matrix-model potential. In addition, A is conjugate to ρ in the thermodynamical sense.

The classical equation of motion for the field A is[†]

$$\partial^2 A(r) = -i\sqrt{4\pi} N \rho_1(\lambda)\delta(x) \tag{7a}$$

where

$$\rho_1(\lambda) \equiv \frac{1}{I}\exp\bigl(-NV(\lambda) + i\sqrt{4\pi}A(\lambda)\bigr) \tag{7b}$$

Note that although the action is nonlocal, the equation of motion is local, but with a new coupling I, to be determined self-consistently.

The boundary condition for the conjugate field A at infinity is,

$$A(r) \approx -i\frac{2N}{\sqrt{4\pi}} \ln|r| + \ const \tag{7c}$$

since that is the electrostatic potential far from the N point-sources (eq.(2)). This boundary condition allowed us to freely integrate by parts the variation of the kinetic term in the action.

† We omit a step here, related to the infrared regulator: the field A should be shifted by a constant, which is the constant potential due to the charged circle in its interior.

We denote a solution of eqs.(7) by

$$A_{classical}(r) \equiv -iA_s(r) \tag{7d}$$

and define the classical charge (eigenvalue) density on the eigenvalue axis as ρ at the classical solution:

$$\rho(\lambda) \equiv \frac{1}{I_s} \exp\bigl(-NV(\lambda) + \sqrt{4\pi} A_s(\lambda)\bigr) \tag{8}$$

Where I_s is the value of $I\{A\}$ at the classical solution.

Henceforth we shall adopt the definition eq.(8) for ρ, instead of eq.(3). From here on we shall work with ρ instead of A_s; They contain the same information, although ρ is defined only on the eigenvalue axis‡ .

Let us assume that the potential V has been chosen to be bounded from below. In that case, as will be shown, $\rho(\lambda)$ is unique or labeled by a discrete index. In section 4, we shall see that quantizing $A(r)$ about the classical configuration determined by ρ, gives rise to a unique string-perturbative expansion for physical quantities in the double scaling limit (abreviated henceforth as *d.s.l.*§), *provided* ρ has support in a single Dyson sea (band) in the planar limit. Thus, the possible discrete non-uniqueness of ρ will be revealed either at the nonperturbative level, or for multiband solutions. However, any *continuous* non-uniqueness of multiband solutions, which results from the freedom to adjust the relative populations of different bands, is eliminated by the nonperturbative effects ('tunneling'), as we shall see.

---

‡ Just as, in the analog 2d electrostatic problem, the charge distribution on a plate together with the boundary condition at infinity, determine the potential throughout an otherwise empty space.

§ In order to properly analyze the d.s.l., a multiple-well potential, bounded from below, must be used. In this paper, when we specialize to a particular $V(\lambda)$, it is the quartic $\lambda^2/2 + g\lambda^4$, which is *not* bounded from below at criticality. This is not important for string perturbation theory; work on nonperturbative effects in which a multiple-well potential *is* used, is currently in progress.

Both the equation of motion and boundary condition for $A_s(r)$ are real, and thus each $\rho(\lambda)$ in the discrete set of solutions is either real, or the complex conjugate of another solution. The possible imaginary parts of solutions ρ can only be revealed nonperturbatively; in any case, all physical quantities are real.

Evaluating the action at the classical solution, we find that the L-dependent prefactor in eq. (5) gets canceled:

$$\begin{aligned} Z_{classical}\{V\} =& const\ \epsilon^{-N} \exp\big(-N^2 \int d\lambda \rho V \\ &+ N^2 \int\int d\lambda d\mu \rho(\lambda)\rho(\mu) \ln|\lambda-\mu| - N\int d\lambda \rho \ln\rho\big) \end{aligned} \tag{5a}$$

The constant depends only on N, and is hence irrelevant.

As we discuss below, the UV divergence in eq (5a) is cancelled by the semiclassical (determinant) factor, whereas the higher quantum corrections to Z are both IR- and UV-finite.

Notice the $O(N)$ correction to the free energy that occurs already at the classical level. This correction has a simple interpretation: It is a combinatorical factor, the entropy of the classical solution (see refs. [40],[41],[42], [43]) . Of course, the sum of all $O(N)$ corrections to the free energy must vanish, since only even powers of N appear in the topological genus expansion[46] .

ρ satisfies the integral equation,

$$V(\lambda) + \frac{1}{N}\ln(\rho(\lambda)I_s) = 2\int d\mu \ln|\lambda-\mu|\rho(\mu) \tag{9a}$$

which follows from eqs.(7), in conjunction with the fact that the $2d$ free Green's function is $\frac{1}{2\pi}\ln|r-r'|$. Upon differentiation w.r.t. λ, eq.(9a) becomes the following one-dimensional integro-differential equation:

$$V'(\lambda) + \frac{1}{N}\frac{\rho'(\lambda)}{\rho(\lambda)} = 2\mathcal{H}(\rho(\lambda)) \tag{9b}$$

Where $\mathcal{H}$ denotes the Hilbert transform:

$$\mathcal{H}(f(\lambda)) \equiv \int \frac{f(\mu)d\mu}{\lambda - \mu} \tag{9c}$$

for any function f*.

In addition to satisfying the $1d$ integro-differential equation, ρ must be normalized to unity (by eq. (8)):

$$\int d\lambda \rho(\lambda) = 1 \tag{9d}$$

The eq. (9b) reduces to the integral equation of ref.46 in the planar ($N \to \infty$) limit; the extra term can be physically understood as due to the variation of the entropy term in the exponent of eq.(5a). Its effect is to smear the edges of the Dyson sea (or seas, for a multi-well potential V), so they become transition regions, rather than singularities as in the $D = 0$ and $D = 1$ collective field theories of Jevicki et. al. We will further discuss the transition regions in section 4[47] .

For $N >> 1$, it is easy to see from eq.(9b) that outside the sea, and beyond the transition regions, $\rho(\lambda)$ is well-approximated by

$$\rho(\lambda) \approx const\ \lambda^{2N} e^{-N[V(\lambda)+O(1/\lambda)]} \tag{9e}$$

Thus, two conclusions can immediately be drawn. Firstly, a normalizable classical ρ exists only if $V(\lambda)$ is bounded from below; and secondly, $\rho(\lambda)$ vanishes nowhere, so that multiple eigenvalue bands ('seas') are related by a tunneling effect. This effect serves to determine the relative population of multiple seas, and is responsible in general for string-nonperturbative effects.

$\star$ Such integrals are understood, here and below, to be principal-valued. Note that we use a nonstandard normalization in our definition of $\mathcal{H}$.

We note that, as our classical solution depends on N and hence includes some higher-genus effects, the terms 'planar' and 'classical' are not synonymous in our formalism. This is also the case for the Das-Jevicki field theory.

Once ρ is known, the corresponding I_s is determined uniquely by eqs.(9); however, I_s drops out of the formalism from here on.

