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# Shielded Coherent Synchrotron Radiation and Its Effect on Very Short Bunches*

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## Abstract

Shielded coherent synchrotron radiation is discussed for two cases: (1) a beam following a circular path midway between two parallel conducting plates, and (2) a beam circulating in a toroidal chamber. Wake fields and the energy radiated are computed for both cases. Under conditions like those of the high-energy bunch compressor of the Next Linear Collider (NLC), in which bunches as short as 40 microns are contemplated, the shielded coherent radiated power is estimated to be small compared to the incoherent power, but can still amount to a few hundred KeV over the compressor arc.

## 1. Introduction

Particles in a bunch following a curved path may radiate coherently at a wavelength comparable to the size of the bunch. The radiated power is proportional to the square of the current, hence proportional in this event to $N^2$, where $N$ is the number of particles in the bunch. The power of incoherent radiation will vary only as $N$, since each particle contributes a power proportional to its own squared current. In typical conditions in electron storage rings, most of the radiation is incoherent, peaking at frequencies far beyond the microwave region. In an experiment using short bunches from a linac, Nakazato et al.,[1] observed coherent radiation, with a clear indication of a transition from $N$ to $N^2$ dependence.

Fortunately for the operation of electron rings, shielding provided by metallic walls of the vacuum chamber greatly reduces the amount of coherent radiation. This effect was recognized in the early days of circular electron accelerators and has been studied theoretically from time to time ever since. In a simple, but relevant model first studied by Schwinger[2] and Nodvick and Saxon,[3] one considers a beam following a circular trajectory in a plane midway between two infinite, parallel, conducting planes, separated by a distance $h = 2g$. According to work of Faltens and Laslett[4] the real part of the longitudinal coupling impedance for this situation should satisfy roughly the condition

$$\max_n \left[\frac{\mathrm{Re}\ Z(n, n\omega_o)}{n}\right] = 300\frac{g}{R} \ \text{ ohms} \ , \tag{1.1}$$

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where $n$ is the azimuthal mode number, $R$ is the trajectory radius, and $\omega_o = \beta c/R$ is the revolution frequency. As we shall see presently, Re$Z/n$ is negligible below a certain threshold, then rises rather sharply to this maximum, and henceforth falls slowly. The threshold can be estimated as

$$n = \pi(R/h)^{3/2} \quad . \tag{1.2}$$

In terms of the frequency $\omega = \omega_o n$, or normalized wavelength $\lambda\!\!\!^{-} = \lambda/2\pi$, the same criterion is

$$\omega h/c = \pi\beta(R/h)^{1/2} \quad , \qquad \lambda\!\!\!^{-} = (h/R)^{1/2}h/\pi \quad . \tag{1.3}$$

For typical $R/h$, with $h$ being the transverse dimension of the vacuum chamber, this threshold is at a much higher frequency than the "waveguide cutoff" for propagation parallel to the plates, which lies near $\omega h/c = 1$.

The radiated power is determined by the impedance and the Fourier spectrum of the bunch. If the bunch is rigid (i.e., maintains a constant form in a rest frame) and the particles radiate coherently, then the energy change per radian on the circular path is

$$dU/d\theta = -q^2\omega_o \sum_{n=-\infty}^{\infty} |\lambda_n|^2 \mathrm{Re}\ Z(n, n\omega_o) \quad , \tag{1.4}$$

where $q$ is the total charge in the bunch and $\lambda_n$ is the Fourier transform of the longitudinal charge distribution $\lambda(\theta)$,

$$\lambda_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\lambda(\theta)d\theta \quad . \tag{1.5}$$

If the bunch has significant Fourier components near the maximum (1.1) of the impedance, the radiated power is quite large, as the following discussion will show. This situation usually does not occur in storage rings, since the bunch is long compared to the wavelength at the maximum impedance. On the other hand, in the last stages of bunch compression in the NLC, the bunches are sufficiently short to produce appreciable coherent radiation, provided that the impedance resembles that of the parallel plate model.

