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Loops in 2d Quantum Gravity*

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ABSTRACT

We argue that existing discussions of discretised random surfaces with boundaries are incomplete, and give the correct versions for Kazakov's multicritical models. This leads to the calculation of self-avoiding loops on the surfaces, and uncovers a new type of phase transition. Applying this technology to strings in one dimension gives indications that the branched polymer phase is absent for c = 1.

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1. Introduction

Over the past year considerable progress has been made in 2d gravity through the use of matrix models. These provide a discretisation of the geometry of a surface that readily lends itself to both numerical and analytical investigation.

The conceptually simplest, and chronologically the first, approach to discretising (Euclidean) 2d gravity is via the techniques of Regge calculus. This consists of triangulating the worldsheet in some regular fashion, and then allowing the lengths of the sides of the triangles to vary. These then become the dynamical degrees of freedom. Computer simulations indicate, however, that these models do not give an interesting continuum limit, and in any case it is hard to imagine a practical way of solving them analytically.

A more useful alternative has been to consider triangulations consisting just of equilateral triangles of-the same size [3]. In this case there are no lengths to be specified, so the only way to vary the geometry of the surface is to allow different numbers of triangles to meet at the different vertices. The curvature concentrated at a vertex at which more (less) than six triangles meet will then be negative (positive). A honeycomb lattice in which all vertices are of order six would correspond to flat space. The area of the surface is proportional to the number of triangles that it is composed of. It is clearly straightforward to extend this approach to "tilings" of the surface that consist of many types of polygon. We then hope that some form of universality makes the behaviour of the continuum limit insensitive to the details of the tiling.

The sum over geometries in the path integral can now be replaced by a sum over triangulations. Moreover the model may be generalised to the case of gravity interacting with matter by allowing a new degree of freedom to reside at the centre of each tile. Interaction terms are then included between neighbouring tiles. Throughout the paper we will refer to theories in which all surface configurations are weighted by a positive quantity in the partition function as being "unitary".

Remarkably, many models of this form may be solved analytically [3]. This is achieved by the observation that any triangulation is dual to some ϕ^3 Feynman diagram, so the sum over surfaces may be replaced by a sum over graphs. The propagators of the Feynman diagrams are chosen to mimic the matter degrees of freedom on the surface.

Suppose that we restrict attention for now to the case in which the gravity is not coupled to any matter degrees of freedom. Then the path integral only requires us to sum over the geometry of the surface. In this case the propagators of the dual Feynman diagrams are trivial, and the diagrams correspond to the perturbative expansion of some 0 + 0 dimensional "field theory".

It has been shown by Brézin, Itsykson, Parisi and Zuber [1] that the field theory that generates these graphs is a model whose variables are Hermitian matrices in 0+0 dimensions. If these matrices are N x N then the $\frac{1}{N}$ expansion is equivalent to an expansion in the genus of the surface. Happily this theory may be solved exactly by transforming the integral over the matrix to an integral over its eigenvalues [1,2]. The Jacobian of this change of variables leads to an additional term in the action, and then the partition function for connected random surfaces reduces to the free energy of a gas of particles with logarithmic interactions, called the "Dyson sea". We thus have a means of explicitly performing the path integral of 2d gravity.

Our main goal in the present paper is to extend this method to sums over triangulations with boundaries. We restrict attention throughout to the case of genus zero, for which we are only concerned with the classical properties of the Dyson sea [1,6,7]. In general, as long as we are not attempting to go beyond the genus expansion, we do not have to worry about the fact that the Dyson sea may reside away from the absolute minimum of the potential. Indeed, many cases of interest have potentials that are unbounded from below. To include nonperturbative effects in the string expansion we must take into account tunneling from the false to the true vacua, but these matters will not concern us here.

The layout of the paper is as follows. In Chapter **2** we argue that the operators

$\mathrm{Tr}\phi^q$ employed by other authors [9,7] do not in fact generate the correct boundaries. Instead they correspond to pathological boundaries which may collapse in on themselves or split into smaller parts. We believe that the correct operators are given by the connected Green's functions G_q^c of the matrix models.

In addition to allowing us to discretise open string worldsheets, these operators can be used to calculate the expected number of non-intersecting loops on a random surface. This is done by considering a worldsheet with a marked path to consist of two bounded surfaces sewn together.

The statistical distribution of the lengths of these loops uncovers a new phase transition, at which the characteristic loop length becomes infinite. We first investigate this phase transition for the simplest case, in which the surface is tiled with squares, and introduce the formalism that we later extend to more intricate tilings.

In Chapter 3 we extend the method of chapter 2 to the case of Kazakov's multicritical models [10,6,7]. This is done with a construction based upon the Coulomb forces of the Dyson sea [1,2], and leads to an extremely rich phase structure. We also find that in cases in which the density of the Dyson sea is monotonically decreasing from its centre to the edge, the asymptotic behaviour of the full Green's functions G_q is qualitatively similar to that of the connected Green's functions G_q^c.

In Chapter 4 we consider non-critical open strings. We start by arguing that Kazakov's solution [11] to this problem overlooks certain pathologies in the triangulation. Though we are unable to provide the correct solution for a worldsheet with an arbitrary number of boundaries, we can perform an analysis for a surface with a single boundary. We find that for unitary models the qualitative behaviour of the surface is the same whether it is generated by full or connected Green's functions. This indicates that Kazakov's pathologies do not affect the universality class of the triangulation.

In Chapter 5 we apply the same techniques to non-critical string theory in one embedding dimension. Lack of understanding of the angular variables [1,3]

restricts our considerations to boundaries lying at a single value of the external dimension. The sewing construction mentioned above then enables us to calculate properties of non-intersecting loops at a fixed slice of embedding space. We find the same phase transition as in the $d = 0$ case, which appears to imply the absence of a branched polymer phase for $d = 1$.

2. Loops in 2d Gravity

To fix our notation we briefly review the matrix model approach to 2d gravity, concentrating for now on the case in which the embedding dimension $d = 0$ [1,2,3].

As discussed in the introduction, the sum over worldsheets may be discretised as a sum over triangulations of various topologies, where we consider the side lengths to be fixed, so that the dynamics is solely reflected by differing numbers of triangles meeting at the vertices.

Now, duality between graphs gives a bijection between triangulations and the Feynman diagrams of ϕ^3 theories, so the path integral may equivalently be expressed as the sum over ϕ^3 Feynman diagrams [3].

In fact we can generalise this construction by considering the surface to be composed not just of triangles, but of many different types of polygon whose sides have a common length. We are then free to give different polygons different weights in the partition function, which may equivalently be viewed as changing their intrinsic areas whilst keeping the lengths of their sides fixed.

The diagrams dual to these "tilings" of the surface will now contain not just ϕ^3 vertices, corresponding to triangles, but also ϕ^4 vertices, corresponding to squares, ϕ^6 vertices corresponding to hexagons, etc.

For the simplest case in which all polygons are squares, we can generate the sum over dual ϕ^4 graphs by considering a Hermitian matrix model with partition

function

$$\int d^{N^2}\phi\, e^{-\mathrm{Tr}(\frac{1}{2}\phi^2+\frac{g}{N}\phi^4)} \tag{2.1}$$

The perturbative expansion of (2.1) in the coupling constant g yields a sum over ϕ^4 Feynman diagrams with a trivial propagator. Each vertex contributes a factor $-g$, and a graph with Euler character χ contains a factor of N^χ, as explained very clearly in [2].

