

# EFFICIENT COMPUTER DETERMINATION OF THE PROPERTIES OF WAVEGUIDE LOADED CAVITIES\*

**Norman M. Kroll and Xin-Tian Lin**

*University of California, San Diego*

*La Jolla, CA 92093*

and

*Stanford Linear Accelerator Center*

*Stanford University, Stanford, CA 94309*

## ABSTRACT

A new method is proposed for computer determination of the resonant frequency and  $Q_{ext}$  of a waveguide loaded cavity which, in principle, requires only a single run of a computer program such as **MAFIA** or **ARGUS**. It is closely related to the method of Kroll and Yu but makes use of the fact that these programs yield stored energy and field strength information as well as resonant frequencies. Application of the method to a set of waveguide models with comparison to analytic results is given. Application to a simple cavity structure with a set of output couplings using MAFIA determined properties is also given.

Submitted to Particle Accelerators.

---

\* Work supported by Department of Energy contracts DE-AC03-76SF00515 and DE-AS03-89ER40527.

In a recent paper, Kroll and Yu<sup>1</sup> (henceforth KY) have described a method for the determination of the  $Q_{ext}$  and resonant frequency of a cavity mode loaded by a waveguide terminated by a matched load. Their method is based on a number of determinations of the mode spectrum of the cavity-waveguide combination with the waveguide terminated by a short<sup>2</sup> at a number of lengths. Under the best of circumstances, it requires that the determination be made at two to four lengths. We describe an extension of their method here which reduces the number of determinations required by a factor two.

## 1. DESCRIPTION OF THE NEW METHOD

Figure 1 (b) of KY provides a typical example of the mode spectrum of the coupled cavity-waveguide system in the neighborhood of a single cavity resonance as a function of the shorted waveguide length. It is reproduced as Fig. 1 here. Since all of the curves refer to a single cavity resonance, they are referred to as branches in KY. The following formula,<sup>3</sup>

$$\tan [k(\omega)D + \chi(u) + \chi'(\omega - u)] = \frac{u}{2Q(\omega - u)} \quad , \quad (1.1)$$

which is an immediate consequence of Eqs. (1.2) and (1.3) of KY , provides, as shown by that paper, an excellent four-parameter representation of these curves in the vicinity of the resonant frequency of the cavity. Here  $k(\omega)$  is equal to  $2\pi/\lambda_g$ , as usual. The parameter  $u$  represents the resonant frequency of the cavity coupled through the output and waveguide to a matched load, while the parameter  $Q$  represents the associated external  $Q$ . The remaining two parameters,  $\chi(u)$  and  $\chi'(u)$ , parameterize the effect of distant cavity resonances. In KY the four parameters are determined by computing four frequency-length pairs in the vicinity of the cavity resonance as identified by inspection of field plots, and requiring that Eq. (1.1) be satisfied for

each pair. A minimum of four computer runs at four different lengths is required to produce four reasonably placed points on a single branch. On the other hand, if the lengths are chosen in the vicinity of an avoided crossing region, where two branches are adequately close to the resonance, only two lengths are required, and two frequency-length pairs are taken from each of the two branches. This reduces the amount of computing required by a factor two as the computer run typically yields all the modes of interest at a specified length in a single run.

The extension of their method, which we report in this paper, replaces two of the frequency length pairs with a determination of the derivative of frequency with respect to length at the remaining two frequency-length pairs. By differentiating Eq. (1.1), we obtain

$$\frac{d\omega}{dD} = -k(\omega) \left[ \frac{\omega D}{kc^2} + \chi'(u) + \frac{2Qu}{4Q^2(\omega - u)^2 + u^2} \right] \quad (1.2)$$

where  $c$  represents the velocity of light. The four parameters are now-determined by requiring that both Eq. (1.1) and Eq. (1.2) be satisfied for the two frequency length pairs. The derivative which appears in Eq. (1.2) is obtained from the computer runs via the formula,

$$\frac{d\omega}{dD} = -\frac{\omega \int (\mu_0 H^2 - \epsilon_0 E^2) dS}{2 \int \mu_0 H^2 dV} \quad , \quad (1.3)$$

which follows from cavity resonator perturbation theory.<sup>4</sup> The volume integral in Eq. (1.3) is proportional to the stored energy of the mode, which is one of the standard outputs of a typical computer run. The surface integral is carried out over the shorting plane at the end of the waveguide. Since the field values are also available as output, it can be readily computed. In the case of standard waveguides, the surface integral can be determined from field values at single points, and there is, of course, no electric field contribution in the case of TE waveguide modes.