In the appendix we use eqs.(9b-d), and properties of the Hilbert transform, to derive the following nonlinear Schrödinger equation:

$$\big(-\frac{1}{N^2}\frac{\partial^2}{\partial\lambda^2} + V_1(\lambda) + \pi^2\rho^2\big)\big(\rho^{-1/2}\big) = 0 \tag{10a}$$

where V_1, henceforth called the 'Schrödinger potential', is the polynomial

$$V_1(\lambda) = \frac{1}{4}(V')^2 + \frac{1}{2N}V'' + P(\lambda) \tag{10b}$$

P is a polynomial whose coefficients are moments of the charge distribution:

$$P(\lambda) \equiv -\int d\mu\rho(\mu)\frac{V'(\mu) - V'(\lambda)}{\mu - \lambda} \tag{10c}$$

This Schrödinger equation is local, except for the coefficients of $P(\lambda)$, which are determined self-consistently. For example, in the case of quartic V,

$$V(\lambda) = \frac{1}{2}\lambda^2 + g\lambda^4 \tag{10d}$$

the polynomial $P(\lambda)$ becomes:

$$P(\lambda) = -1 - 4g(m_2 + \lambda^2) \tag{10e}$$

where the eigenvalue moments are defines as

$$m_n = \int \lambda^n\rho(\lambda)d\lambda \tag{10f}$$

and we have used the symmetry $\rho(\lambda) = \rho(-\lambda)$, which follows from eq. (9b) and the symmetry of $V(\lambda)^\dagger$. The nonlinear Schrödinger equation has several points of interest, which we now discuss. Firstly, in it $1/N$ plays the role of Planck's constant, and the tunneling effects implied by eq.(9e), can now be seen to be similar to quantum mechanical tunneling. This is demonstrated in section 4, using the WKB approximation. In the exterior of the Dyson sea(s), ρ is exponentially suppressed for large N, and (10a) approximates a linear Schrödinger equation. a strange feature of this correspondence is that the 'wavefunction' here is $1/\sqrt{\rho}$, the inverse of the intuitive $\sqrt{\rho}$. Another feature, probably closely related, is that the V'' term in our Schrödinger potential V_1, has the opposite sign compared with the $D=1$ potential arising from the Marinari-Parisi reformulation [24]‡.

The transition regions interpolate between the WKB solutions outside and inside a given Dyson sea, similar to the role of the Airy function in ordinary (linear) WKB (see section 4). In the sea interior, the ρ^2 term in (10a) is no longer negligible, and in fact there the leading WKB approximation is

$$\rho(\lambda) \approx \frac{1}{\pi}\sqrt{-V_1(\lambda)} \quad (\ \textit{inside a sea}\ ) \tag{10g}$$

which is just the generalization of Wigner's semicircle law to arbitrary potential V [46] .

In the planar limit and outside the Dyson sea, $V_1(\lambda)$ has a physical interpretation: $4V_1(\lambda)$ is the square of the gradient of the effective potential, that is, the square of the total force exerted on a single eigenvalue, due to the potential $V(\lambda)$ and the repulsion of the other $(N-1)$ eigenvalues.

† For simplicity, we assume throughout that $V(\lambda) = V(-\lambda)$. Actually, this only implies a symmetric ρ if $\rho(\lambda)$ is unique, but we will ignore this complication here, since the entire analysis can be easily redone without the symmetry assumption, and besides the possible nonsymmetry will not affect perturbation theory.

‡ The relation between the two facts is, that in order to get the Marinari-Parisi $D=1$ Schrödinger equation, one chooses as wavefunction $\psi\{M\} = \exp(-\frac{N}{2}\,\mathrm{tr}\,V(M))$, with M the original random matrix. Thus, if the Van der Monde were to be ignored, the one-particle wave-function would be the *square root* of ρ. If we choose the inverse wave function, the sign of V'' in the Schrödinger potential agrees with ours, rather than with that of Marinari and Parisi.

As stated in the introduction, the Schrödinger equation is weaker than the integro-differential equation. This is because (10a) is a second-order differential equation, so for given V_1 its solution $\rho(\lambda)$ has two free continuous real parameters. By eqs.(10), V_1 has one additional unknown, m_2; but we must impose the two self-consistency conditions, for m_2 and for $m_0 = 1$. This leaves us with a *single undetermined real parameter*§ . However, as is proven in the appendix, the integro-differential equation allows no zero modes that preserve the normalization condition (9d). Hence, the free real parameter can only assume discrete values.

3. Quantum Corrections.

We next address the quantum corrections to $Z_{classical}$ (eq. (5a)), needed in order to regain the full partition sum Z in eq. (5). Let us separate the field A into its classical and quantum pieces,

$$A(r) = -iA_s(r) + A_q(r) \tag{11}$$

and also separate out the quadratic part of the action:

$$S\{A\} = S_{classical} + S_I\{A_q\} + \frac{1}{2}\int\int d^2r d^2r' A_q(r) K(r,r') A_q(r') \tag{12}$$

Here S_I is the interacting piece, consisting of the terms of order three and higher in A_q. K is the inverse propagator of the quantum field in the background of the classical solution:

§ Note that the usual WKB procedure for bound states, does not apply for the Dyson sea. In the usual procedure, one imposes that the wavefunction component which blows up exponentially at spatial infinity, vanishes. But here the wavefunction is $\psi = \rho^{-1/2}$, so in fact ρ normalizability *requires* ψ to blow up exponentially, and no useful information results from this condition.

$$K(r,r') = -(\partial_r)^2\delta(r-r') + 4\pi N\delta(x)\delta(x')\{\rho(\lambda)\delta(\lambda-\lambda') - \rho(\lambda)\rho(\lambda')\} \tag{13}$$

The only zero mode of K is the constant function¶, so we fix that by defining our space of configurations A_q to satisfy $A_q(r) \to 0$ as $|r| \to \infty$. This renders K nonsingular*.

Now, S_I depends only on the value of A_q on the $x=0$ axis, so we define the one-dimensional field

$$q(\lambda) \equiv \sqrt{4\pi}\big[A_q(\lambda 0) - \int \rho(\mu)d\mu A_q(\mu 0)\big] \tag{14}$$

and denote

$$S_i\{q\} \equiv S_I\{A_q\} \tag{15}$$

We then have, by eqs.(6),(6a),(8),(11),(12) and (14):

$$S_i\{q\} = -N\ln\big(\int \rho(\lambda)d\lambda e^{iq(\lambda)}\big)_{NQ} \tag{16}$$

where NQ denotes the non-quadratic part. The full partition function, including quantum corrections, is given by

$$Z\{V\} = \sum Z_{classical}\{V\}\big(\frac{\det K}{\det(-\partial^2)}\big)^{-1/2}\langle e^{-S_i\{q\}}\rangle \tag{17}$$

where the expectation value is a Gaussian average, i.e. evaluated via Wick's theorem, the sum is over the discrete set of classical solutions, the functional determinants are subject to the boundary condition $\lim_{|r|\to\infty} A_q(r) = 0$, and for each classical solution, $Z_{classical}$ is given by eq.(5a).

¶ This is equivalent to the fact, proven in the appendix, that $\rho(\lambda)$ has no normalization-preserving zero modes.

∗ This choice of boundary condition follows from the fact that both $A(r)$ and $-iA_s(r)$ satisfy the boundary condition (7c).