Section 2 gives definitions and formulas for the impedance, and derivations of expressions for the energy loss and wake field. The impedance for the toroidal chamber is taken from Ref. 5; that for the parallel plates is obtained by taking a limit of a formula in the same paper. Numerical examples are presented in Section 3 for parameters appropriate to four different designs of the NLC bunch compressor.

The notation and point of view will be the same as in Ref. 5. The reader may refer to that paper for details of technique not discussed here, in particular, for the numerical treatment of high-order Bessel functions.

## 2. Impedance, Wake Field, and Energy Loss

We work in cylindrical coordinates $(r, \theta, z)$, and suppose that the centroid of the bunch follows an orbit in the plane $z = 0$. For the present discussion, the orbit is circular; slightly different ideas may be required for single-pass orbits. It is convenient to make a Fourier analysis of the longitudinal electric field with respect to $\theta$ and the time $t$:

$$E_\theta(r,\theta,z,t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \sum_{n=-\infty}^{\infty} e^{in\theta}\, E_{\theta n}(r,z,\omega) \quad . \tag{2.1}$$

Since we are interested primarily in longitudinal effects, we consider the average of this field over $r$ and $z$, weighted with the transverse charge distribution of the beam. Specializing to the case of a rigid bunch, we suppose that the charge and current densities have the form

$$\rho(r,\theta,z,t) = q\lambda(\theta - \omega_o t) f(r,z) \quad , \qquad J_\theta(r,\theta,z,t) = \omega_o r \rho(r,\theta,z,t) \quad , \tag{2.2}$$

where

$$\int_0^{2\pi} \lambda(\theta) d\theta = 1 \quad , \qquad \int r dr \int dz f(r,z) = 1 \quad . \tag{2.3}$$

Here and in the following, the $(r,z)$ integrals extend over the support of $f$. The field written without arguments $(r,z)$ will be understood as the transverse average,

$$E_\theta(\theta,t) = \int r dr \int dz f(r,z) E_\theta(r,\theta,z,t) \quad . \tag{2.4}$$

The Fourier transform of $\lambda(\theta - \omega_o t)$, defined in analogy to (2.1), is $\lambda_n\, \delta(\omega - \omega_o n)$. The corresponding transform of the current is

$$I_\theta(n,\omega) = \int dr \int dz J_\theta(r,n,z,\omega) = \omega_o q \lambda_n \delta(\omega - \omega_o n) \quad . \tag{2.5}$$

If the environment of the beam has no longitudinal inhomogeneity, as in the models treated below, then there exists a complex function $Z(n,\omega)$, the longitudinal coupling impedance, such that

$$-2\pi R E_\theta(n,\omega) = Z(n,\omega) I_\theta(n,\omega) \quad . \tag{2.6}$$

If the environment is inhomogeneous, perhaps because of cavities in the vacuum chamber, this single impedance function is replaced by a matrix, $Z(n,n',\omega)$, since many harmonics of the current contribute to the excitation of one harmonic of the field.

The use of the weighted average (2.4) in the definition of the impedance is not conventional; usually, the simple average or merely the value of the field at the center of the beam is used. The weighted average seems quite natural, however, and we shall see that it leads to a cleaner derivation of the formula for energy loss than would otherwise be possible. If $f(r,z) = W(r)H(z)$, where $rW(r)$ and $H(z)$ are rectangular step functions or delta functions, then the simple average coincides with (2.4). In Ref. 5, $f$ had such a form and the impedance was defined with the simple average; consequently, the impedance obtained in Ref. 5 can be used in our present formulas.

The wake field $E_\theta(\omega_o(t+\tau),t)$ is defined as the field on the trajectory at an angular distance $\omega_o\tau$ in front of the bunch center. Taking the weighted average of (2.1) over $r$ and $z$, and introducing (2.6) and (2.5), we see that the wake field is given by the Fourier transform of the impedance, weighted by the transform of the bunch:

$$\mathcal{V}(\omega_o\tau) = -2\pi R E_\theta(\omega_o(t+\tau),t) = \omega_o q \sum_{n=-\infty}^{\infty} e^{in\omega_o\tau}\lambda_n Z(n,n\omega_o) \ . \tag{2.7}$$

The function $\mathcal{V}$ is sometimes called the *wake potential.* It is the "wake voltage per turn"; a positive value of $\mathcal{V}(\omega_o\tau)$ means energy loss by particles at a distance $\omega_o\tau$ from the center of the bunch.