If we allow each square in the dual tiling to have unit area, then a surface with area A and Euler character χ is weighted by a factor of $(-g)^A N^\chi$. Thus $-ln(-g)$ plays the role of a cosmological constant.

As we take the limit N $\rightarrow \infty$, the genus zero contribution will clearly dominate, so we can write the partition function for the sum over squarings of connected genus zero surfaces as

$$Z = \lim_{N\rightarrow\infty} \frac{1}{N^2} \ln \int d^{N^2}\phi\, e^{-\mathrm{Tr}(\frac{1}{2}\phi^2+\frac{g}{N}\phi^4)} \tag{2.2}$$

where as usual the logarithm subtracts out disconnected graphs. Since the integrand depends only upon the eigenvalues α_i of ϕ, we may write (2.2) as an integral over the α_i's. This change of variables gives us an additional Jacobian and leads to the expression

$$Z = \ln \int \prod_i d\alpha_i\, e^{-N\left(\sum_i \frac{1}{2}\alpha_i^2+g\alpha_i^4-\frac{1}{N}\sum_{i\neq j}\ln(\alpha_i-\alpha_j)\right)} \tag{2.3}$$

Now in the limit N $\rightarrow \infty$ the integral is dominated by the saddle point, and Z may be found exactly, as discussed in [1,2]. Expanding this solution perturbatively in g gives

$$Z = \sum_A (-g)^A Z_A \tag{2.4}$$

where Z_A is the partition function for surfaces of fixed area A. We can show that $Z_A \sim 48^A A^{-7/2}$ as $A \rightarrow \infty$ [1], so this series is convergent for g $\geq g_c \equiv -\frac{1}{48}$.

Taking $g \to g_c$ leads to a continuum limit with the characteristic area diverging as $\ln(g - g_c)$. Recently it was observed [6,7,8] that by taking $g \to g_c$ and $N \to \infty$ simultaneously, we obtain contributions to all orders in the genus expansion,

The preceeding discussion corresponds to the theory of 2d gravity with no boundaries. A natural generalisation of this is clearly to allow an arbitrary number of boundaries of arbitrary intrinsic length. In addition to providing a model for non-critical open string theory, consideration of worldsheets with boundaries enables us to calculate properties of closed non-intersecting paths on the random surface.

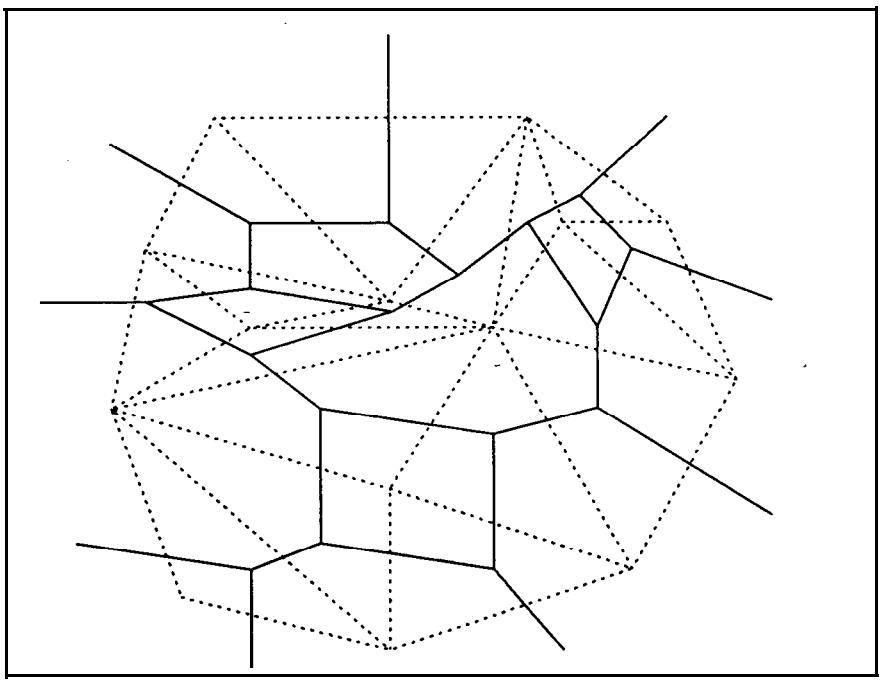
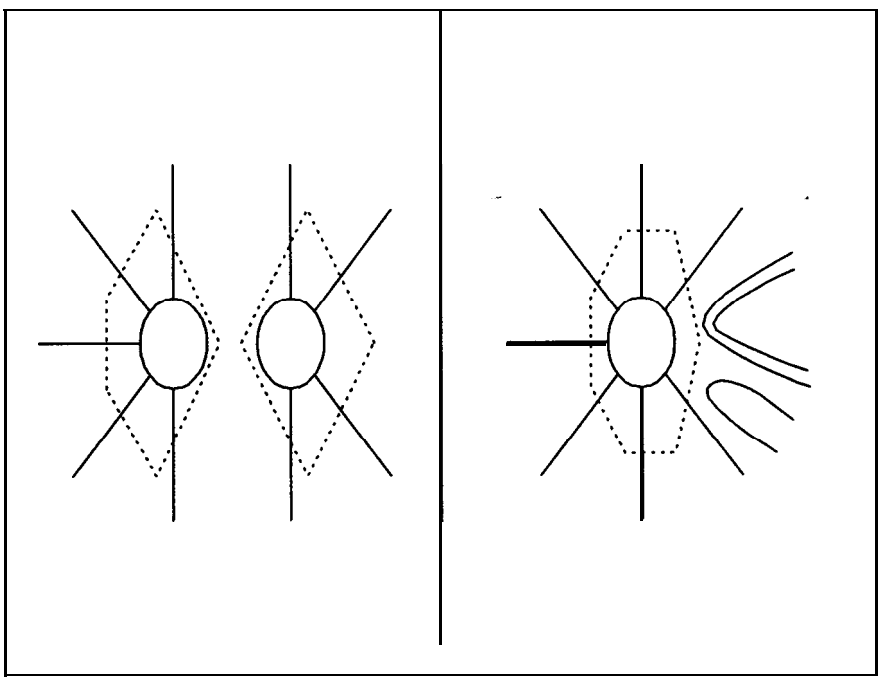


Figure 1. A triangulation with boundaries and its dual ϕ^3 diagram.

To integrate over surfaces with a boundary requires us to sum over triangulations (or more general tilings) with an edge, of the form shown in dotted lines in Figure 1. The dual diagram to this is drawn in solid lines. Thus we want to consider Feynman diagrams with some number of external legs. The matrix model Green's functions

$$G_q \equiv \frac{1}{N^{q/2-1}} \langle \mathrm{Tr}\ \phi^q \rangle \tag{2.5}$$

generate in perturbation theory precisely diagrams of this form, and the dependence on N ensures that surfaces of Euler character χ are correctly weighted by N^{χ}. We may be tempted to identify these Green's functions as the partition function of a random surface with a single boundary of length q, with the normalisation corresponding to the subtraction of vacuum bubbles. However, more careful analysis reveals this reasoning to be incomplete. There are various diagrammatic contributions to G_q whose dual graphs correspond to pathological triangulations of bounded surfaces. Consider first contributions to G_q of the form shown in Figure 2. These graphs obviously correspond to two distinct surfaces with two distinct boundaries. Even worse, graphs of the form shown in Figure 3 correspond to surfaces with boundaries of length shorter than q.