If it is clear from an examination of the field plots associated with a single computer run that two branches from the same cavity mode are present, as is often the case, then that run is sufficient to determine the properties of the resonance. If only one branch is recognizable, then a second run at a different length will provide a second point on the same branch, and the determination of cavity properties can be made from the two runs. This latter procedure may also be preferable when the frequency separation of the points on the two branches is too large.

## 2. IMPLEMENTATION FORMULAS

The parameter determination procedure described in Sec. 2 of KY is readily adapted to our new method by taking limits as the frequencies approach one another in pairs. For this purpose, we follow KY by using  $\psi = k(\omega)D$  and  $v = \omega/2Q$  in place of  $D$  and  $Q$  and summarize the resultant procedure below. We also need to replace  $d\omega/dD$  by

$$\frac{d\psi}{d\omega}(\omega) = \frac{k(\omega)}{\frac{d\omega}{dD}(\omega)} + \frac{\omega D}{k(\omega)c^2} \quad (2.1)$$

From the computed points and an assumed value of  $\chi'(u)$ , we define

$$B_i = \left[ \frac{d\psi}{d\omega}(\omega_i) + \chi'(u) \right]^{-1}, \quad (2.2)$$

$$B_{12} = (\omega_1 - \omega_2) \cot [\psi_1 - \psi_2 + \chi'(u)(\omega_1 - \omega_2)] \quad , \quad (2.3)$$

and

$$A = \frac{B_1 - B_{12}}{\omega_1 - \omega_2} \quad . \quad (2.4)$$

From these, we obtain

$$v = \frac{-B_1}{1 + A^2} \quad (2.5)$$

and

$$u = \omega_1 - Av \quad (2.6)$$

as functions of  $x'(u)$ . We then determine  $\chi'(u)$  numerically as the root of

$$\frac{d\psi}{d\omega}(\omega_2) + \chi'(u) + \frac{v}{(\omega_2 - u)^2 + v^2} = 0 \quad . \quad (2.7)$$

In this method,  $\chi(u)$  is not needed to determine the other parameters. It is useful, however, if one wishes to use Eq. (1.1) to generate frequency-length curves and may be obtained in an obvious manner.

### 3. APPLICATION TO AN ANALYTIC WAVEGUIDE MODEL

We first tested this procedure on the analytic waveguide model introduced in Sec. 3 of KY. The model consists of an analytic approximation to a cavity of square cross-section formed by inserting a zero thickness symmetric inductive iris into a rectangular waveguide. It allows one to determine  $u$ ,  $Q$ , frequency-length pairs, and  $d\psi/d\omega$  from analytic formulas. It therefore enables us to compare  $u$  and  $Q$  obtained by applying our new method at a single waveguide length to the analytically obtained values and also to examine the dependence of the results upon the length chosen.

The results obtained for a set of seven iris openings are shown in Fig. 2 for  $Q$ , and in Fig. 3 for frequency. In these figures,  $d/a$  represents the ratio of the iris opening to the waveguide width. Calculations were performed at a large number of closely spaced lengths so that the dependence of the computed  $Q$  and frequency upon the length could be displayed as continuous curves. In carrying out the procedure, it is also necessary to choose the pair of modes to be employed. For this calculation, we simply used the pair with the smallest frequency separation with the proviso that

both lie significantly below the frequency of the next higher cavity resonance. It is clear from inspection of Fig. 1, which is also based upon the analytic waveguide model, that the calculation will begin with the lowest frequency branch pair at the smallest length and switch successively and discontinuously to the next higher branch as the length is increased. The discontinuities in the curves shown arise from this effect.

Ideally, one would wish for the curves to coincide precisely with the horizontal dotted lines, which represent the exact values. One may judge the reliability of the method by noting the deviation from this state of affairs. We note the following: The relative accuracy increases with increasing  $Q$ . The agreement and constancy of the curve arising from the branch 1-branch 2 combination is significantly poorer than that arising from the higher branch combinations. Since one usually has a rough idea of where the resonant frequency is, one can avoid this combination by choosing a sufficiently large waveguide length. Agreement is best at the avoided crossing region. It is our view that the agreement is good enough for most practical purposes at any waveguide length, but since it is easy to avoid a lowest branch pair evaluation, it is better to do so.