If the beam environment is not homogeneous, the formula replacing (2.7) is

$$\mathcal{V}(\omega_o\tau,t) = \omega_o q \sum_{n=-\infty}^{\infty} e^{in\omega_o\tau} \sum_{n'=-\infty}^{\infty} Z(n,n',\omega_o n')\lambda_{n'}e^{i(n-n')\omega_o t} \ . \tag{2.8}$$

average over one period $T = 2\pi/\omega_o$ is given by formula (2.7), since the averaging sets $n = n'$.

Let us now compute the radiated power. The change in energy in time $dt$ is the work done by the field $E_\theta$,

$$\begin{aligned} dU &= \int\int\int E_\theta(r,\theta,z,t)\left[\rho(r,\theta,z,t)rdrd\theta dz\right]\left[\omega_o r dt\right] \\ &= q\omega_o \int\int\int r^2 drdz f(r,z)E_\theta(r,\theta,z,t)\lambda(\theta-\omega_o t)d\theta dt \ . \end{aligned} \tag{2.9}$$

For the integration over $r$, we note that $r^2$ varies almost linearly over the extent of the beam, which is tiny compared to $R$. Thus, $r^2 \approx Rr$ and the average (2.4) appears. The power is

$$\begin{aligned} dU/dt &= q\omega_o R \int d\theta E_\theta(\theta,t)\lambda(\theta-\omega_o t) \\ &= q\omega_o R \int d\theta \int d\omega e^{-i\omega t}\sum_n e^{in\theta}E_\theta(n,\omega)\sum_{n'} e^{in'(\theta-\omega_o t)}\lambda_{n'} \ . \end{aligned} \tag{2.10}$$

If we now substitute (2.6), and carry out the $\theta$ and $\omega$ integrals we obtain

$$dU/dt = -(q\omega_o)^2 \sum_{n=-\infty}^{\infty} |\lambda_n|^2 \mathrm{Re}\ Z(n,n\omega_o) \ . \tag{2.11}$$

We have used the properties $\lambda_{-n} = \lambda_n^*$ and $Z(-n,-n\omega_o) = Z(n,n\omega_o)^*$. Notice that the latter follows directly from the definition (2.6) and the corresponding reflection property of $E_\theta(n,\omega)$ and $I(n,\omega)$. For a delta function bunch, $\lambda_n = 1/2\pi$ and

$$dU/dt = -I^2 \sum_{n=-\infty}^{\infty} \text{Re } Z(n, n\omega_o) \quad , \tag{2.12}$$

possible impedances, "connected in series."

It remains to give formulas for the coupling impedance for the two models to be explored. In the first model we have infinite, parallel, perfectly conducting plates, separated by a distance $h = 2g$; the beam circulates in the median plane between the plates. As in Ref. 5, the boundary conditions on the plates are enforced by expanding the fields in Fourier series in $z$; for instance,

$$E_\theta(r, n, z, \omega) = \sum_{p=1}^{\infty} \sin \alpha_p(z+g) E_\theta(r, n, p, \omega) \quad , \tag{2.13}$$

where $\alpha_p = \pi p/h$. Correspondingly, the $z$-dependence of charge and current densities must be represented by Fourier series. The $r$-dependence of fields is expressed in terms of Bessel functions of order $n$.

In order to obtain the impedance in a convenient analytic form, we choose a simple form for the transverse beam profile:

$$f(r,z) = W(r)H(z) \quad , \qquad W(r) = \delta(r-R)/R \quad . \tag{2.14}$$

With a little numerical work, one could accomodate a function $W(r)$ with a finite width. The impedance comes out in terms of the Fourier transform of $H(z)$. Again, to get a simple formula, we take $H(z)$ to be a rectangular step function, symmetrical about $z = 0$, but any other choice could be treated easily.