Figures 2 and 3. Some contributions to G_q with pathological dual triangulations.

In order to perform the correct sum we must subtract these contributions from G_q, leaving only the diagrams dual to genuine triangulations with boundaries. This subtraction corresponds precisely to removing all disconnected Green's functions

from G,, which just leaves us with G_q^c, the connected Green's function with q external particles. So we conclude that the partition function for the sum over all triangulations with a single boundary of length q is given by G_q^c.

In Chapter 4 we briefly describe how a physical interpretation may be given to surfaces with a boundary generated by the full Green's function G,.

David [5] has claimed that different types of boundary, generated by full Green's functions, connected Green's functions, or one-particle-irreducible Green's functions just lead to an excluded volume effect and do not change the continuum behaviour. His argument was based on considering a surface with a single boundary of fixed length, and calculating the statistical distribution of the area. However, we will see that when the length of the boundary too is allowed to vary, different types of Green's function lead to different macroscopic behaviours, at least in the case of non-unitary models.

We now turn to the actual calculation of G_q^c. It is important to realise that in planar theories the relationship between the-Green's functions and. the connected Green's functions is not simply exponentiation of the generating functionals. As described in [1], following equation (31), for even potentials the correct relation is given by solving the combinatoric problem of clustering $2q$ points on a circle into r_1 pairs, r_2 quadruplets, . . . in all possible ways. This gives G_{2q} in terms of G_{2q}^c

$$G_{2q} = \sum_{\substack{r_n \geq 0 \\ \sum_n 2nr_n = 2q}} \frac{(2q)!}{(2q - \sum_n r_n)!} \frac{(G_2^c)^{r_1}}{r_1!} \frac{(G_4^c)^{r_2}}{r_2!} \cdots \frac{(G_{2n}^c)^{r_n}}{r_n!} \cdots \tag{2.6}$$

This relationship holds for any even potential. As discussed in [1] we may express (2.6) by the algebraic relation

$$z(j) = j\psi(z(j)) \tag{2.7}$$

where $z(j) = j\phi(j)$, and $\phi(j)$ and $\psi(j)$ are the generating functionals of Green's

functions and connected Green's functions respectively.

$$\phi(j) = \sum_{0}^{\infty} G_{2q} j^{2q} \qquad \psi(j) = 1 + \sum_{1}^{\infty} G^{c}_{2q} j^{2q} \tag{2.8}$$

The algorithm for finding G^c_q is then straightforward. Once we know the G_q's or equivalently the generating functional $\phi(j)$, we know z(j) explicitly in terms of j. Write this as

$$z(j) = \mathcal{W}(j) \tag{2.9}$$

where $\mathcal{W}$ is some known function. Then from (2.7) we substitute j = $\frac{z}{\psi}$ into (2.9) to give

$$z = \mathcal{W}(\frac{z}{\psi}) \tag{2.10}$$

This equation may then be solved to yield $\psi(z)$ explicitly in terms of z. We thee have the generating functional for the G^c_q's.

Now we show how to calculate the expectation value of the number of closed non-intersecting paths on a random surface. Imagine taking two triangulated surfaces, each with a (non-pathological) boundary of length q, and sewing them together along their boundaries. This yields a closed surface with a labelled non-intersecting path of length q. The path is drawn around the edges of the plaquettes since each external leg of a Feynman graph is dual to an edge of the corresponding tiling.

The partition function for the sum over surfaces of area A with a labelled q-cycle is

$$\sum_{\text{Surfaces S}} \#_{q-\text{cycles on S}} = Z_A \left\langle \#_{q-\text{cycles}} \right\rangle_A \tag{2.11}$$

where the subscript A denotes ,a restriction to graphs of area A. But by considering

the sewing construction above, this partition function may also be written

$$\frac{1}{2} \times \frac{1}{q} \times \sum_{0<B<A} G_q^c(B) G_q^c(A - B) \tag{2.12}$$

where the symmetry factors of $\frac{1}{2}$ and $\frac{1}{q}$ correspond to swapping the two sewed surfaces and rotating around the seam respectively. $G_q^c(A)$ refers to the connected Green's function with A vertices and no factor of $(-g)^A$, so we have for example

$$G_q^c = \sum_A (-g)^A G_q^c(A) \tag{2.13}$$

Now, for any observable of a random surface, the expectation values at fixed area and fixed cosmological constant are related by

$$Z \langle observable \rangle = \sum_A (-g)^A Z_A \langle observable \rangle_A \tag{2.14}$$

so for fixed cosmological constant,

$$\begin{aligned} \langle \#_{q-\text{cycles}} \rangle &= \frac{1}{2qZ} \sum_A \sum_{0<B<A} (-g)^A G_q^c(B) G_q^c(A - B) \\ &= \frac{1}{2qZ} G_q^c G_q^c \end{aligned} \tag{2.15}$$

To compute fixed area quantities such as $\langle \#_{q-\text{cycles}} \rangle_A$ we can simply extract the coefficient of $(-g)^A$ from this expression.

It is crucial to the above arguments that we use the connected Green's functions G_q^c since sewing together two G_q's may lead to disconnected surfaces. We may wonder what the effect would be if we replaced the G_q^c's with a more restricted set of Green's functions. An obvious class to consider would be the one particle irreducible functions. These would actually lead to smoother non-intersecting paths on the surface, as will be discussed later.

Since our interest is primarily in the continuum limits of these tilings, we must obviously consider the asymptotic behaviour of G_q^c as $q \to \infty$. In principle, this behaviour could be extracted by solving the implicit equation (2.7) for $\psi(z)$ and then performing a Taylor expansion. However, even in the simplest case of a purely quartic potential in $d = 0$, the implicit equation is cubic and in general this method is impractical.

Instead we take as an ansatz the behaviour $G_q^c \sim \frac{\alpha^q}{q^\beta}$ as $q \to \infty$ where α and β are some constants. Then from (2.8) we will have

$$\psi(z) \sim \sum_q \frac{(\alpha z)^q}{q^\beta} \qquad \text{as } z \to 1/\alpha \tag{2.16}$$

and so, for β non-integral,

$$\psi \sim (z - 1/\alpha)^{\beta-1} \qquad \text{as } z \to 1/\alpha \tag{2.17}$$

where here and throughout the paper we suppress irrelevant analytic terms when writing asymptotic relationships. When β is an integer, this behaviour contains logarithmic corrections.

We see that the singularity structure of ψ thus determines the asymptotic behaviour of G_q^c, and so to determine the latter we look for the singularity of ψ nearest to the origin. This corresponds to finding the smallest value of z for which $\psi(z)$ or one of its derivatives becomes infinite.