The two-point function, to be used in the Wick expansion, is

$$\langle q(\lambda)q(\lambda')\rangle = 4\pi\big(H(\lambda,\lambda') + \frac{1}{4\pi N}\big) \tag{18}$$

where $H(\lambda,\lambda')$ is the restriction to the eigenvalue axis of $H(r,r')$; the latter is defined uniquely by the following properties —

$$\partial_r^2 H(r,r') = 4\pi N\rho(\lambda)\delta(x)H(r,r') + \delta(r-r') \tag{19a}$$

$$H(r,r') = H(r',r) \tag{19b}$$

$$H(r,r') \approx H(\infty,r') + O(\frac{1}{|r|}) \tag{19c}$$

as $|r| \to \infty$, where $H(\infty,r')$ is a finite function of r'.

By integrating eq.(19a) over r with measure $\int d^2r$ and using (19c), we find a fourth property:

$$\int H(\lambda,\lambda')\rho(\lambda)d\lambda = -\frac{1}{4\pi N} \tag{19d}$$

For actual calculations, only the one-dimensional restriction $H(\lambda,\lambda')$ is required. For $\lambda' \approx \lambda$, H is dominated by the logarithmic singularity of the free $2d$ Green's function:

$$H(\lambda,\lambda') \approx \frac{1}{2\pi}\ln|\lambda-\lambda'| + \ regular \quad (\lambda' \approx \lambda) \tag{19e}$$

Equation (17) is our central tool for evaluating physical quantities using the conjugate-field formalism. We shall refer to the last, Wick-expanded factor as the 'Feynman diagrams', since they go beyond the semiclassical determinant factor* .

$\star$ In the sense that they involve the interaction piece of the quantum action.

The only remaining divergences in eq.(17) are UV ones, and they appear in two places: the ϵ^{-N} factor in $Z_{classical}$ (see eq.(5a)), and the determinant factor†. But these two divergences cancel, as we now show. Using eq.(13) and formally expanding in powers of the free massless propagator, we find:

$$\begin{aligned}\ln\Big[\frac{\det K}{\det(-\partial^2)}\Big]^{-\frac{1}{2}} &= -\frac{1}{2}\,\mathrm{tr}\ln\big[K/(-\partial^2)\big] \\ &= -2\pi N\int d\lambda\rho(\lambda)\Big(\frac{1}{-\partial^2}\Big)_{\lambda\lambda} + \ UV\ finite\end{aligned} \tag{20}$$

This is ill defined. But if we use the UV regularization eq.(2a), the $\delta(\lambda-\lambda')$ term in $K(r,r')$ is smeared, and (20) becomes

$$\ln\Big[\frac{\det K}{\det(-\partial^2)}\Big]^{-\frac{1}{2}} = N\ln\epsilon + \ UV\ finite \tag{20a}$$

and hence the singular ϵ-dependence drops out of eq.(17), as claimed.

The remainder of this paper deals mostly with properties of the functions $\rho(\lambda)$ and $H(\lambda,\lambda')$, especially in the double scaling limit, and with the summation of the Feynman graphs in eq.(17). The evaluation of $\det K$, which is done using heat-kernel methods, will be reported on elsewhere.

To get an idea what the expressions for Feynman graphs look like, we record the lowest-order terms in the Wick expansion:

$$\begin{aligned}\ln\langle e^{-S_i\{q\}}\rangle &= \ln\langle\Big[\int\rho(\lambda)d\lambda e^{iq(\lambda)}\Big]^N\exp\Big(\frac{N}{2}\int\rho(\lambda)d\lambda q(\lambda)^2\Big)\rangle \\ &= \frac{N}{8}\Big\{\int d\tau\Big[\langle q(\tau)^2\rangle-\int d\tau'\langle q(\tau')^2\rangle\Big]^2 \\ &\quad -2\int\int d\tau d\tau'\langle q(\tau)q(\tau')\rangle^2+\ldots\Big\}\end{aligned} \tag{21}$$

In eq.(21), the leftmost equality is exact, and we have introduced a useful new

---

† Superficially, it appears that the normal-ordering contributions to the Feynman-graph factor are also divergent, due to (19e); but these normal-ordering divergences manifestly cancel to all orders, as we shall show below.

variable $\tau(\lambda)$:

$$\tau(\lambda) \equiv \int_{-\infty}^{\lambda} \rho(\mu) d\mu \tag{21a}$$

which ranges from 0 to 1. We will sometimes denote the arguments of a function of λ, by τ instead* .

The terms displayed in the expansion on RHS of (21), are those involving only two propagators. Using eq.(18), we rewrite these Feynman-graph contributions to the free energy, thus:

$$\begin{aligned}\ln\langle e^{-S_i\{q\}}\rangle =& 2\pi^2 N\{\int d\tau[H(\tau,\tau) - \int d\tau' H(\tau',\tau')]^2 + \frac{1}{8\pi^2 N^2} \\ &- 2\int\int d\tau d\tau' H(\tau,\tau')^2 + O(H^3)\}\end{aligned} \tag{21b}$$

The first term in the curly brackets is the normal-ordering contribution to this order. $H(\tau,\tau)$ is logarithmically divergent, but this divergence is τ-independent, so $H(\tau,\tau) - \int d\tau' H(\tau',\tau')$ is UV finite† . The other two-propagator graph in (21b) is manifestly finite. Note that the Feynman-graph contributions to the free energy are not local. Indeed, we started from a nonlocal action for the A field, so this is to be expected – the cluster expansion does not hold.

In order to evaluate Feynman graphs, we must solve for the propagator H. Before doing so, however, let us show how to sum the normal-ordering contributions to all orders. This is easy to do: from the leftmost equality in (21), we obtain

$$\langle e^{-S_i\{q\}}\rangle = \langle\{\int d\tau \exp(-w(\tau)) : e^{iq(\tau)} :\}^N \exp[\frac{N}{2}\int d\tau : q(\tau)^2 :]\rangle \tag{22}$$

where w is a finite function, defined as follows:

$$w(\tau) \equiv w_1(\tau) - \int d\tau' w_1(\tau') \tag{22a}$$

$\star$ In deriving the expansion (21), we made use of the fact that $\int d\tau q(\tau) = 0$, by eqs.(14),(21a).
$\dagger$ This can be shown rigorously by making use again of the UV regularization procedure, eq. (2a).

$$w_1(\tau(\lambda)) \equiv \lim_{\lambda' \to \lambda} (2\pi H(\lambda, \lambda') - \ln|\lambda - \lambda'|) \tag{22b}$$

$w_1(\tau)$ is finite by virtue of eq.(19e).

We now turn to the task of computing $H(\lambda, \lambda')$. As was the case with the $2d$ equation of motion (7), the two-dimensional differential equation (19a) can be converted into a *one-dimensional* integro-differential equation, by treating the RHS as a source term. This new equation is

$$\frac{\partial}{\partial \lambda} H(\lambda, \lambda') = \frac{1}{2\pi} \frac{1}{\lambda - \lambda'} + 2N \mathcal{H}_\lambda \big(\rho(\lambda) H(\lambda, \lambda')\big) \tag{23}$$

where the principal value is understood in the first term, and the subscript to the Hilbert transform indicates that the transform acts on λ, with λ' held fixed. When the conditions (19b-c) are imposed, eq.(23) has a unique solution.