With the beam profile as stated, the impedance for the parallel-plate problem is given by

$$\begin{aligned} \frac{Z(n,\omega)}{n} = & \frac{2\pi^2 Z_o}{\beta} \frac{R}{h} \sum_{p(odd)\geq 1} \Lambda_p \Big[ \frac{\beta\omega R}{nc} J_n'(\gamma_p R)\big((J_n'(\gamma_p R) + iY_n'(\gamma_p R)\big) \\ & + \big(\frac{\alpha_p}{\gamma_p}\big)^2 J_n(\gamma_p R)\big(J_n(\gamma_p R) + iY_n(\gamma_p R)\big) \Big] \quad , \end{aligned} \tag{2.15}$$

where

$$\gamma_p^2 = (\omega/c)^2 - \alpha_p^2 \; , \quad \Lambda_p = (\sin x/x)^2 \; , \quad x = \pi p \delta h/2h \quad , \tag{2.16}$$

the vertical size of the beam being $\delta h$. The impedance is in ohms with $Z_o = 120\,\pi\ \Omega$.

In the second model the vacuum chamber is a torus of rectangular cross section. The cross section has height $h = 2g$ and width $w$. The inner and outer torus radii are $a$ and $b$, respectively. The radial wave functions are expressed in terms of cross products of Bessel functions, defined as follows:

$$\begin{aligned} p_n(x,y) &= J_n(x)Y_n(y) - Y_n(x)J_n(y) \ , \\ s_n(x,y) &= J_n'(x)Y_n'(y) - Y_n'(x)J_n'(y) \ . \end{aligned} \tag{2.17}$$

For perfectly conducting walls, the impedance of the torus is given by

$$\begin{aligned} \frac{Z(n,\omega)}{n} = \frac{2i\pi^2 Z_o}{\beta} \frac{R}{h} \sum_{p(odd)\geq 1} \Lambda_p \Bigg[ & \frac{\beta\omega R}{nc} \frac{s_n(\gamma_p b, \gamma_p R) s_n(\gamma_p R, \gamma_p a)}{s_n(\gamma_p b, \gamma_p a)} \\ + & \left(\frac{\alpha_p}{\gamma_p}\right)^2 \frac{p_n(\gamma_p b, \gamma_p R) p_n(\gamma_p R, \gamma_p a)}{p_n(\gamma_p b, \gamma_p a)} \Bigg] \ . \end{aligned} \tag{2.18}$$

The expression (2.18) has poles corresponding to the resonances of the closed, perfectly conducting structure, and is imaginary wherever it is finite. When wall resistance is introduced, the function acquires a real part and the poles are replaced by sharp peaks in the real part. Our calculations include wall resistance, following the theory given in Ref. 5.

The formula (2.15) for parallel plates can be derived immediately from a result of Ref. 5; namely, the formula for impedance of a beam circulating in a cylindrical pillbox of radius b. One merely takes the limit for $b \to \infty$. This is best done using the form appropriate to the low-frequency region, in which Bessel functions are replaced by modified Bessel functions [see Eq. (4.3) of Ref. 5]. The latter have simple exponential behavior in the relevant limit. Analytic continuation through the upper half $\omega$ plane produces the high-frequency form (2.15).

To close this section we recall the well known formula for the total power from incoherent synchrotron radiation.[6] For a single electron,

$$\frac{dU}{dt} = \frac{1}{4\pi\epsilon_o} \frac{2e^2\beta^4\gamma^4}{3R^2} \ . \tag{2.19}$$

## 3. Numerical Examples of Wake Field and Energy Loss

For calculations, we take a Gaussian bunch with length $\sigma$:

$$\lambda(\theta) = \frac{1}{\sqrt{2\pi}} \frac{R}{\sigma} \exp\left[-\frac{1}{2}\left(\frac{R\theta}{\sigma}\right)^2\right] , \qquad \lambda_n = \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left(\frac{n\sigma}{R}\right)^2\right] \ . \tag{3.1}$$

