SLAC–PUB–5331
August 1990
(A/T)

LONG-TERM BEHAVIOR OF PARTICLE ORBITS IN STORAGE RINGS*

Robert L. WARNOCK

Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309

In a large proton storage ring the particles circulate on narrowly defined orbits for many hours, and undergo about 10^{12} interactions with nonlinear magnetic fields. The nonlinear fields include significant high multipoles from inevitable errors in the fabrication of superconducting magnets. In a first approximation one neglects collective and dissipative effects, and assumes that magnetic fields are constant. The motion is then Hamiltonian, being perturbed harmonic motion in three degrees of freedom. Since the nonlinear perturbation depends periodically on the independent variable (azimuthal location in the ring), the phase space effectively has seven dimensions. Thus, we deal with a generic difficulty of nonlinear mechanics in a phase space of dimension greater than four: the complement of the set of invariant surfaces is a connected and dense set, associated with nonlinear resonances.[1,2] Arbitrarily close to any initial condition there is an orbit that can visit remote regions of phase space by moving along resonances. The task of a stability study is to show that this "diffusion" along resonances proceeds so slowly that the orbit stays within certain limits during a finite but sufficiently long time.

Diffusion has been observed directly in experiments with beam scrapers at the CERN SPS.[3] When a scraper is pushed into the beam, it produces an immediate drop in beam current, and then a decay of current with $1/e$ decay time τ_1. After the scraper is pulled outward to a new location still within the beam, the beam current stays constant for a while, and then decays with a longer decay time τ_2. The period of constant current is interpreted as the time it takes for the beam to diffuse outward to reach the new scraper location. When the scraper is returned to its first position, the original behavior at that position is seen again, with a decay time close to τ_1.

Generally speaking, mechanisms that increase the density of nonlinear resonances tend to increase the rate of diffusion. For instance, the introduction of synchrotron oscillations or magnet power supply ripple adds closely spaced sidebands to the principal resonances of betatron motion. The rate of diffusion drops when the r.f. is turned off to produce a coasting beam. When power supply ripple is deliberately increased at the SPS, by adding ripple comparable in magnitude to that normally present, beam lifetime suffers drastically. Continuing efforts to make power supplies smoother have led to improved performance of the SPS. Simulations by Furman and Schmidt[4] confirm the importance of ripple in long-term behavior, particularly when the ripple frequency is low.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

Invited talk presented at the XII International Conference on Particle and Nuclei, Cambridge, Mass, June 25–29, 1990

The chief theoretical tool for study of these questions is "tracking", which is numerical integration of Hamilton's equations by an algorithm that guarantees the symplectic character of the resulting time evolution map.[5] Tracking is quite informative, but it has its limitations. Because of computational expense, rather few orbits can be followed for a time comparable to the desired storage time of the beam. Furthermore, a machine like the SSC must be treated as a statistical object, since small-bore superconducting magnets have an unpredictable scatter in strengths of multipoles. It is therefore desirable to study a statistical sample of different machines, and many different initial conditions for the orbits in each machine.

For the status of tracking at the SSC, the reader may consult reports by Yan and Schachinger[6] and by Garavaglia, Kaufmann, Stiening, and Ritson.[7] These authors make "survival plots", which give the number of turns that an orbit survives versus the initial transverse amplitude of the orbit. As the initial amplitude is increased, particle loss appears to be negligible for 10^6 turns or more until a rather clear "dynamic aperture" is reached. For instance, in Yan's studies at the 2 TeV injection energy with a 4 cm magnet bore, this aperture occurs at about $x_o = (\beta_x/\beta_y)^{1/2} y_o = 5.3$ mm (with $\dot{x}_o = \dot{y}_o = 0$). At $x_o \approx 6$ mm, many particles are lost in less than 10^5 turns. The number of turns required for injection is 10^7.