It is possible to derive from (23) a $1d$ *differential equation* for H, similarly to the procedure that lead from eqs.(9) to eqs.(10). Namely, eq.(23) is differentiated w.r.t. λ, then used again, and the properties of the Hilbert transform (listed in the appendix) are used. In addition, eq.(9b) for ρ is used. The result of these manipulations is‡:

$$\begin{aligned} \{\frac{\partial^2}{\partial \tau^2} + 4\pi^2 N^2\} H(\lambda, \lambda') = & - \pi N \delta(\tau - \tau') + \frac{1}{\rho^2(\lambda)} Q(\lambda|\lambda') \\ & - \frac{1}{2\pi \rho^2(\lambda)} \frac{\partial}{\partial \tau'} (\frac{\rho(\lambda')}{\lambda - \lambda'}) \end{aligned} \tag{24}$$

where principle value is again understood in the last term. Here $Q(\lambda|\lambda')$ is a polynomial in λ, with coefficients that are one-sided moments§ of H with measure $\int d\tau$:

$$Q(\lambda|\lambda') \equiv \frac{N}{2\pi} \frac{V'(\lambda) - V'(\lambda')}{\lambda - \lambda'} + 2N^2 \int \rho(\mu) d\mu \frac{V'(\lambda) - V'(\mu)}{\lambda - \mu} H(\mu, \lambda') \tag{24a}$$

and these moments are to be determined self-consistently, just as for the moments m_n of ρ, which entered the Schrödinger eq.(10a) through the polynomial $P(\lambda)$.

‡ We use the τ variable, defined in (21a), with $\tau' = \tau(\lambda')$, $\tau = \tau(\lambda)$.

§ That is, moments w.r.t. one of the two variables.

Denote these one-sided moments as follows:

$$h_n(\mu) \equiv \int \lambda^n \rho(\lambda) d\lambda H(\lambda, \mu) \tag{24b}$$

By (19d) we have,

$$h_0(\mu) = -\frac{1}{4\pi N} \tag{24c}$$

Restricting again to the quartic potential, we obtain:

$$Q(\lambda|\lambda') = \frac{1}{2\pi} 4gN\{[\lambda'^2 + 4\pi N h_2(\lambda')] + \lambda[\lambda' + 4\pi N h_1(\lambda')]\} \tag{25}$$

The LHS and first term on RHS of (24) constitute the Green's equation for the correlator of a quantum-mechanical harmonic oscillator, but the other, nonlocal source terms on the RHS spoil this simple picture. As with the eigenvalue density, the $1d$ differential equation is somewhat weaker than the $1d$ integro-differential equation: the latter, however, is equivalent to the $2d$ Green's equation (19a), once the boundary condition (19c) is imposed.

The equation (24) is linear, and so can be readily solved in terms of Q (which however is itself unknown). The general solution is:

$$\begin{aligned} H(\lambda, \lambda') =& A(\lambda') \cos 2\pi N\tau + B(\lambda') \sin 2\pi N\tau - \frac{1}{2}\theta(\tau - \tau') \sin 2\pi N(\tau - \tau') \\ &+ \frac{1}{4\pi^2 N} \int_0^\lambda \frac{d\mu}{\rho(\mu)} \sin 2\pi N(\tau - \tau(\mu))[2\pi Q(\mu|\lambda') + \frac{\partial}{\partial \tau'}(\frac{\rho(\lambda')}{\lambda' - \mu})] \end{aligned} \tag{26}$$

where A,B are free functions and θ is the step function. The symmetry of $H(\lambda, \lambda')$ determines A and B up to three real constants; these constants, as well as $Q(\lambda|\lambda')$, can be determined by making use of eq.(23).

The formalism for evaluating the classical solution(s), propagator and quantum corrections is rather complicated for finite N. Fortunately, however, massive simplifications occur in the $\frac{1}{N}$ expansion, and the formalism simplifies even further when the couplings approach criticality and the d.s.l. (double scaling limit) is taken. We next turn to a discussion of this limit.

4. The Double Scaling Limit.

In this section, we will describe the procedure for performing the double scaling limit (*d.s.l.*) in our conjugate-field formalism. We leave out many details, to be included in a forthcoming publication.

Let us specialize to the quartic potential, eq.(10d), and therefore to the $k = 2$ critical model (pure gravity). The plan of the section is as follows. In part 4.a, the nature of the planar limit and the *d.s.l.* for the model is reviewed. Then the integro-differential and nonlinear Schrödinger equations for a classical solution, ρ, are used to find the string-perturbative expansions for the moments m_n of ρ (defined in (10f)). These expansions are unique, and it is seen they are unique for any critical potential, provided attention is restricted to classical solutions that are single-band in the planar limit. In addition, the WKB approximation for ρ in the region exterior to the Dyson sea, is found.

In part 4.b we discuss the perturbative corrections to ρ inside the sea, and study the details of $\rho(\lambda)$ in the transition regions at the edges of the sea. In part 4.c we describe how the same techniques, when applied to the propagator H, yield the perturbative expansion for $H(\lambda, \lambda')$ in various regions of the (λ, λ') plane. The formulae of section 3 for quantum corrections are seen to greatly simplify in the *d.s.l.*, allowing the perturbative series for physical quantities (specific heat, etc.) to be found. In particular, the normal-ordered Feynman diagram expansion terminates after a small number of terms.

Since the critical quartic potential is unbounded from below, all our results up to this point were obtained by continuing from the well defined $g > 0$ regime, to $g \approx g_c < 0$. This procedure is satisfactory only for string perturbation theory. In part 4.d, we look at the Schrödinger potential V_1 *directly* for $g \approx g_c$; V_1 is seen to acquire a small 'second sea' in the transition region. This sea is finite in shape when V_1 and λ are appropriately double-scaled.

Our formalism is ill defined in this critical coupling regime, like pure gravity itself, since there is no normalizable $\rho(\lambda)$. Nevertheless, we investigate the behavior of ρ in the new transition region, and find that the second sea has a population suppressed by the expected $\exp(-const\ \frac{1}{\kappa})$, where κ is the string coupling. The constant in the exponential agrees with that obtained in other approaches. Thus, ρ can be thought of as an instanton. As such, it has two unusual aspects. Firstly, the tunneling factor occurs in the *field* configuration as well as in the classical action. Secondly, a single conjugate-field configuration seems to describe both the non-tunneling and tunneling eigenvalue configurations⋆ . This configuration, however, should be viewed only as a warm-up exercise for non-perturbative calculations in our formalism. The meaningful calculations are to be done for a potential V bounded from below, and would thus most likely not apply to any model with $k = 2$ behavior [29].

4.a. WKB and Perturbation Theory for ρ.

From eqs.(10), we find the Schrödinger potential for the case of quartic V†:

$$V_1(\lambda) = 4g^2(\lambda^2 - a^2)(\lambda^2 - b^2)^2 - 4g\delta m_2 + \frac{1}{2N}(1 + 12g\lambda^2) \tag{27a}$$

where:

$$a^2 = \frac{1}{6g}(\sqrt{1 + 48g} - 1) \tag{27b}$$

$$b^2 = -\frac{1}{4g} - \frac{1}{2}a^2 \tag{27c}$$

and δm_2 is the deviation of the second moment m_2 from its planar-limit value:

$$\delta m_2 \equiv m_2 + \frac{1}{36g} - \frac{a^2}{144g}(1 + 48g) \tag{27d}$$

It will shortly be seen that δm_2 is $O(\frac{1}{N})$ for fixed g, in the planar limit.

⋆ We thank S. Shenker for discussions concerning this latter point.

† Our notation, in particular our definition of $a(g)$, differs slightly from that of ref. [46] .