A wave moving with phase velocity equal to the particle velocity at $r = R$, i.e., with frequency $\omega = \omega_o n$, has wavelength $2\pi R/n$. The relevant length for our discussion is wavelength over $2\pi$,

$$\lambda\!\!\!^{-} = R/n \ . \tag{3.2}$$

**Table I:** Energy losses for four versions of the NLC bunch compressor. The values of energy loss $\Delta U$ are for a bunch containing $N = 2 \cdot 10^{10}$ electrons.

| Version | V.1 | V.2 | V.3 | V.4 |
|---|---|---|---|---|
| $R$(m) | 84.218 | 213.6 | 149.5 | 106.8 |
| $\Delta\theta$(degrees) | 10.89 | 180 | 180 | 180 |
| $\sigma_i$(microns) | 460 | 460 | 460 | 460 |
| $\sigma_f$(microns) | 37 | 86 | 61 | 44 |
| $\Delta U$(plates) MeV | 0.176 | 0.164 | 0.827 | 2.08 |
| $\Delta U$ (torus) MeV | 0.123 | 0.0121 | 0.337 | 1.30 |
| $\Delta U$(incoherent) MeV | 2.19 | 14.2 | 20.4 | 28.5 |

In terms of $\lambda\!\!\!^{-}$ the spectral density of the bunch that appears in in the power formula (2.11) is

$$|\lambda_n|^2 = \frac{1}{(2\pi)^2} \exp\left[-\left(\frac{\sigma}{\lambda\!\!\!^{-}}\right)^2\right] \quad . \tag{3.3}$$

We illustrate with values of $\sigma$ and $R$ from four different conceptual designs for the NLC high-energy (16.2 GeV) bunch compressor [7]. Since the expected beam pipe radius is about 1 cm, the plate separation or torus height $h$ will be 2 cm, and the torus width $w$ also 2 cm. The beam height $\delta h$, not a significant parameter, will be 1 mm. Table I shows the compressor parameters, including the initial and final bunch lengths, $\sigma_i$ and $\sigma_f$, and the total deflection angle $\Delta\theta$ of the compressor arc.

Figure 1 shows a typical graph of the real part of $Z(n, n\omega_o)/n$, for the parallel-plate model including only the $p = 1$ term in Eq. (2.15); the parameters are $R = 149.5$ m and $h = 2$ cm, for Version 3 of Table I. At the frequencies of the plot, the higher axial modes $p = 3, 5, \cdots$ make a negligible contribution. The Faltens-Laslett estimate (1.1) of the peak value is confirmed. According to the estimate (1.2), Re$Z/n$ should first have significant magnitude around $n = 2 \cdot 10^6$. Indeed, it first reaches half-maximum at about that point. Figure 2 shows Re$Z$ plotted as a function of $\lambda\!\!\!^{-}$. From the graph and (2.11), (3.3), we see that significant power will be radiated only for bunch lengths less than 90 $\mu$ or so with $R$ and $h$ as in Version 3. For longer bunches the relevant impedance is too low.

Figure 3 shows Re$Z/n$ for an aluminum toroidal chamber with parameters for Version 3. The beam is centered in the chamber. The threshold at which the impedance