Following the procedure described below (2.8) leads to an implicit equation defining ψ in terms of z. We can write this equation as

$$H(\psi, z) = 0 \tag{2.18}$$

where H is some function of two variables. The partial derivatives of ψ with respect

to z can then be expressed as

$$\frac{\partial\psi}{\partial z} = -H_z/H_\psi$$
$$\frac{\partial^2\psi}{\partial z^2} = \frac{H_z(H_\psi H_{\psi z} - H_z H_{\psi\psi}) - H_\psi(H_\psi H_{zz} - H_z H_{\psi z})}{H_\psi{}^3} \qquad \text{etc.} \tag{2.19}$$

Typically there will be no solution of $H = 0$ with finite z and infinite ψ. Also, the partial derivatives of H will generally be finite for finite (ψ, z). Hence our tactic will be to search for points in the (ψ, z) plane at which $H_\psi = 0$, and simultaneously $H = 0$. These are potentially singular points. It must then be checked that they do indeed yield an infinite value for a derivative of ψ.

The solution of (2.18) generally leads to $\psi(z)$ being defined on a many-sheeted Riemann surface. It is only one of these sheets that satisfies (2.8). However, the procedure outlined above indiscriminately yields the singular points on all of the sheets, and so in general care must be taken to include contributions only from the physical sheet. This will be discussed in detail in the next chapter.

To illustrate our procedure we apply it to ϕ^4 theory in $d = 0$ defined by (2.2). In this case we have from [1]

$$H(\psi, z) = 3(1 - a^2)\psi^2(\psi - 1) \text{ t } 9a^4z^2\psi - a^2z^2[9a^2z^2 + (2 + a^2)^2] \tag{2.20}$$

where a and g are related by $12ga^4 + a^2 - 1 = 0$, so the critical point corresponding t o g = - & h a s u^2 = 2. From (2.20)

$$H_\psi = 3(1 - a^2)(3\psi^2 - 2\psi) + 9a^4z^2 \tag{2.21}$$

We see that $H = 0$ indeed allows no solutions with z finite and ψ infinite, and further that all partial derivatives of H remain finite for finite (ψ, z). Solving

$H_\psi = 0$ and $H = 0$ simultaneously yields

$$\psi = \frac{2a^2}{3(a^2-1)} \qquad z^2 = \frac{4}{9a^2(a^2-1)} \tag{2.22}$$

or

$$\psi = \frac{2+a^2}{3} \qquad z^2 = \frac{(a^2-1)(2+a^2)}{9a^2} \tag{2.23}$$

However, solution (2.23) also has $H_z = 0$, and in fact gives finite values for $\frac{\partial\psi}{\partial z}$ and all higher derivatives.

Since we have found just one pair of branch points for $\psi(z)$, which is even, these must occur on every sheet of the Riemann surface on which $\psi(z)$ is defined. In particular they occur on the "physical" sheet defined by (2.8) and so contribute to the asymptotic behaviour of G_q^c.

Away from criticality, ie for $a^2 \neq 2$, solution (2.22) has $H_z \neq 0$, and so $\frac{\partial\psi}{\partial z}$ is infinite. At the critical point $a^2 = 2$ solution (2.22) also has $H_z = 0$. This means that $\frac{\partial\psi}{\partial z}$ is finite but $\frac{\partial^2\psi}{\partial z^2}$ is still infinite. Thus the value of β changes at this point.

We have then, from the value of z in (2.22)

$$\alpha^2 = \frac{9a^2(a^2-1)}{4} \tag{2.24}$$

and so

$$G_{2q}^c \sim \frac{1}{q^\beta}\left[\frac{9a^2(a^2-1)}{4}\right]^q \qquad \text{as } q \to \infty \tag{2.25}$$

where $1 < \beta < 2$ for $a^2 \neq 2$ and $2 < \beta < 3$ for $a^2 = 2$. Closer analysis of the behaviour of H around the critical point shows that $\beta = \frac{3}{2}$ away from criticality and $\beta = \frac{5}{2}$ at criticality. This result will be derived in a more transparent way in the next chapter.

We may check (2.25) by using a saddle point approximation to sum the exact expression given by [1]

$$G^c_{2q} = \frac{(3q-1)!3^{1-q}}{(q-1)!(2q-1)!}\sum_{k=0}^{\infty}\frac{(-12g)^k(2k+q-1)!}{(k-q+1)!(k+2q)!} \tag{2.26}$$

this yields, away from criticality,

$$G^c_{2q} \sim 9\sqrt{\frac{3}{2\pi}}\,\frac{2-a^2}{a^2-1}\left[\frac{9a^2(a^2-1)}{q^{3/2}4}\right]^q \qquad \text{as } q\to\infty \tag{2.27}$$

which also shows $\beta = \frac{3}{2}$. At criticality, the saddle point moves off to infinity and (2.27) becomes invalid. However, the equation following (37) in [1]

$$G^c_{2q} = -\frac{\alpha^{2q}}{3^q}(a^2-1)^{q-1}A_q[3q(a^2-2)-2(a^2-1)] \tag{2.28}$$

tells us that $\beta = \frac{5}{2}$. This verifies our result.

Substituting the asymptotic behaviour of G^c_{2q} into our previous expression for $\langle \#_{q-\text{cycles}}\rangle$ we have

$$\langle \#_{q-\text{cycles}}\rangle \sim \frac{\alpha^{2q}}{q^{2\beta+1}} \qquad \text{as } q\to\infty \tag{2.29}$$

Apparently this implies some sort of phase transition at $\alpha = 1$. For $\alpha < 1$, $\langle \#_{q-\text{cycles}}\rangle$ is finite as $q\to\infty$ and in fact $\langle \sum_0^\infty q^r \#_{q-\text{cycles}}\rangle$ is finite for any power r. For $\alpha > 1$ though, these quantities are all infinite. We may thus take $\langle \#_{\text{cycles}}\rangle \equiv \langle \sum_0^\infty \#_{q-\text{cycles}}\rangle$ for example to be the order parameter.

Where does this phase transition occur? We might expect it to occur at the usual critical point, but this is not the case. Considering the ϕ^4 example, $\alpha^2 = \frac{9}{p}{}^2(a^2-1)$, so the new phase transition occurs at $a^2 = 4/3$, ie at $g = -1/64 \equiv g_{loop}$.

We therefore conclude that there is a new "long loop" phase in this model in which the typical length of a non-intersecting path on the surface is infinite, but the characteristic area is still finite. The phase diagram for ϕ^4 theory is shown in Figure 4.

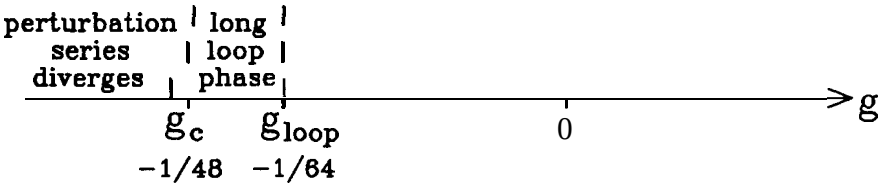


Figure 4. The phase diagram for the ϕ^4 model.

We will see in the next chapter that, for unitary theories, the existence of the long loop phase is independent of the nature of the tiling. So, for example, if we had chosen to tile the surface with octagons and dodecagons instead of squares, we would have found the same qualitative behaviour. It is gratifying that our intuitions about universality are thus far respected.