## 4. A COMPUTER APPLICATION

As a test of the method in a computational context, we applied the MAFIA<sup>5</sup> code to the model illustrated in Fig. 4. It is the same as the analytic model of the previous section except that here the iris is of finite thickness. Since we are going to study the mode which is independent of waveguide height (the dimension perpendicular to the plane of Fig. 4), the problem is essentially a two-dimensional one. Nevertheless, it proved to be advantageous to use MAFIA rather than one of the two-dimensional

codes available to us. The principal reasons were: first, that it produces a sequence of modes in a single run, and second, that its boundary condition capability was sufficiently flexible to allow us to study the mode in which we were interested. We avoided the appearance of height-dependent modes by choosing the waveguide height to be sufficiently small (0.01 in) to push the frequency of such modes above the range specified for the mode search.

As noted in Sec. 1, the analysis of the computer output begins with an examination of the field plots. To illustrate the procedure, we exhibit in Fig. 5 electric field plots of the first four modes for the case  $d/a = 0.5$  at  $D = 2.0$  in. Since we are studying the lowest cavity mode, which is symmetric with respect to the center of the iris, a symmetry boundary condition is imposed there, and only half of the cavity-waveguide combination is shown. Examination of the plots shows unambiguously that the third mode is closest to the cavity resonance because the field in the cavity exhibits the expected pattern and because its amplitude relative to the field in the waveguide is largest. Next, it is apparent from the field patterns in the cavity that modes 2 and 4 are different branches of the same resonance. The frequency separation of mode 2 from mode 3 is much smaller than that of mode 4 from mode 3 (533 versus 1710 MHz). Furthermore, the relative cavity excitation is clearly higher in mode 2 than in mode 4. We conclude from the above that the two branches nearest the resonance are clearly identified, and hence that a single length determination of cavity properties using modes 2 and 3 can proceed. The result obtained is frequency = 8769.07 MHz and  $Q = 34.542$ .

In order to investigate the stability of results with respect to the length chosen, we carried out computations at a number of different lengths. We also investigated a number of different iris opening ratios  $d/a$ . The results are exhibited graphically for  $Q$  in Fig. 6 and for frequency in Fig. 7. Note that the points arising from the first two

modes are displayed with a different symbol from those arising from modes 2 and 3. The horizontal dotted lines represent arithmetic averages of the determinations for each  $d/a$  value, and are inserted to assist an assessment of the variation. The exact values are, of course, not known for this model, and there is no reason to expect that the average has any special validity. The pattern of fluctuation follows closely that of the physically very similar analytic waveguide model. We therefore believe the fluctuations to be due to the inherent limitations of the theory and not to computer-generated fluctuations. We also consider the values found at the larger lengths using modes 2 and 3 to be the more reliable, as suggested by the discussion at the end of Sec. 3.

Although single-length two-branch determinations proved to be satisfactory for this model at all lengths tested, it was noted at the end of Sec. 1 that there may be circumstances in which a two-length single-branch determination is preferable. Accordingly, we used the data obtained to investigate the length dependence of the above results to also test their consistency with two-length single-branch determinations. We were also able to compare results with those obtained from the KY method. A representative set of comparisons is shown in Table I. The method employed for each case should be clear from the branch and length lists.

The agreement among the various methods is seen to be excellent and shows that Eq. (1.1) must represent the situation near resonance very well for this model. It also shows that MAFIA yields sufficiently accurate values for the stored energy integral and boundary values of the fields to yield an accurate value of  $d\omega/dD$  via Eq. (1.3).

The number of lattice points used in carrying out these calculations varied with waveguide length as care was taken to see that the lattice remained fixed in regions common to cases which differed only in waveguide length. For the case illustrated in



**Table I**

$d/a$	Frequency (MHz)	Q	Branches	Lengths (inches)
0.5	8769.07	34.54	2, 3	2.0
	8769.09	34.52	2	1.3, 1.9
	8768.93	34.63	2, 3	2.0, 2.1
	8768.84	34.59	2	1.3, 1.6, 1.9, 2.0
0.65	8405.01	10.41	2, 3	2.0
	8407.99	10.39	2	1.5, 2.0
	8403.95	10.34	2, 3	1.5, 2.0
0.70	8266.35	7.54	2, 3	2.0
	8270.48	7.48	2	1.5, 2.0
	8263.87	7.48	2, 3	1.5, 2.0
0.75	8109.44	5.57	2, 3	2.1
	8114.20	5.47	2	1.5, 2.1
	8109.52	5.55	2, 3	1.5, 2.1

Fig. 5, the number of lattice points in the plane of the figure was 7896. Increased density was used in the vicinity of the iris. Because of the two-dimensional character of the problem, only two points were used for the direction perpendicular to the plane of Fig. 5. The reliability of this procedure was confirmed with a simple rectangular cavity.