For $g > 0$, b^2 is negative, so $V_1(\lambda)$ vanishes only at two points, which are $\lambda = \pm a$ in the planar limit; these points are the edges of the Dyson sea. We define b such that $\mathrm{Im}\, b > 0$. The $k = 2$ critical point is at

$$g = g_c = -1/48 \tag{28a}$$

and for $g \approx g_c$, b^2 is positive, and in fact $b^2 \approx a^2 \approx 8$. As explained above, we will employ throughout most of section 4 (except part 4.d) the well-defined procedure of calculating at $g > 0$ and then continuing to criticality. The double scaling limit for this theory, consists in simultaneously letting $N \to \infty$, $g \to g_c$ while holding the string coupling fixed [1-8] :

$$g_{string} = \kappa \equiv \frac{1}{N}(g - g_c)^{-5/4} \tag{28b}$$

Next, consider our nonlinear Schrödinger equation, (10a), in a region of λ outside the sea; more precisely, $|\lambda| > a$, and $|\lambda \pm a|$ are held fixed as N increases. The $\pi^2\rho^2$ (nonlinear) term is then exponentially suppressed, by virtue of eq.(9e), and in consequence $\rho^{-1/2}$ approximately satisfies a *linear* Schrödinger equation:

$$\big[-\frac{1}{N^2}\frac{\partial^2}{\partial\lambda^2} + V_1(\lambda)\big](\rho^{-1/2}) \approx 0 \quad (\ \mathit{outside\ sea}\ ) \tag{29a}$$

This is easily solved via an asymptotic WKB expansion: ($V_1 > 0$ outside sea)

$$\begin{aligned} \rho(\lambda) = &|\sqrt{V_1(\lambda)}|\exp\big(-2N\int^{\lambda} d\mu\sqrt{V_1(\mu)}\big)\{const\ + O(\frac{1}{N}) \\ &+ O(e^{-2N\int^{\lambda} d\mu\sqrt{V_1(\mu)}})\} \quad (\ \mathit{outside\ sea}\ ) \end{aligned} \tag{29b}$$

where $V_1(\lambda)$ is given by eq.(27a)‡ . The corrections on the RHS of eq.(29b) are of two kinds: the $O(\frac{1}{N})$ corrections constitute the usual, linear-WKB asymptotic

‡ It is easy to check that (29b) agrees with eq.(9e).

expansion, whereas the exponentially-suppressed corrections are due to the nonlinearities of the exact (10a). Since our $V(\lambda)$ is symmetric, so is V_1, and the branch we choose for $\sqrt{V_1(\mu)}$ in the exponent, is as follows:

$$\mathrm{sgn}\sqrt{V_1(\lambda)} \equiv \mathrm{sgn}\lambda \qquad (29c)$$

This choice ensures that $\rho(\lambda)$ is symmetric, and in addition can be continued to an analytic function throughout the complex λ plane, except for a cut along the Dyson sea $(-a, a)$.

Next, we take $|\lambda| >> 1$. The exponentially suppressed corrections in (29b) can be neglected, and we obtain from eqs.(27a),(29b) an asymptotic expansion for the logarithmic derivative of ρ:

$$\begin{aligned}\frac{\rho'(\lambda)}{N\rho(\lambda)} \approx & \frac{2\delta m_2}{(\lambda^2-b^2)\sqrt{\lambda^2-a^2}} - \frac{1}{4gN}\frac{1+12g\lambda^2}{(\lambda^2-b^2)\sqrt{\lambda^2-a^2}} \\ & + \frac{1}{N}\left(\frac{\lambda}{\lambda^2-a^2} + \frac{2\lambda}{\lambda^2-b^2}\right) + O\left(\frac{1}{N^2}\right)\end{aligned} \qquad (29d)$$

This expression can now be expanded as a Laurent series in $\frac{1}{\lambda}$, and compared with the corresponding expansion resulting from eqs.(9) to yield the eigenvalue moments, m_n. The easiest way to compare the two series term by term, is to continue both of them to complex λ, with $\sqrt{\lambda^2 - a^2}$ defined as described after (29c). We then multiply both (9b) and (29d) by λ^n, n any integer, and integrate over $d\lambda$ along a closed contour with large $|\lambda|$. For negative n, the coefficients agree identically. In addition, all odd moments vanish by symmetry[§], whilst for even, positive n values we find:

$$\begin{aligned}m_n = & \frac{4g}{\pi}\int_{-a}^{a}\lambda^n d\lambda(\lambda^2-b^2)\sqrt{a^2-\lambda^2} + \frac{1}{8\pi gN}\int_{-a}^{a}\lambda^n d\lambda\frac{8gN\delta m_2-(1+12g\lambda^2)}{(\lambda^2-b^2)\sqrt{a^2-\lambda^2}} \\ & + \frac{a^n}{2N} + d_0 b^n + O\left(\frac{1}{N^2}\right) \quad (\ n\ even\ )\end{aligned} \qquad (30a)$$

§ This symmetry can be shown to hold to all orders in $1/N$ perturbation theory.

with

$$d_0 = \frac{1}{2N} + \frac{1}{b\sqrt{b^2 - a^2}}(\frac{\delta m_2}{2} - \frac{1+12gb^2}{16gN}) \tag{30b}$$

For $n = 0$ and 2, this just reproduces eqs.(9d) and (27d) respectively, up to $O(1/N^2)$ terms, so we gain no new information. For higher n, eqs.(30) give us all the moments in terms of δm_2, to order $1/N$. How is δm_2 to be determined, then? it is clear that d_0 must vanish to this order in $1/N$, since at $N >> 1$ the support of $\rho(\lambda)$ is the Dyson sea, or very near it, and the b^n term on the RHS of (30a) cannot occur for such a distribution. thus $d_0 = O(\frac{1}{N^2})$, which gives us the requisite missing information¶ * . —

$$N\delta m_2 = \frac{1+12gb^2}{8g} - b\sqrt{b^2 - a^2} + O(\frac{1}{N}) \tag{30c}$$

so δm_2 is indeed $O(1/N)$. Substituting this back into (30a-b) yields all nonvanishing moments m_n, to order $1/N$.

It is straightforward to extend this technique to any desired order in $1/N$, and to take the *d.s.l.* limit (28b), as well. We thus see that the perturbative genus expansion for $\rho(\lambda)$, or at least for the set of all its moments, is unique and can be easily determined, as claimed. Furthermore, the technique extends to any potential V, as long as we have a planar limit of ρ to expand about. When this limit is restricted to have a single band, it is unique, and so the perturbative expansion about it will also be unique.

4.b. Sea Interior and Transition Region. Consider the sea interior, namely the region $|\lambda| < a$ with $a - |\lambda|$ fixed (in either the planar- or the double-scaling limits). The nonlinear Schrödinger equation (10a) can be solved via the

¶ The $\frac{a^n}{2N}$ contribution in (30a), comes from the edges of the sea; the remaining two terms come from its bulk.

∗ If $d_0 \neq 0$, $\mathcal{H}(\rho(\lambda))$ must have poles at $\lambda = \pm b$, to first order in $1/N$. This is actually impossible for *any* $\rho(\lambda)$ for which the Hilbert transform is well-defined, since b is imaginary.

WKB approximation in this region, as was done in the exterior region. In the interior, however, the linearized WKB is of no use. This is because the planar limit we wish to expand about is given by eq.(10g), and thus the nonlinearity is crucial here.