As mentioned before we may try replacing G_q^c in $\langle \#_{q-\text{cycles}} \rangle$ by some more restricted type of Green's function. The first natural candidate for this is the one-particle-irreducible function. What does this replacement mean in terms of the boundary of a triangulation? Well, since every propagator in the matrix model is dual to the edge of a triangle, a 1PI graph corresponds to a bounded triangulation that cannot be split into two parts by unsewing a single edge.

This means that the boundary can never come within one lattice separation of another part of itself, and so the restriction corresponds to a smoothing of jagged paths.

To calculate the effect of this smoothing on the long loop phase transition, write the generating functional of 1PI vertex functions as

$$\Gamma(x) = \Gamma_2 x^2 + \sum_{q=2}^{\infty} \Gamma_{2q} x^{2q} \tag{2.30}$$

As shown in [1], $\Gamma(x)$ may be determined for ϕ^4 theory from the condition $H = 0$

where

$$H(\Gamma, x) = 3x^2(1-a^2)(1+\Gamma)^2 + 9a^4\Gamma\left(1+\Gamma-\frac{\Gamma^2}{x^2}\right) - a^2\Gamma(2+a^2)^2 \qquad (2.31)$$

Solving $H = 0$ and $H_\Gamma = 0$ simultaneously shows that the phase transition occurs at

$$a^2 \text{ x}1.732 \qquad \text{i} \quad \text{e} \qquad g \approx \frac{1}{49.2^{1/2}} \qquad (2.32)$$

So the effect of smoothing paths is to shift the long loop phase transition toward the usual critical point, as we might have expected.

A natural extension to this might appear to be to calculate the n-particle irreducible graphs, corresponding to successively smoother boundaries. Sadly this is not possible since by considering the edgemost vertices of a graph, we see that no ϕ^n graph with external legs can be $(n-1)$-particle-irreducible. Any further smoothing of the boundary thus requires more subtle techniques.

3. Multicritical Models

So far we have dealt explicitly only with the simplest tiling of surfaces, ie a squaring, corresponding to a ϕ^4 potential. Now we extend our considerations to more complicated potentials [6,7,10] which will lead to a direct connection between the long loop phase transition and Coulomb forces of the Dyson sea,.

For simplicity, we restrict attention to even potentials, so that the relationships between G_q^c and G_q derived before are valid. This restriction of course implies that $G_q^c = 0$ for odd q, so when we write $G_q^c \sim \alpha^q/q^\beta$ we really mean $G_{2q}^c \sim \alpha^{2q}/q^\beta$. This shows that the sign of α is irrelevant.

Consider the random surface partition function given by

$$Z = \lim_{N\to\infty} \frac{1}{N^2} \ln \int d^{N^2}\phi e^{-N\mathrm{Tr}V(\phi/\sqrt{N})} \qquad (3.1)$$

where we let the potential $V(\phi)$ have the form

$$V(\phi) = \frac{1}{2}\phi^2 + g(\xi_4\phi^4 + \xi_6\phi^6 + \ldots + \xi_{2r}\phi^{2r}) \tag{3.2}$$

When we perform the Feynman diagram expansion of this we can again interpret $-ln(-g)$ as the cosmological constant, and now tiled surfaces are weighted by an additional factor of

$$\xi_4^{\#\text{squares}}\xi_6^{\#\text{hexagons}}\ldots\xi_{2r}^{\#2r-\text{gons}} \tag{3.3}$$

The powers of N are chosen to ensure that a surface of Euler character χ contributes with a factor of N^{χ}.

Now introduce the function [1,2]

$$F(\lambda) \equiv \left\langle \mathrm{Tr}\left(\frac{1}{\lambda - \phi}\right)\right\rangle \tag{3.4}$$

By Taylor expanding this and comparing with (2.8) we see that

$$\phi(j) = \frac{1}{j}F(\frac{1}{j}) \tag{3.5}$$

and hence from the definition $z(j) = j\phi(j)$ we have

$$z(j) = F(\frac{1}{j}) \tag{3.6}$$

From (2.7) we substitute $j = z/\psi$ into this to give the implicit equation $z = F(\psi/z)$. In our previous notation we write this as $H = 0$ where we may choose H to have the form

$$H(\psi, z) = F_0\left(\frac{\psi}{z}\right) - z \tag{3.7}$$

Now, in the large N limit, as shown by [2], $F(X)$ can be found by considering the Schwinger-Dyson equations derived from translation of ϕ in the integral in

(3.1). These give us

$$F(\lambda)^2 = V'(\lambda)F(\lambda) - \left\langle \frac{V'(\lambda) - V'(\phi)}{\lambda - \phi} \right\rangle \tag{3.8}$$

The last term is a polynomial in λ of degree $2(r-1)$ whose coefficients are combinations of $\langle \mathrm{Tr}\phi^2 \rangle$, $\langle \mathrm{Tr}\phi^4 \rangle$ etc. In principle these can be determined as explained in [2]. From (3.8) we see that the function *F(X)* is generally of the form

$$F(\lambda) = \frac{1}{2}\left\{ V'(\lambda) \ \mathrm{t} \ \sqrt{R(\lambda)} \right\} \tag{3.9}$$

where R(X) is a polynomial.

The properties of *F(X)* may be given a statistical mechanical interpretation in the following way. Recall that the method used in [1] to solve the model (3.1) is to reduce the integral over ϕ to an integral over a;, the eigenvalues of ϕ, by writing

$$d\phi = \prod_i da; \prod_{i<j} (\alpha_i - \alpha_j)^2 dU \tag{3.10}$$

where U is the unitary matrix diagonalising ϕ. Since the integrand of (3.1) depends only upon the eigenvalues of ϕ we have, up to a constant,

$$Z = \ln \int \prod_i d\alpha_i e^{-N\left(\sum_i V(\alpha_i) - \frac{1}{N}\sum_{i\neq j} \ln(\alpha_i - \alpha_j)\right)} \tag{3.11}$$

But this is just the free energy of N bosonic particles at temperature $\frac{1}{N}$ in a one dimensional potential V [2,6,7]. Each has charge $\frac{1}{\sqrt{N}}$ giving rise to logarithmic repulsions.

In the limit N $\rightarrow \infty$, the number of particles increases, the charge on each decreases, and the system "freezes" as the temperature drops and statistical fluctuations are suppressed. Then Z is just the classical energy of a gas of logarithmically interacting particles in the potential V. We refer to this as the "Dyson sea". Generally the particles, or eigenvalues, inhabit the region in which F has a discontinuity across the real axis [1].

Assuming that the Dyson sea is confined to a single interval $[-2a, 2a]$, (3.8) gives

$$F(X) = \frac{1}{2}\{V'(X) - P(\lambda)\sqrt{\lambda^2 - 4a^2}\} \tag{3.12}$$

where P(X) is an even polynomial of degree $2(r-1)$. We see from this that the domain of F is actually a 2-sheeted Riemann surface, with the sheets sewn together along $[-2a, 2a]$.

Now, as $\lambda \to \infty$, $F \to 0$ or ∞ depending upon the sheet. So from $H = 0$ we see that there can be no nonzero finite values of z at which $\psi \to \infty$.