## REFERENCES

1. N. M. Kroll and D. U. L. Yu, to be published in *Particle Accelerators*, **34** (1990); SLAC-PUB-5171 (1990).
2. We note here that the short at the end of the waveguide can be replaced by an open; that is, one may require that the tangential component of the magnetic field, rather than that of the electric field, vanish there. The implementation procedures, and indeed all implementation formulas, remain the same.
3. We have experimented with replacing  $\chi'(\omega-u)$  in Eq. (1.1) with  $\chi_1[k(\omega)-k(u)]$ , with  $\chi_1$  replacing  $\chi'$  as a parameter to be determined. The latter form has the formal advantage of leading to results independent of the choice of waveguide origin. While the results obtained by the two forms have been of comparable quality, no consistent advantage of the latter form has been found in practice.
- 4. J. C. Slater, *Microwave Electronics* (Van Nostrand, New York, 1950) (Bell Telephone Laboratory Series) Ch. 4, Sec. 4.7. Equation (1.3) is an immediate consequence of Eq. (7.1) of the reference. If one employs an open, rather than a short, boundary condition on the waveguide termination as discussed in Ref. 2, then the sign of the RHS of Eq. (1.3) must be reversed. Apart from that, all procedures and formulas are the same for the two boundary conditions.
5. T. Weiland et al., DESY Report M-86-07 (1986).

## FIGURE CAPTIONS

- 1) A typical example of relative frequency  $f$  plotted as a function of relative length  $r$ . The frequency  $f$  is normalized to the frequency of the uncoupled cavity mode, and the waveguide length is normalized to one-half the cutoff wavelength. The cavity mode is sufficiently isolated, so that other cavity modes do not have a significant effect over the frequency range shown.
- 2) Computed values of  $Q_{ext}$  for the analytic waveguide model determined by the procedure described in Sec. 2. The results are plotted as a function of the waveguide length chosen for carrying out the procedure. The curves associated with different ordinate scales refer to different iris openings. The different sections of the curve for a particular iris opening arise from the branch pair selection procedure discussed in the text. The horizontal dotted lines represent analytically determined exact values.
- 3) Computed values of resonant frequency  $U$  for the analytic waveguide model normalized to the resonant frequency of the uncoupled cavity mode. For other details, refer to the Fig. 2 caption.
- 4) Test cavity for computer application.
- 5) Electric field plots for the first four coupled cavity-waveguide modes of the test cavity for  $d/a = 0.5$  and waveguide length  $D = 2.0$  in.
- 6) Computed values of  $Q_{ext}$  for the test cavity. Each value is determined from a single MAFIA run at the waveguide length and iris opening indicated. The figure exhibits the consistency of results for different choices of waveguide length. The “diamond” points were determined from mode 1 and mode 2 data, the “x”

points from modes 2 and 3. The horizontal dotted lines represent the arithmetic average of all points computed at each iris opening, and is shown to facilitate assessment of the variation.

- 7) Computed values of the loaded resonant frequency for the test cavity. For other details, refer to the Fig. 6 caption.