The correct procedure is as follows. Defining new variables**,

$$t \equiv N \int_{-a}^{\lambda} \sqrt{-V_1(\mu)}\, d\mu \tag{31a}$$

$$\rho(\lambda) \equiv \frac{1}{\pi}\sqrt{-V_1(\lambda)}\, f(t)^{-2} \tag{31b}$$

we find the differential equation,

$$f'' + f - f^3 = O(\frac{1}{N^2}) \tag{31c}$$

In the planar limit, the solutions of (31c) are elliptic functions.

These planar solutions have two free real parameters; this is just the ambiguity discussed at the end of section 2, and is resolved by the integro-differential equation and the consistency conditions. Let us see how this works. Since we expect $f(t) \to 1$ in the planar limit (by (10g) and (31b)), we need only consider $f \approx 1$; then (31c) informs us that

$$f(t) = 1 + \epsilon \cos 2\bar{t} + (\frac{3}{4} - \frac{1}{4}\cos 4\bar{t})\epsilon^2 + O(\epsilon^3) + O(\frac{1}{N^2}) \tag{32a}$$

where ϵ is a small unknown oscillation amplitude, and $\bar{t} = t + \varphi$, with φ the constant phase of the oscillation. By employing the consistency conditions for $m_0 = 1$ and

** Recall that $V_1 < 0$ in the sea interior.

m_2, it can be shown that

$$\epsilon = O(\frac{1}{N}) \tag{32b}$$

and this turns out to mean that the oscillations can be ignore in the *d.s.l.* ; thus we may use

$$f \approx 1 \tag{32c}$$

Next, we study the transition regions at the edges of the sea. By symmetry, it suffices to investigate the $\lambda \approx a$ transition region. Since $V_1(\lambda)$ has a first-order zero at $\lambda = a$ in the planar limit, we may approximate it by a linear function in the transition region, since the width of this region vanishes in the planar limit. We rescale λ and ρ as follows:

$$\lambda - a = N^{-2/3}[8ag^2(b^2 - a^2)^2]^{-1/3} y \tag{33a}$$

$$\rho = \frac{1}{\pi} N^{-1/3}[8ag^2(b^2 - a^2)^2]^{1/3} \eta(y)^{-2} \tag{33b}$$

The eq.(10a) then becomes

$$(-\frac{\partial^2}{\partial y^2} + y + \eta^{-4})\eta \approx 0 \tag{33c}$$

The boundary condition for this differential equation is furnished by the approximate solution inside the sea, eqs.(31b) and (32c), which become in terms of the rescaled variables,

$$\eta(y) \approx (-y)^{-1/4} \quad at\ y < 0,\ |y| >> 1 \tag{33d}$$

As $N \to \infty$, the rescaled equation (33c) becomes exact, and (33d) becomes an exact boundary condition in the $y \to \infty$ limit. On the exterior side of the transition region, the asymptotic behavior is

$$\eta(y) \approx const\ y^{-1/4} e^{(2/3)y^{3/2}} \quad at\ y > 0,\ |y| >> 1 \tag{33e}$$

In agreement with the exterior WKB solution, (29b).

For fixed coupling g, the width of the transition region is $O(N^{-2/3})$. When g is, instead, continued to the critical point g_c in accordance with the *d.s.l.*, we find from eqs.(27)

$$b^2 - a^2 = O((g - g_c)^{1/2}) \tag{34a}$$

so by (33a) and (28b), the width of the transition region is of order $(g - g_c)^{1/2}\kappa^{2/3}$. Using eq.(33b), we also find that the normalized eigenvalue population in the transition region, is of order

$$\int\limits_{transition} \rho d\lambda = O(\frac{1}{N}), \tag{34b}$$

either for g fixed, or in the *d.s.l.*. This means that of the original N matrix eigenvalues, on the order of *one* eigenvalue are likely to inhabit the transition region.

4.c. Quantum Corrections in d.s.l. In part 4.a, we used both the $1d$ differential equation and the integro-differential equation for $\rho(\lambda)$, to find the perturbative expansion for the moments m_n of ρ. A similar procedure can be employed for the propagator $H(\lambda, \lambda')$, by using the $1d$ Green's equation (24), and the corresponding integro-differential equation (23). In this case, one finds $1/N$ expansions for the one-sided moments $h_n(\lambda)$, defined in (24b). The moments h_1 and h_2 can then be substituted in eq.(25). We find, to the leading approximation for large N,

$$Q(\lambda|\lambda') \approx \frac{1}{2\pi}\frac{\partial}{\partial\tau'}\big(\rho(\lambda')\frac{\lambda + \lambda'}{b^2 - \lambda'^2}\big) \tag{35}$$

This, in turn, can be used in eq.(26). The unknown functions A,B are determined as explained in section 3.

The results of this analysis, are as follows*. When λ and λ' are both interior

$\star$ The details will be presented in a separate publication, as will the computation of the determinant factor $\det K$.

to the sea, and $\lambda \neq \lambda'$, we have

$$H(\lambda, \lambda') = O(\frac{1}{N}) \quad |\lambda| < a,\ |\lambda'| < a \tag{36a}$$

The function $w_1(\lambda)$ (eqs.(22)), the regular piece of $H(\lambda, \lambda')$ as $\lambda' \to \lambda$, is $O(1)$, and in the sea interior it is approximated thus:

$$w_1(\lambda) \approx \ln \rho(\lambda) + \ const \quad (|\lambda| < a) \tag{36b}$$

In other regions of λ and λ', these functions have different behaviors. For instance, when λ,λ' are both in the same transition region,

$$H(\lambda, \lambda') = O(1) \quad (\ \lambda \approx a,\ \lambda' \approx a\ ) \tag{36c}$$

However, the contribution to Feynman diagrams from vertices in a transition region, is still suppressed by a power of $1/N$ for each such vertex, due to eq.(34b). Combining the behavior of ρ and H in the various regions with eq.(22), we find that only the first few Feynman diagrams survive in the *d.s.l.*. We are referring to diagrams resulting from contractions among normal-ordered vertices; recall that an infinite number of normal-ordering contractions have been summed to obtain eq.(22).

4.d. Instanton Configuration in Direct d.s.l. Rather than continuing $V(\lambda)$ to criticality *after* solving for ρ and H, it is instructive to attempt taking the *d.s.l.* directly. This will demonstrate what nonperturbative effects look like in the conjugate-field formalism, although a trustworthy nonperturbative calculation requires a potential $V(\lambda)$ bounded from below.

When we use the quartic potential (10d) with negative g, b^2 is positive (part 4.a). As g approaches g_c from the origin along the real axis ($g \approx g_c$, $g > g_c =$

$-1/48$), b approaches a thus:

$$b > a \quad , \; b - a = O((g - g_c)^{1/2}) \tag{37}$$

Thus by (27a), V_1 has two small 'seas', concave regions just outside the main Dyson sea, where it is negative. Unlike $V(\lambda)$, V_1 *is* bounded from below. Thus we can attempt to solve the nonlinear Schrödinger equation (10a) near criticality, ignoring the fact that the solution will not solve the integro-differential equation†. We shall continue to use eq.(30c) for δm_2, in eq.(27a). the justification is that, assuming oscillations inside the sea are still suppressed (eq.(32c)), substitution of (31b) in (10f) for $n = 0$ and $n = 2$ indeed yields (30c), at least to the approximation needed to take the *d.s.l.*‡§ .