Also, all derivatives of $F(X)$ are finite except at $\lambda^2 = 4a^2$, so away from the points $\psi/z = \pm 2a$ the partial derivatives of $H(\psi, z)$ are finite. There are thus just two possible types of contribution to the asymptotic behaviour of G_q^c. The first of these is from the zeros of H_ψ, and the second from the critical points $\psi/z = \pm 2a$.

As mentioned before, the equation $H(\psi, z) = 0$ defines $\psi(z)$ as a function on a non-trivial Riemann surface. We are only concerned with the "physical" sheet that satisfies (2.8), ie $\psi(z) \sim 1 + O(z)$. It is the branch points on this sheet that determine the asymptotic behaviour of G_q^c. Searching for the zeros of H_ψ, however, locates the branch points of $\psi(z)$ on every sheet of the surface. We will show later that all types of branch point can actually be made to occur on the physical sheet of ψ, but first we discuss in detail their contributions to the asymptotic behaviour of G_q^c.

Proceeding via the method described in the last chapter, we seek points with $H_\psi = 0$. But $H_\psi = \frac{1}{z}F'\left(\frac{\psi}{z}\right)$, so these are just stationary points of F. Further, where $H_\psi = 0$ we have $H_z = -\frac{\psi}{z^2}F'\frac{\psi}{z} - 1 = -1 \neq 0$, so at these points $d\psi/dz \sim \infty$ and hence $1 < \beta < 2$.

In fact we can calculate β exactly at an nth order stationary point of $F(X)$. Suppose that $F(X) \sim (A - \lambda)^n + B$ as $\lambda \to A$ where A and B are constants. Then

since $z = F(X)$ we have

$$\frac{d\lambda}{dz} \sim \frac{1}{(A-\lambda)^{n-1}} \tag{3.13}$$

and so

$$\lambda \sim (z-B)^{1/n} \tag{3.14}$$

up to irrelevant analytic terms in z. So we see that at a stationary point of $F(X)$ with $F' = F'' = \ldots = F^{(n-1)} = 0$ and $F^{(n)} \neq 0$ the contribution to the asymptotic behaviour of G_q^c (for q even) is given by

$$\alpha = \frac{1}{F(X)} \qquad \beta = \frac{n+1}{n} \tag{3.15}$$

This reproduces $\beta = 3/2$ for the simplest stationary points, agreeing with our previous result for a ϕ^4 potential.

We may be concerned that it can be arranged for $F(X) = 0$ at a stationary point. However, this can not occur for a branch point on the physical sheet of $\psi(z)$ and hence will not determine the behaviour of G_q^c.

Now we turn to the second type of contribution, that from the points $\lambda = \frac{\psi}{z} = \pm 2a$. The case of $F'(2a) = 0$ will not correspond to a branch point on the physical sheet of+(z). We thus assume in the following that $F'(2a) \neq 0$.

Let the behaviour of F near the edge of the Dyson sea be given by

$$F(X) \sim (\lambda - 2a)^{m-\frac{1}{2}} P(\lambda) + Q(\lambda) \tag{3.16}$$

where P and Q are polynomials in λ and $P(2a) \neq 0$. In general, $m = 1, 2, 3, \ldots$ with $m = 1$ corresponding to a non-critical theory, $m = 2$ to pure gravity at criticality, and higher values of m to the multicritical models. This definition of m

agrees with that used in [6]. From (3.16) we have that

$$F'(\lambda), F''(\lambda), \ldots F^{(m-1)}(\lambda) \tag{3.17}$$

are finite as $\lambda \to 2a$, and

$$F^{(m)}(\lambda) \sim \frac{1}{\sqrt{\lambda - 2a}} \qquad \text{as } \lambda \to 2a \tag{3.18}$$

Since z = F(X) this means that

$$\frac{dz}{d\lambda}, \frac{d^2z}{d\lambda^2}, \ldots \frac{d^{m-1}z}{d\lambda^{m-1}} \tag{3.19}$$

are finite as $\lambda \to 2a$ while

$$\frac{d^m z}{d\lambda^m} \sim \frac{1}{\sqrt{\lambda - 2a}} \tag{3.20}$$

The asymptotic behaviour of G_q^c is determined by the singularities of $\psi(z)$, or equivalently the singularities of X(z), and to determine the nature of these we need to know the derivatives $\frac{d\lambda}{dz}, \frac{d^2\lambda}{dz^2}$ etc. These can be derived by using the relationship

$$\frac{d^p\lambda}{dz^p} = \left(\frac{1}{dz/d\lambda}\frac{d}{d\lambda}\right)^p \lambda \tag{3.21}$$

or equivalently

$$\begin{aligned} \frac{d\lambda}{dz} &= \frac{1}{dz/d\lambda} \\ \frac{d^2\lambda}{dz^2} &= -\frac{d^2z/d\lambda^2}{(dz/d\lambda)^3} \\ \frac{d^3\lambda}{dz^3} &= \frac{3(d^2z/d\lambda^2)^2 - (dz/d\lambda)(d^3z/d\lambda^3)}{(dz/d\lambda)^5} \qquad \text{etc.} \end{aligned} \tag{3.22}$$

Using (3.21) recursively we see from (3.20) that for m = 1 all derivatives of λ are finite at $\lambda = 2a$, so this point does not contribute to the asymptotic behaviour of G_q^c.

For $m > 1$, we see from (3.22) that the first non-regular derivative is

$$\begin{aligned} \frac{d^m\lambda}{dz^m} &\sim \frac{1}{\sqrt{\lambda - 2a}} \qquad \text{as } \lambda \to 2a \\ &\sim \frac{1}{\sqrt{z - F(2a)}} \qquad \text{since } F'(2a) \neq 0 \end{aligned} \tag{3.23}$$

So up to analytic terms, $\lambda(z) \sim (z - F(2a))^{m-\frac{1}{2}}$. This shows that the contribution to the asymptotic behaviour of G_q^c (for even q) is

$$\alpha = \frac{1}{F(2a)} \qquad \beta = m + \frac{1}{2} \tag{3.24}$$

agreeing with the behaviour at $g = g_c$ for the ϕ^4 potential considered before.

We now turn to the question of which of the branch points of $\psi(z)$ actually occur on the physical sheet. For this it is useful to recall the physical interpretation of F(X) in terms of the Dyson sea.

First rewrite the integral over ϕ in (3.4) as an integral over the eigenvalues α_i of ϕ as shown in (3.10). Then performing the integral over α_1 gives

$$u(\lambda) = -\frac{1}{\pi}\mathrm{Im}\, F(\lambda + i\epsilon) \qquad \text{for } \lambda \in \mathrm{R} \tag{3.25}$$

where $u(\lambda)$ is the density of eigenvalues, or the charge density of the Dyson sea. The Coulomb potential C(X) due to the sea is given by

$$C'(X) = \int_{-2a}^{2a} d\mu \frac{-2u(\mu)}{\lambda - \mu} \tag{3.26}$$

$$= -\frac{2}{\pi}\mathrm{Im}\ \mathrm{P}\int_{-\infty}^{\infty} d\mu \frac{F(\mu + i\epsilon)}{\mu - \lambda} \tag{3.27}$$

where P denotes the principle valued part of the integral. Because F(X) is analytic in the upper-half plane, we have

$$\mathrm{Im}\ \mathrm{P}\int_{-\infty}^{\infty} d\mu \frac{F(\mu + i\epsilon)}{\mu - \lambda} = \pi \mathrm{Re}\, F(\lambda) \tag{3.28}$$

Substituting this into (3.27) gives with (3.25)

$$F(\lambda) = -\frac{1}{2}C'(\lambda) - i\pi u(\lambda) \tag{3.29}$$

on the branch in which the root in (3.12) is taken to have the same sign as λ, and

$$F(\lambda) = \frac{1}{2}[2V'(\lambda) + C'(\lambda)] + i\pi u(\lambda) \tag{3.30}$$

on the other branch. Henceforth we refer to (3.29) as the "physical" and to (3.30) as the "unphysical" branch of F(X).