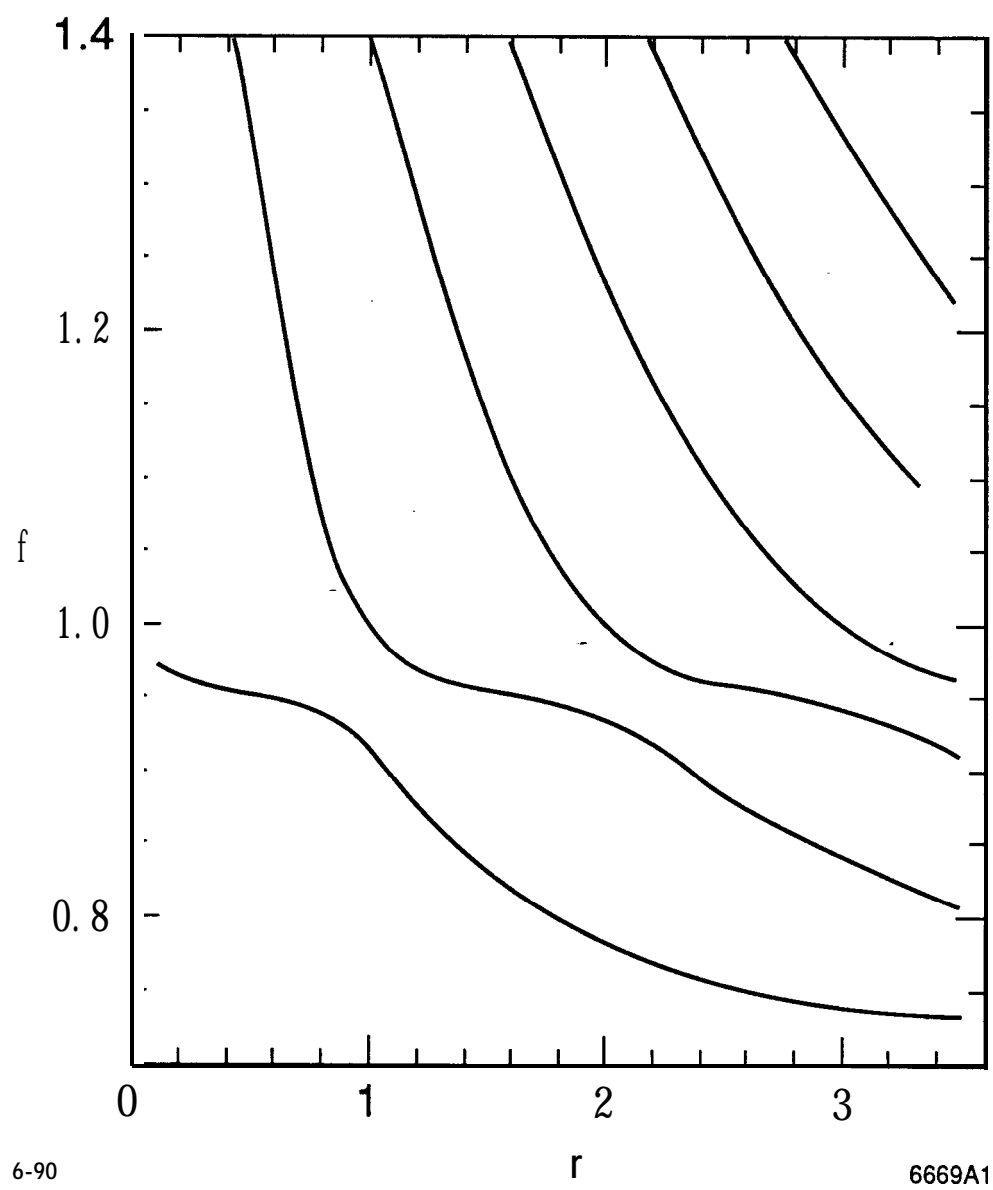


Fig. 1

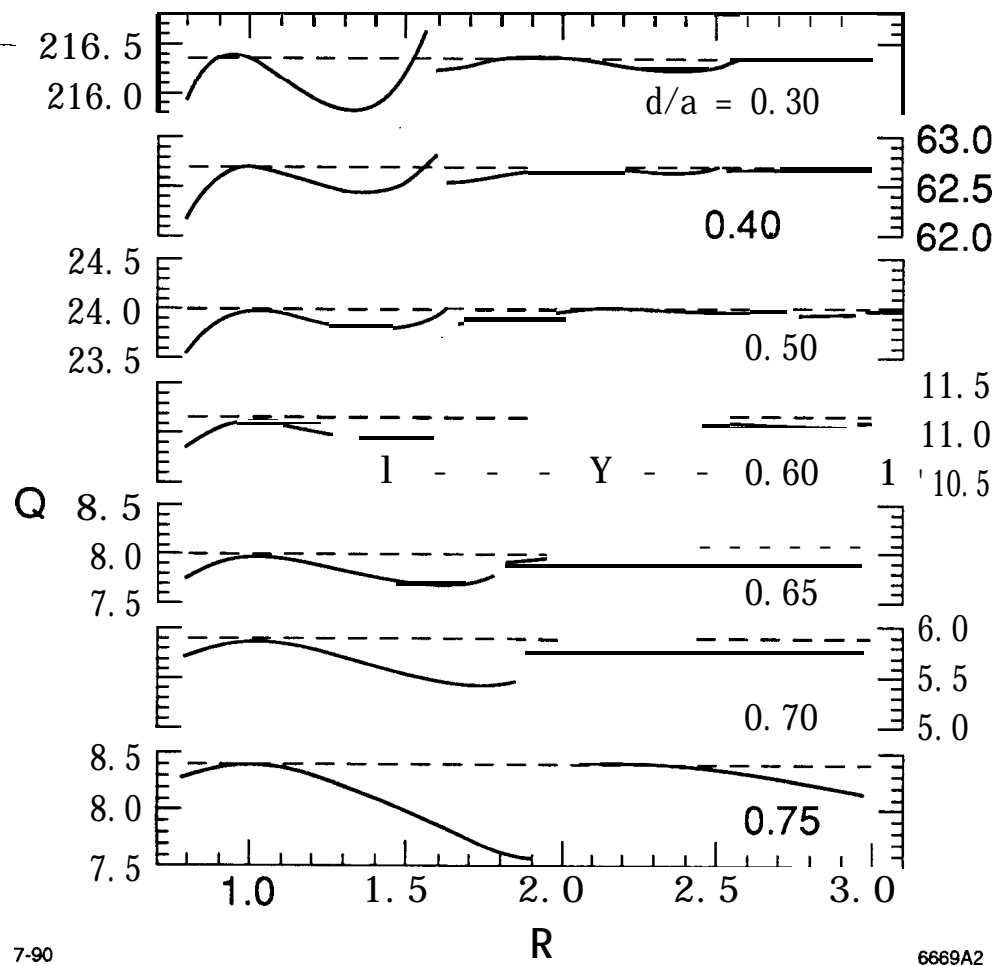