We concentrate on the behavior of $V_1(\lambda)$,$\rho(\lambda)$ in the new transition region; we invoke symmetry again and concentrate on the $\lambda \approx a$ region. To that end, we magnify the region via a new rescaling, different from (33):

$$\lambda - 2\sqrt{2} = 4\sqrt{3}(g - g_c)^{1/2} x \tag{38a}$$

$$\rho = \frac{1}{\pi}(g - g_c)^{3/4} \zeta(x)^{-2} \tag{38b}$$

where $2\sqrt{2}$ is the value of a at $g = g_c$. The Schrödinger equation in this region assumes the form,

$$[-\frac{1}{48}\kappa^2 \frac{\partial^2}{\partial x^2} + (\frac{64\sqrt{8}}{\sqrt{3}}(x + \sqrt{2})(x - \frac{1}{\sqrt{2}})^2 - \kappa\frac{2\sqrt{2}}{3^{1/4}}) + \zeta^{-4}]\zeta \approx 0 \tag{38c}$$

This becomes exact in the *d.s.l.*. The old transition region occurs at $x + \sqrt{2} = O(\kappa^{2/3})$, and is thus part of the new transition region. We are assuming that the string coupling κ is small, in order to isolate the leading nonperturbative effect.

† This is similar to the Marinari-Parisi approach to rendering the even-k models well-defined.

‡ This argument, as opposed to the one in 4.a, does not use information about large $|\lambda|$. Such information cannot be trusted, as V is unbounded from below there.

§ The transition region contributes $O(1/N)$ to both m_0 and m_2, so to eliminate these contributions we computed the subtracted $m_2 - a^2 m_0$.

To the right of the old transition region, V_1 is positive for

$$-\sqrt{2} < x < \frac{1}{\sqrt{2}}$$

and ρ is exponentially suppressed. The extra new minimum of V_1 is at $x = 1/\sqrt{2}$, and the new sea surrounding it, where $V_1 < 0$, has width $\Delta x = O(\kappa^{1/2})$ and depth $O(\kappa(g-g_c)^{3/2})$.

The resulting *d.s.l.* solution for ρ in the new transition region, has the following properties. Between $x \approx -\sqrt{2}$ and $x \approx \frac{1}{\sqrt{2}}$, $V_1 > 0$ and ρ tunnels in accordance with (29b). ρ is thus suppressed in the small new sea, with the WKB suppression factor (up to prefactors which can be found)

$$\Lambda^2 \equiv \exp\Big(-\frac{4\sqrt{6}}{5\kappa}(48)^{5/4}\Big) \tag{39a}$$

which is precisely the tunneling factor appearing in the nonperturbative ambiguity for pure gravity, using the various previous approaches [19]* .

In the new sea, we find that $\zeta(x)$ has the following form:

$$\zeta(x) \approx (\frac{3}{\kappa})^{1/4}\sqrt{8}e^{-2\sqrt{2}z^2}\Big(d_1\Lambda^{-1} - d_2\kappa^{1/2}\Lambda\int\limits_0^z e^{4\sqrt{2}\bar{z}^2}d\bar{z}\Big) \tag{39b}$$

where z is yet a *third* rescaled eigenvalue variable, appropriate for the extra sea:

$$x - \frac{1}{\sqrt{2}} \equiv \kappa^{1/2}(48)^{-3/8}z \tag{39c}$$

When the *d.s.l.* limit is taken, κ small and z held fixed, the corrections to (39b) are higher powers of κ. In eq.(39b), d_1,d_2 are two pure numbers, obtained by matching (39b) at large and negative z with the WKB approximation, (29b), which holds between the two seas.

$\star$ In this connection, note that our normalization for κ differs from the one usually employed in the literature.

The configuration given by eqs.(38-39) can be interpretted as an instanton. For well-defined models, there exist similar instantons which are true classical solutions of the conjugate-field theory.

In the configuration discussed above, the center $z = 0$ of the new sea is a local minimum of ρ. ρ then increases with z for $z > 0$. When z is sufficiently large, one exits the new sea and enters another $V_1 > 0$ region, where the exponentially suppressed terms in eq.(29b) dominate for a while; this allows ρ to continue to increase. Eventually the dominant term will again dominate, but if ρ^2 has by then increased to become comparable in magnitude to V_1, a nonperturbative solution of (10a) is needed. We need not worry about this exterior region, however, since in this model ρ is not a trustworthy configuration there.

5. Conclusions.

We have presented a new field-theory formulation of $D = 0$ matrix models. The field is conjugate to the Jevicki-Sakita collective field, i.e. conjugate to the density of matrix eigenvalues. The theory is two dimensional, with an eigenvalue coordinate and an auxiliary coordinate that can be eliminated from the formalism. The action is nonlocal, but the equation of motion is local, except for a self-consistency condition. There is a unique or discretely labeled classical solution for any well-defined potential. The equation for the classical eigenvalue distribution is a modified version of the planar integral equation of Bessis, Itzykson and Zuber, with an entropy term that smoothes the edges of the Dyson sea and introduces higher-genus effects already at the classical level. Single-band classical solutions are perturbatively unique. The classical distribution also satisfies a nonlinear Schrödinger equation, with a potential similar to, but different from the one appearing in the Marinari-Parisi $D = 1$ reformulation.

The classical solutions, and the quantum corrections about them, are systematically calculable, and all divergences (UV- and IR-) cancel manifestly to all orders.

The normal-ordering graphs can be summed exactly to all orders. In the double scaling limit, the formalism simplified drastically.

The conjugate-field formalism can be used to systematically compute string-nonperturbative effects. We demonstrate this for the ill-defined, but simple, case of $k = 2$ realized with a quartic potential. In this case, the classical distribution contains two small seas on either side of the main Dyson sea. The population of the new seas is exponentially suppressed by the same tunneling factor as appears in other approaches. A more complete presentation of the conjugate-field formalism will be presented in a forthcoming publication.

APPENDIX

In this appendix, we derive the Schrödinger equation (10) from the integro-differential equation, eq.(9b), and prove that a classical solution does not have any normalization preserving zero modes.

First, we list a few useful properties of the Hilbert transform (which in our normalization is given by eq.(9c)). For any function $f(\lambda)$,

$$\mathcal{H}(f'(\lambda)) = \frac{d}{d\lambda}\mathcal{H}(f(\lambda)) \qquad (A.1)$$

Also, for any two functions u and v, one easily finds*:

$$\mathcal{H}(u\mathcal{H}(v) + v\mathcal{H}(u)) = \mathcal{H}(u)\mathcal{H}(v) - \pi^2 uv \qquad (A.2)$$

Actually, for the present derivation, we only need the degenerate form of this identity when $v = u$:

$$2\mathcal{H}(u\mathcal{H}(u)) = \mathcal{H}(u)^2 - \pi^2 u^2 \qquad (A.3)$$

But the more general (A.2) is needed to derive the $1d$ differential equation for the propagator $H(\lambda, \lambda')$, eq.(24), from the corresponding integro-differential equation

$\star$ By using the fact that the real and imaginary parts of an analytic function, having all its singularities in the lower half of the complex plane, are related by the Hilbert transform.