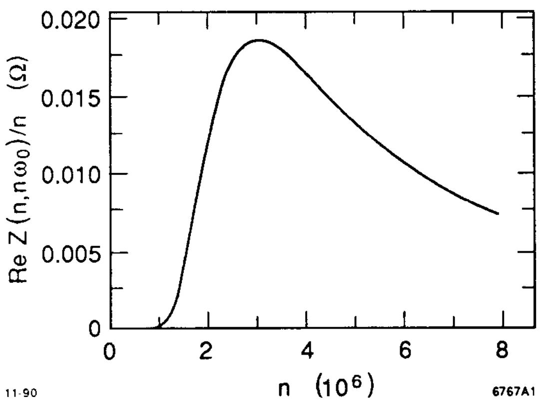


**Figure 1:** $\mathrm{Re}Z(n, n\omega_o)/n$ versus $n$ for the parallel-plate model, $R = 149.5$m, $h = 2$cm

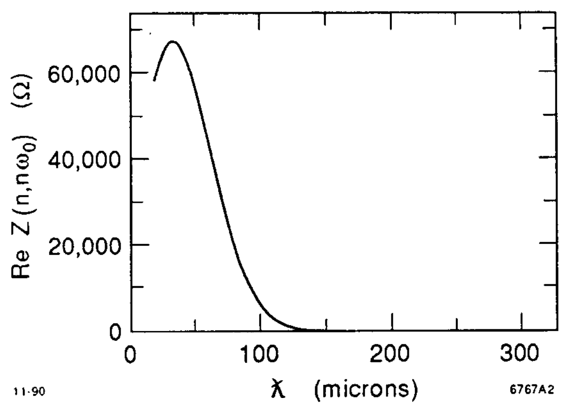


**Figure 2:** $\mathrm{Re}Z(n, n\omega_o)/n$ versus $\lambda\!\!\!^{-}$ for the parallel-plate model, $R = 149.5$m, $h = 2$cm.

is first appreciable is a bit higher than in the parallel-plate model, but the maximum occurs at about the same point. The resistive wall theory is somewhat defective, in that Re$Z$ is negative at some distance from either resonance peak. In the calculation, we put Re$Z = 0$ wherever the theory gives negative values.

To estimate the total energy loss in the arc, we assume that the energy loss per unit angle of deflection at a particular bunch length is the same as in our steady state model running at the same (fixed) bunch length. Repeating the steady state calculation for many bunch lengths, we find a curve for $dU/d\theta$ versus $\sigma$. Since $\sigma$ decreases in the compressor almost linearly with $\theta$, this is equivalent to knowing $dU/d\theta$ as a function of $\theta$. Integration with respect to $\theta$ produces the figures for total energy loss shown in Table I. The values are in MeV per particle, supposing that the bunch contains $N = 2 \cdot 10^{10}$ electrons. The conductivity of aluminum is used for the toroidal chamber, whereas the parallel plates are perfectly conducting. For comparison, values for incoherent radiant energy are listed. For that, we assume that the energy radiated per unit angle is $(1/\omega_o)\, dU/dt$, with $dU/dt$ given by Eq. (2.19).

The curve of $dU/d\theta$ for Version 3 of the compressor is shown in Figure 4, for the parallel-plate model. In all four versions the energy loss is sharply concentrated near the end of the arc, since elsewhere the impedance is too small at wavelengths within the bunch spectrum $|\lambda_n|^2$. The corresponding curve for the resistive toroidal chamber has a similar form, but is concentrated still closer to the end of the arc, due to the higher threshold of the impedance. The high-threshold effect is especially pronounced in Version 2 of the compressor, which has relatively large values of $R$ and $\sigma_f$.

For a given final bunch length $\sigma_f$, one can reduce the coherent radiation by making $R$ as large as possible and the transverse dimensions of the chamber as small as possible. This raises the threshold (2.2) for the onset of a large impedance, and allows the bunch spectrum (3.3) to cut off the radiation.

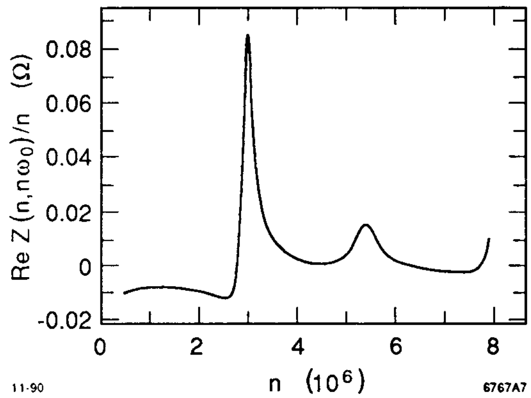


**Figure 3:** $\mathrm{Re}Z(n, n\omega_o)/n$ versus $n$ for aluminum toroidal chamber, $R = 149.5\mathrm{m}, h = w = 2\mathrm{cm}$.