Fig. 2

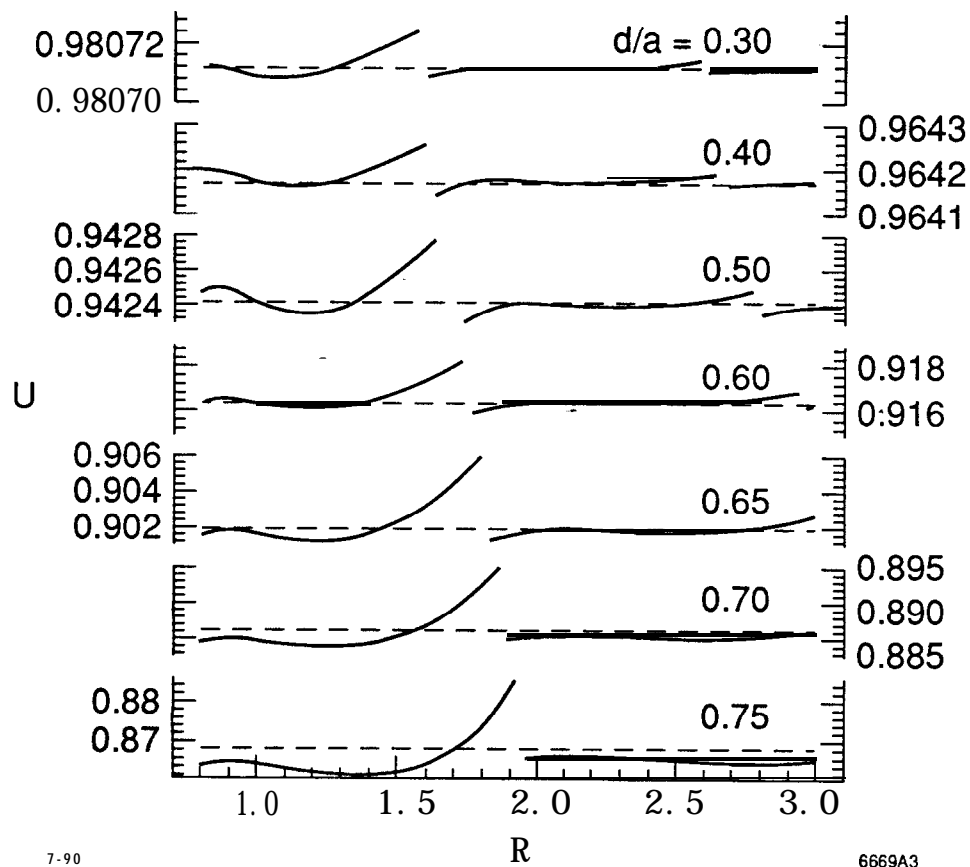


Fig. 3

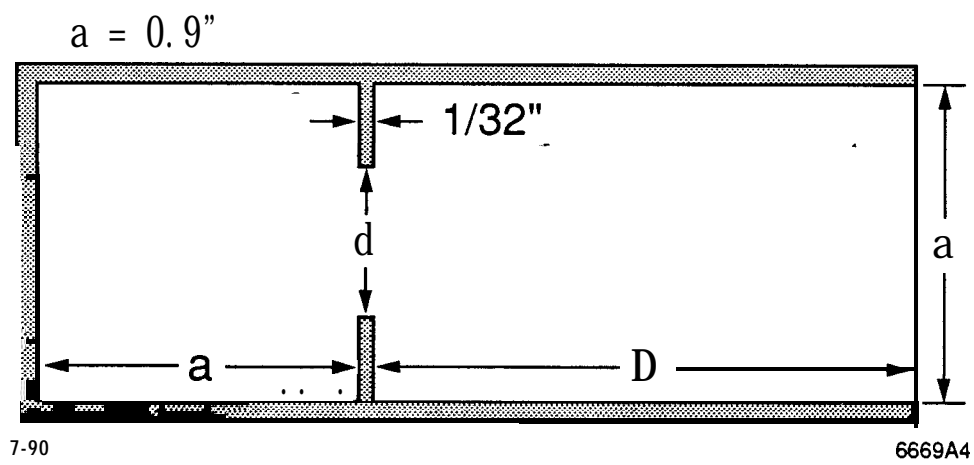
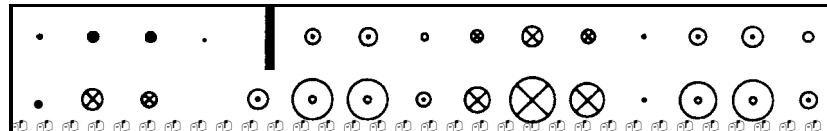
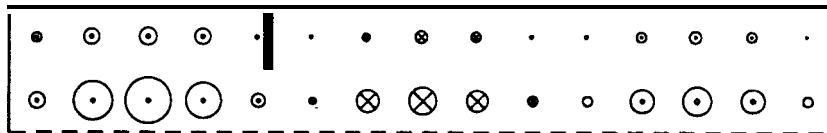