Outside the Dyson sea, ie in the region $\lambda \in \mathbf{R}\backslash[-2a, 2a]$, the density of eigenvalues will be zero, so $u(\lambda)$ = 0. The Coulomb force will be monotonic decreasing in this region and will have the form shown in Figure 5. This shows that the physical branch (3.29) will have no stationary points on the real axis outside the support of $u(\lambda)$. It may, however, have stationary points off the real axis.

Now consider the unphysical branch (3.30). For some range of the parameters in the potential, *V'(X)* and $2V'(\lambda)$ + C'(X) will be as shown in Figure 6. This shows that for a potential of order $2r$, for some range of {g, ξ_i} there will be $2r - 2$ stationary points of F(X) with $\lambda \in \mathbf{R}\backslash[-2a, 2a]$. But as shown in the appendix, there are precisely 2r − 2 solutions of F'(X) = 0 in the complex plane. So in some region of the space of potentials, all stationary points of F(X) will be given by this construction.

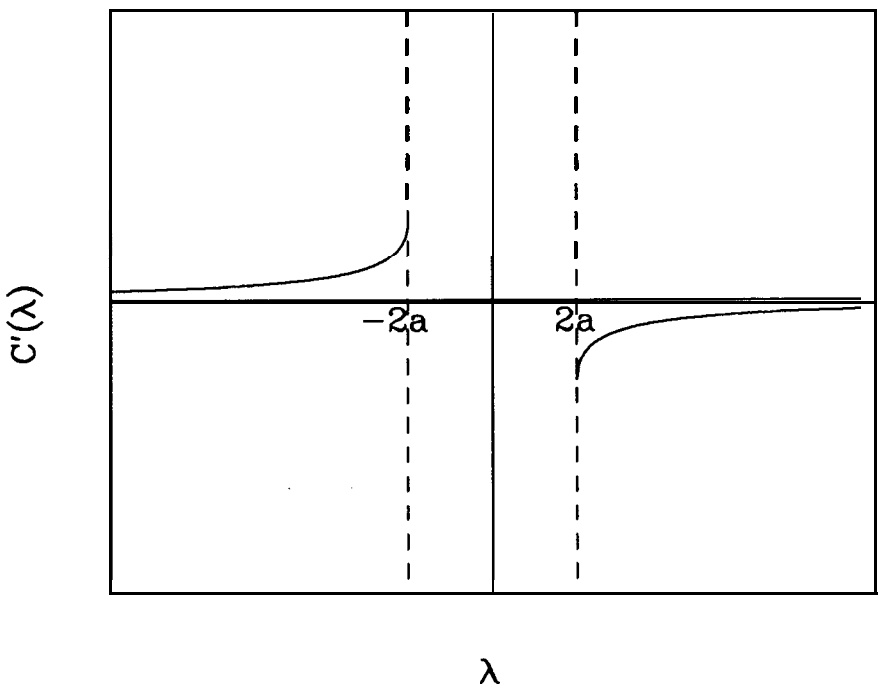


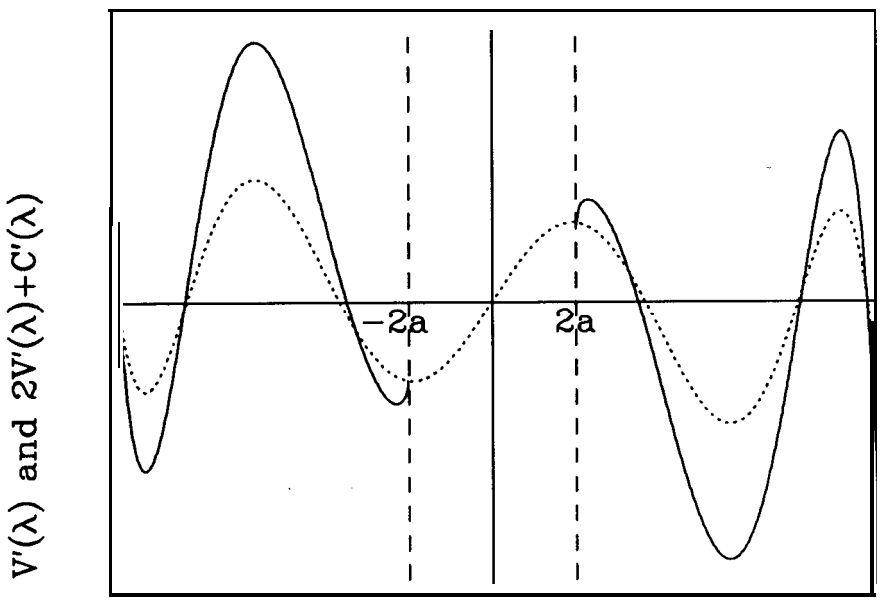
Figure 5. The Coulomb force C'(X) outside the Dyson sea.

As explained before we are interested in the values of F(X) at these points. We may interpret these physically as being the (locally) extremal values of the force felt by a test particle with the same mass as the particles in the Dyson sea and half their electric charge. Note that these values of F(X) are real.

The physical sheet of $\psi(z)$ is defined by $\psi(z) \sim 1 + O(z)$ as z $\rightarrow$ 0. This corresponds to $\lambda(z) \sim \frac{1}{z}$ as z $\rightarrow$ 0. Hence for small z, the physical branch of X(z) is given by inverting the physical branch of F.

As we increase $|z|$ this branch defines X(z) in a continuous and single-valued way until a singularity is reached. This "first" singularity gives the largest value of $|\alpha|$ and hence dominates the asymptotic behaviour of G_q^c.

There are three possibilities for the location of this dominant singularity. Firstly, if g is not at criticality, it may be a stationary point on the unphysical sheet of F. Secondly, if the theory is critical, it may lie at the singular point of F at the edge of the Dyson sea. Thirdly, it can be a stationary point on the physical sheet of F.



A

Figure 6. ***V'(X),*** shown dotted, and $2V'(\lambda)$ + C'(X), shown solid.

We now show by construction that each of these cases is possible. The first case can be achieved by choosing the parameter space {g, ξ_i} so that $2V'(\lambda)$ + $C'(\lambda)$ has the form shown in Figure 6. Then as argued before, all the stationary points of F lie on the real axis of the unphysical branch and it is the pair lying closest to the Dyson sea that determine the behaviour of G_q^c.

The second case is obtained from the first case by varying {g, ξ_i} in such a way that some stationary points of F on the unphysical branch (3.30) move to the edge of the Dyson sea.