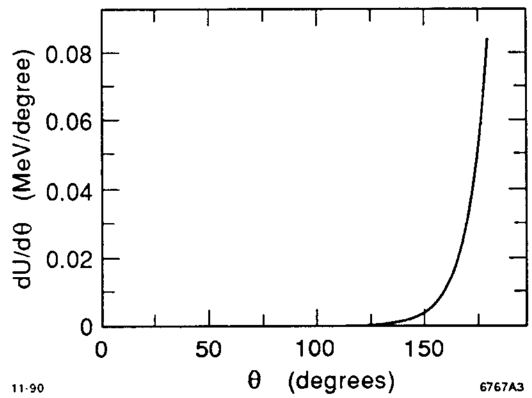


**Figure 4:** $dU/d\theta$ for the parallel-plate model in Version 3 of the bunch compressor.

In Figure 5 we show the wake voltage per turn, as defined by (2.7), for the parallel-plate problem and Version 3 of the compressor. Particles within one $\sigma$ of the bunch center lose energy, whereas those around two $\sigma$ on either side gain energy. The peak wake voltage per unit length is comparable to typical wakes in the SLAC linac structure, which amount to a few volts per picocoulomb over a 3.5 cm cell.

Corresponding results for the toroidal model are displayed in Figure 6. Within two $\sigma$ of the center the behavior is similar, although the voltage is somewhat lower due to the higher threshold of the torus impedance. The persistent oscillations well beyond the bunch length can be understood if we refer to the limiting case of infinite conductivity. As that limit is approached, the peak of $\mathrm{Re}Z(n, n\omega_o)$ narrows to a delta function, and the wake field has the form $\exp(in\omega_o\tau)$ with essentially a single value of $n$. In fact, the oscillations in Figure 6 have almost exactly the period of such a single exponential, if we choose $n$ to have the value at the peak of the impedance; (in units of bunch length, the wavelength of the oscillation should be $2\pi R/n\sigma$, which in the present example is about five). If we increase the $Q$ of the system by increasing the conductivity, we find numerically that the wake indeed approaches the expected single oscillatory exponential. In Figure 6 the oscillations before and after the bunch die off relatively quickly, because a band of $n$ values is involved. In Ref. 5, Section 7, it was shown that the band width is given approximately as

$$\frac{\Delta n}{n} = \frac{2R}{w}\frac{1}{Q} \quad . \tag{3.4}$$

The result (3.4) arises from the peculiar nature of the dispersion relation $\omega(n)$ of the resonances of this structure. In the $(\omega, n)$-plane, the dispersion curve is almost parallel to the synchronism line $\omega = \omega_0 n$ at the point where the two cross. This means that a rather broad range of $n$-values can be simultaneously synchronous and resonant when the system has resistive walls.

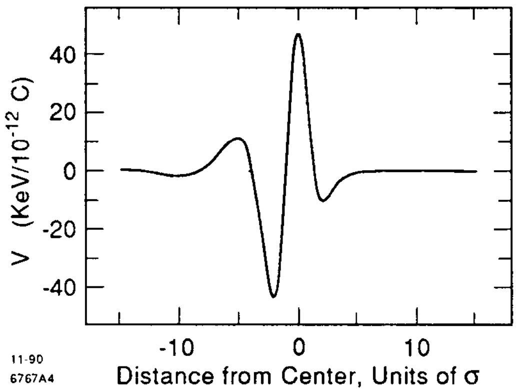


**Figure 5:** Wake voltage per turn for the parallel-plate model, $R = 149.5\text{m}$, $h = 2\text{cm}$, bunch length $\sigma = 61\mu\text{m}$.

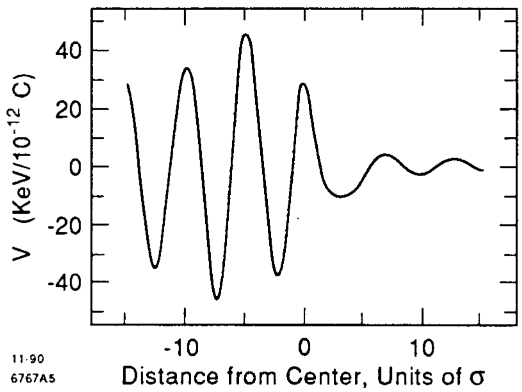


**Figure 6:** Wake voltage per turn for the toroidal chamber, $R = 149.5$ m, $h = w = 2$ cm, bunch length $\sigma = 61\mu\text{m}$.

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