To accomplish faster tracking there is some interest in using special computers that could exploit particular features of the problem.[8] Another idea is to summarize information from a tracking code in an explicit formula for the time evolution map, say a map representing a full turn. One iteration of a full-turn map should take much less time than element-by-element tracking through a full turn. The map can be represented as a power series in Cartesian phase space coordinates,[9] or as a Fourier series in angle coordinates with action-dependent coefficients.[10] Such maps are only approximately symplectic. To enforce the symplectic condition, one can replace the power series map by a sequence of momentum kicks and phase space rotations, in such a way that the resulting symplectic map agrees with the power series to a certain approximation.[11] To create a symplectic map in angle-action coordinates one can construct a generating function of the canonical transformation corresponding to time evolution.[10] This generator can be obtained as a Fourier series from data provided by a tracking code. The symplectic map induced by the generator can be iterated almost as fast as the corresponding explicit Fourier map.[12] At present, the method of maps looks promising, but it needs more development to become thoroughly convenient and reliable.

Even if speed could be greatly increased, the method of tracking as practiced is deficient in that it gives no insight about underlying mechanisms of instability. It merely answers the question of whether a particular orbit is stable or not. Actually, information from tracking, employed more ambitiously, ought to tell us a great deal more about the system.

One way to seek insight is to compute Liapunov characteristics from tracking data.[2] These serve to detect exponential divergence of nearby trajectories, a sign of chaotic behavior. One hopes that fast divergence from nearby orbits will give an "early warning" of instability of an orbit. It appears that this method can clearly identify regions of stable and unstable behavior, but for the crucial problem of locating the transition from one to the other the results may be expensive in computation time and somewhat ambiguous.[13] Further work is needed to

develop the Liapunov technique, particularly in the form based on the tangent map to the time evolution map.[14,15]

In a longer view of the problem, one would like to have a way of making definitive statements about stability for any orbit starting in a large region of phase space. A method which can, in principle, lead to such statements is being studied by R. Ruth and the author.[16] We determine a canonical transformation to new action-angle variables $(\mathbf{J}, \mathbf{\Phi})$ such that the action $\mathbf{J}$ is nearly constant. By examining the residual temporal variation of $\mathbf{J}$, which is a sensitive indicator of unstable behavior, one can set bounds on the motion for a finite but very long time interval. This bound holds for any initial condition in an open region. The idea behind the bound is elementary: if the maximum change in $\mathbf{J}$ during N_o turns is $\delta\mathbf{J}$, then the change during qN_o turns cannot be larger than $q\delta\mathbf{J}$. If $\delta\mathbf{J}$ is so small that $q\delta\mathbf{J}$ is an acceptable excursion in phase space for large q, we have the means to get a bound for a large number of turns qN_o in terms of information from a much smaller number N_o. For instance, in one example we derived bounds for 10^8 turns from tracking data on 10^4 turns, the latter being required for many initial conditions.

Determination of the canonical transformation is equivalent to finding a family of nearly invariant tori. In order that the bounds hold for a time comparable to the desired beam storage time, it is necessary that the tori be invariant to high accuracy, say to one part in 10^6 or better. A new method allows one to find such tori by fitting Fourier series to tracking data on nonresonant orbits.[17] This nonperturbative method is quite robust and relatively inexpensive. To find the change of $\mathbf{J}$ in time we find the change in the original variables by tracking, then use the canonical transform to find the induced change of $\mathbf{J}$.

In a first test the method has been applied to two-dimensional betatron motion in a model lattice containing strong sextupoles. In a region of strong nonlinearity (around 20% smear) it was possible to claim stability for times comparable to storage times in proton rings. For real proton rings, it will be necessary to include the third degree of freedom for synchrotron oscillations. The formalism easily accomodates that extension, but the computational problems are more severe and have yet to be investigated. In any case, the program offers a prospect that was previously lacking in applications of nonlinear mechanics, the possibility of setting practically useful bounds that are rigorous in principle, and "almost rigorous" (i.e, very reliable) in numerical realizations. Previously, long-term bounds were obtained mathematically in the Nekhoroshev Theorem,[18] but not for a range of parameters of practical interest; the nonlinear perturbation strength had to be absurdly small to prove stability for suitable time intervals.

ACKNOWLEDGEMENT

I wish to thank J.Bisognano for inviting me to participate in PANIC XII, and for continuing interest in my work.

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