(23). We will not go through this latter derivation, but the procedure parallels that for ρ.

Differentiating eq.(9b) w.r.t. λ and using (A.1), we obtain

$$V'' + \frac{d}{d\lambda}(\frac{1}{N}\frac{\rho'}{\rho}) = 2\mathcal{H}(\rho') \tag{A.4}$$

Now we use (9b) again to rewrite ρ' on the RHS of (A.4), and also use eq.(A.3). Upon rearranging terms and using the definition (9c), the nonlinear Schrödinger equation, given by eqs.(10), is obtained.

Next, let $\delta\rho$ be an infinitesimal zero-mode of ρ which preserves the normalization condition (9d). Thus

$$\int d\lambda \delta\rho(\lambda) = 0 \tag{A.5}$$

and by varying (9b) we find a linear integro-differential equation for ρ:

$$\frac{1}{N}(\frac{\delta\rho}{\rho})' = 2\mathcal{H}(\delta\rho) \tag{A.6}$$

Let us define the function $\eta(r)$ in the two-dimensional space, as follows: on the eigenvalue axis, it is defined as

$$\eta(\lambda) \equiv \frac{\delta\rho}{\rho} \tag{A.7}$$

and off that axis, it is defined by continuation of (A.6):

$$\eta(r) = const\ + 2N \int \eta(\mu)\rho(\mu) \ln|r - r'| d\mu \tag{A.8}$$

where $r' = (\mu, 0)$ runs over the eigenvalue axis in the integral. The additive constant in eq.(A.8) is determined by the condition (A.5), which we rewrite as

$$\int \eta(\lambda)\rho(\lambda)d\lambda = 0 \tag{A.9}$$

From (A.8) and (A.9) we find that

$$\eta(r) \to const \ + O(\frac{1}{|r|}) \quad as\ |r| \to \infty \tag{A.10}$$

But by eq.(A.8), η satisfies the $2d$ linear differential equation

$$\partial^2\eta(r) = 4\pi N\delta(x)\rho(\lambda)\eta(r) \tag{A.11}$$

where $r = (\lambda, x)$. This differential equation has a unique solution subject to the boundary condition (A.10). We have thus proven that a classical solution has no normalization-preserving zero modes.

ACKNOWLEDGEMENTS

I would like to thank Adrian Cooper, Lenny Susskind and Larus Thorlacius for useful discussions and comments, and Stephen Shenker for discussions concerning instantons and eigenvalue tunneling.

REFERENCES

1. M. Douglas and S. Shenker, *Nucl.Phys.* **B335**(1990), 635 .
2. D.J. Gross and A.A. Migdal, *Phys.Rev.Lett.* **64**(1990), 717.
3. E. Brezin and V.A. Kazakov, *Phys.Lett.* **B236**(1990), 144.
4. E. Brezin, V.A. Kazakov and A.B. Zamolodchikov,, *Nucl.Phys.* **B338**(1990), 673.
5. D.J. Gross and N. Miljkovic, *Phys.Lett.* **B238**(1990), 217 .
6. P. Ginsparg and J. Zinn-Justin, *Phys.Lett.* **B240**(1990), 333.
7. D.J. Gross and I.R. Klebanov, *Nucl.Phys.* **B344**(1990), 475 .
8. I.R. Klebanov and R.B. Wilkinson, Princeton preprint PUPT-1188(1990).
9. V. Knizhnik, A. Polyakov and A. Zamolodchikov, *Mod.Phys.Lett.* **A3**(1988), 819
10. F. David, *Mod.Phys.Lett.* **A3**(1988), 1651;
    J. Distler and H. Kawai, *Nucl.Phys.* **B321**(1989), 509.
11. J. Polchinski, Texas preprint UTTG-19-90.
12. J. Polchinski, *Nucl.Phys.* **B346**(1990), 253 .
13. Z. Yang, *Phys.Lett.* **B255**(1991), 215 .
14. A. Gupta, S.P. Trivedi and M.B. Wise, *Nucl.Phys.* **B340**(1990), 475.
15. I. Kostov, *Phys.Lett.* **B215**(1988), 499.
16. D.J. Gross, I.R. Klebanov and M.J. Newman, *Nucl.Phys.* **B350**(1991), 621.
17. M. Bershadsky and I.R. Klebanov, *Phys.Rev.Lett.* **65**(1990), 3088.
18. S. Ben-Menahem, *Nucl.Phys.* **B364**(1991), 681.
19. S. Shenker, Rutgers preprint RU-90-47(90), and references within.
20. J.Luis Miramontes and J.S. Guillen, CERN-TH. 6323/91.

21. F. David, *Mod.Phys.Lett.* **A5**(1990), 1019

22. F. David, *Nucl.Phys.* **B348**(1991), 507

23. J. Greensite and M.B. Halpern, *Nucl.Phys.* **B348**(1991), 507

24. E. Marinari and G. Parisi, *Phys.Lett.* **B240**(1990), 375

25. J. Ambjorn, J. Greensite and S. Varsted, *Phys.Lett.* **B249**(1990), 411

26. M. Karliner and A.A. Migdal, *Mod.Phys.Lett.* **A5**(1990), 2565

27. J. Ambjorn, C.V. Johnson and T. Morris, Southampton and Niels Bohr Inst. Preprint SHEP 90/91-29, NBI-HE-91-27.

28. S. Dalley, C.V. Johnson and T. Morris, SHEP 90/91-28 and SHEP 90/91-35.

29. K. Demeterfi, N. Deo, S. Jain and C.-I. Tan, *Phys.Rev.* **D42**(1990), 4105.

30. G. Bhanot, G. Mandal and O. Narayan, *Phys.Lett.* **B251**(1990), 388.

31. J. Feinberg, TECHNION-PH-92-1.

32. S.R. Das and A. Jevicki, *Mod.Phys.Lett.* **A5**(1990), 1639.

33. K. Demeterfi, A. Jevicki and J.P. Rodrigues, *Mod.Phys.Lett.* **A6**(1991), 3199 , and references therein.

34. D. Karabali and B. Sakita, *Int.J.Mod.Phys.* **A6**(1991), 5079.

35. A. Jevicki, BROWN-HET-807.

36. J. Avan and A. Jevicki, *Phys.Lett.* **B272**(1991), 17.

37. A.M. Sengupta and S.R. Wadia, *Int.J.Mod.Phys*, **A6**(1991), 1961 .

38. G. Moore, RU-91-12.

39. D.J. Gross and I.R. Klebanov, *Nucl.Phys.* **B359**(1991), 3.

40. A. Jevicki and B. Sakita, *Nucl.Phys.* **B185**(1981), 89.

41. A. Jevicki, *Nucl.Phys.* **B146**(1978), 77.

42. B. Sakita, "Quantum Theory of Many-Variable Systems and Fields", World Scientific, Singapore, 1985.

43. J.D. Cohn and S.P. de Alwis, COLO-HEP-247, IASSNS-HEP-91/7.

44. A. Jevicki, BROWN-HET-777.

45. O. Lechtenfeld, IASSNS-HEP-91/86.

46. D. Bessis, C. Itzykson and J.B. Zuber, *Advances in Applied Math.* **1**(1980), 109.

47. For a discussion of the transition regions using the standard Gel'fand-Dikii formalism, see: M.J. Bowick and E. Brezin, *Phys.Lett.* **B268**(1991), 21.