Fig. 4

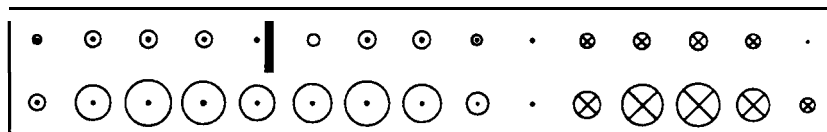




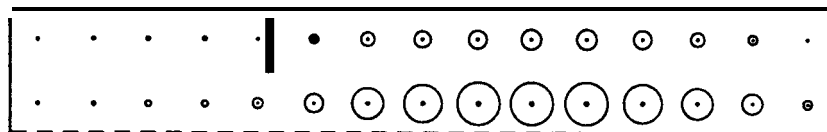
Mode 4 - Frequency = 10688.57 MHz



Mode 3 Frequency = 8978.10 MHz



Mode 2 Frequency = 8445.13 MHz



7-90

Mode 1 Frequency = 7125.13 MHz 6669A5

Fig. 5

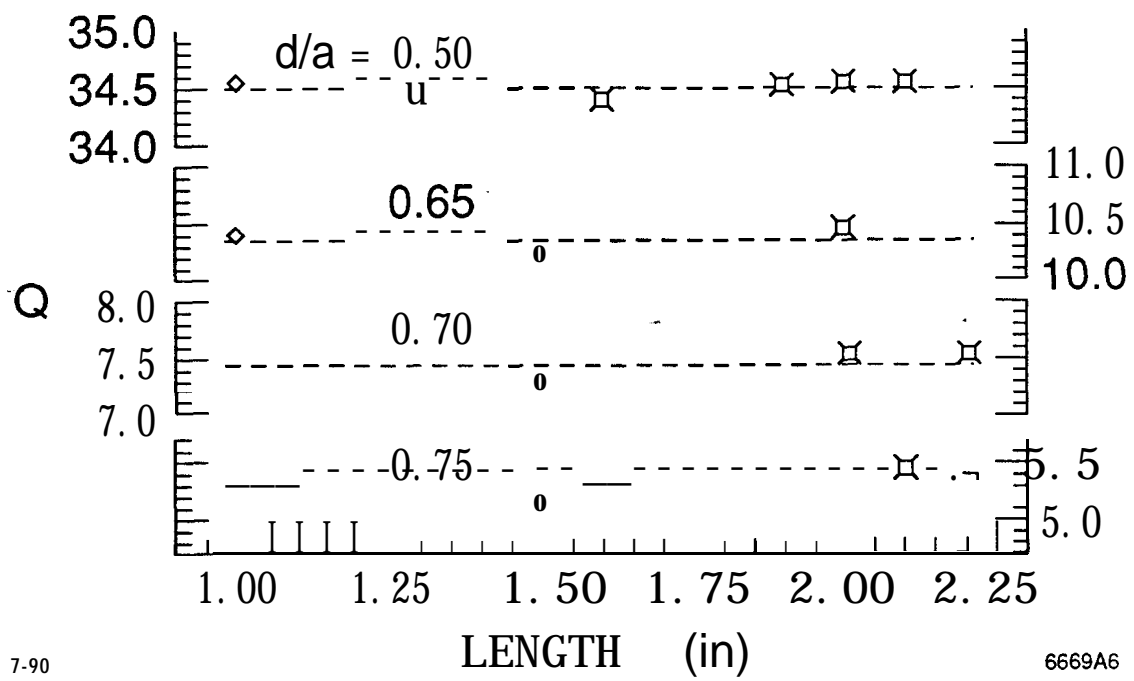


Fig. 6

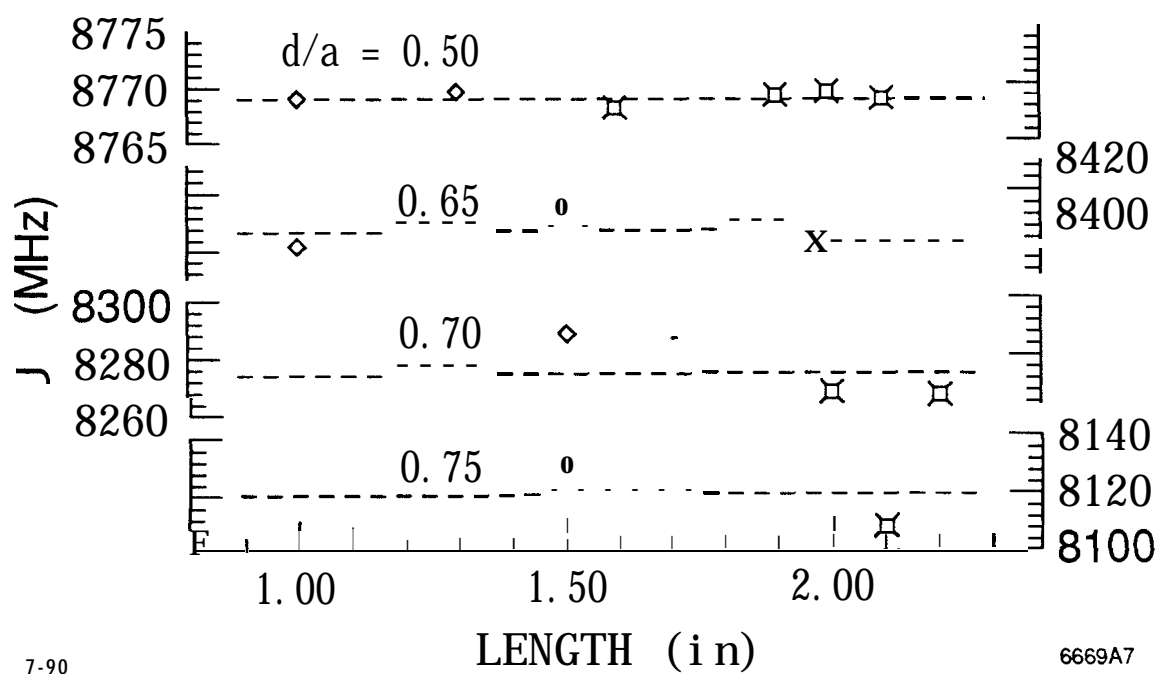


Fig. 7