The third case can be achieved by considering a limiting potential V such that the distribution of eigenvalues is given by

$$u(\mu) = \frac{1}{2}\{\delta(\mu - 2a) + \delta(\mu + 2a)\}$$

Using the expression

$$F(X) = \int_{\infty}^{\infty} d\mu \frac{u(\lambda)}{\lambda - \mu} \tag{3.31}$$

this gives

$$F(\lambda) = \frac{\lambda}{\lambda^2 - 4a^2}$$

Solving z = F(X) then yields

$$\lambda(z) = \frac{1}{2z} + \sqrt{\frac{1}{4z^2} + 4a^2}$$

where we've used $\psi(0)$ = 1. We see from this that the contributing branch points are at z = $\pm\frac{i}{4a}$.

We can change the potential a little from this limiting value to give one of the form (3.2) with the Dyson sea confined to a single interval. The behaviour of F away from the sea will then change only slightly, and in particular the dominant singularities of X(z) will still be stationary points of F on the physical branch (3.29).

Note that this construction leads to a distribution $u(\mu)$ that is non-monotonic from the centre to the edge of the Dyson sea. In general, by considering (3.31) to define a Coulomb force in the complex plane, we can see that stationary points of F can occur on the physical branch (3.29) only for non-monotonic distributions $u(\mu)$.

We now discuss the physical interpretation of a complex α. Writing $\alpha = Re^{i\theta}$ we have

$$G_q^c \sim \frac{R^q}{q^\beta} \cos \theta q \qquad \text{as } q \to \infty \tag{3.32}$$

or more precisely since we're dealing with an even potential

$$G^c_{2q+1} = 0$$
$$G^c_{2q} \sim \frac{R^{2q}}{q^\beta} \cos 2\theta q \qquad \text{as } q \to \infty \tag{3.33}$$

For $R > 1$ this signals some sort of "non-unitary-long-loop" phase, in which q-cycles of quantised length $n\pi/\theta$ are dominant. For want of a better name we may call the case with $R < 1$ a "non-unitary-short-loop" phase.

For unitary theories we can say much more about the dominant singularity of $\psi(z)$. For any theory with real weights, the G^c_q's will be real. But recall that the generating functional $\psi(z)$ is defined as

$$\psi(z) = 1 + \sum_1^\infty G^c_{2q} z^{2q} \tag{3.34}$$

so for z $\in$ R we must have $\psi(z) \in$ R.

Now restrict attention to unitary theories. For these the asymptotic behaviour $G^c_q \sim \frac{\alpha^q}{q^\beta}$ clearly requires that $\alpha \in$ R. The dominant singularity of ψ will then be

$$\psi(z) \sim (z - \frac{1}{\alpha})^{\beta-1} \qquad \text{as } z \to \frac{1}{a!} \tag{3.35}$$

and so the dominant singularity of ψ(z) must lie on the real line. The reality of G^c_q then implies that $\psi(z) \in$ R at this point, and so $\frac{\psi}{z} \in$ R.

This shows that for unitary theories in d = 0 the stationary value of F(X) determining α must occur for $\lambda \in$ R and must have F(X) $\in$ R. Considering the behaviour of F on the real line, as shown in Figures 5 and 6, we see that for a unitary theory there will be just one such stationary point.

Since we expect that all unitary tilings of the surface will lie in the same universality class, it is wise to check that they all have a long loop phase, and

that by choosing to tile the surface with, say, octagons instead of squares, the long loop phase does not vanish. This is easily done by considering the unitary theory at criticality. Then $\alpha = 1/F(2a) = 2/V'(2a)$. Now, the limiting critical unitary potential giving the greatest value of $V'(2a)$ will be the one behaving as $\frac{1}{2}\phi^2$ for $|\phi| < 2a$ and dropping off steeply outside this region. This will have a = 1, so $V'(2a) = 2$. A actual critical unitary potential will thus have $V'(2a) < 2$, and hence the theory will have $\alpha > 1$ at criticality. This shows that the long loop phase transition, at which $\alpha = 1$ will indeed occur in the region in which perturbation theory converges.

This argument shows that loop considerations do not break down the universality class of unitary theories, ie all unitary theories have a long loop phase with β as given in Chapter 2.

To conclude this section we note that the asymptotic behaviour of the full Green's functions G_q can be obtained even more simply from F(X). Recall that

$$\phi(j) = \sum_{q=0}^{\infty} G_q j^q \tag{3.36}$$

so that if

$$G_q \sim \frac{\tilde{\alpha}^q}{q^{\tilde{\beta}}} \qquad \text{as } q \to \infty \tag{3.37}$$

then

$$\phi(j) \sim (j - \frac{1}{\tilde{\alpha}})^{\tilde{\beta}-1} \qquad \text{as } j \to \frac{1}{\tilde{\alpha}} \tag{3.38}$$

But since F(X) $\equiv \frac{1}{\lambda}\phi\left(\frac{1}{\lambda}\right)$ this will mean F(X) $\sim (\lambda - \tilde{\alpha})^{\tilde{\beta}-1}$ as $\lambda \to \tilde{\alpha}$. So to determine the asymptotic behaviour of G_q we examine the singularities of F(X). The location of the singularity closest to the origin gives $\tilde{\alpha}$ and the order of the singularity gives $\tilde{\beta}$. But from (3.16) we see that the only singularity occurs at $\lambda = 2a$ and near this point F has behaviour F(X) $\sim (\lambda - 2a)^{m-1/2}$ for a multicritical model. Hence we always have $\tilde{\alpha} = 2a$ and $\tilde{\beta} = m + \frac{1}{2}$.

We can easily see that both at and away from criticality this agrees with the exact expression for ϕ^4 theory

$$G_{2q} = \frac{(2q)!}{q!(q+2)!} a^{2q} \left[2q + 2 - qa^2\right] \tag{3.39}$$

derived in [1].

For (multi) critical theories with monotonic eigenvalue distribution, since the dominant singularity is at the edge of the Dyson sea we see that $\beta = \tilde{\beta}$ and so the asymptotic behaviour of G_q^c is similar to that of G_q up to a redefinition of α. In particular this holds for unitary theories, which correspond to pure gravity.

The dubious reader may be concerned that we can arrange for $\alpha > \tilde{\alpha}$ by choosing the potential *V(X)* appropriately. This corresponds to G_q^c growing faster with q than G_q. We must remember though that this only occurs in non-unitary theories, so some diagrams-contribute a negative weight and there is no objection to the connected Green's functions being larger in magnitude than the full Green's, functions.

4. Non-critical Open Strings

Now that we have an understanding of the correct boundary operators in the matrix models, we can attempt to use them to construct an open string theory. Kazakov [11] has claimed to have discovered an exactly soluble open string theory in $d = 0$ by choosing a particular potential for the one-matrix model. His sum over tilings is generated by the partition function

$$\int d^{N^2}\phi \; e^{\mathrm{Tr}\left[-\frac{1}{2}\phi^2 + \frac{g}{4}\phi^4 - \frac{\gamma}{g}\ln(\kappa^2 - g\phi^2)\right]} \tag{4.1}$$

Expansion of the logarithm in powers of κ yields vertices of arbitrarily high order. Kazakov interprets these as being dual to boundaries of the worldsheet.

However, his discussion fails to mention three undesirable types of graph that may be generated by (4.1). Firstly, as described before, the "hole" vertices may be self-contracted. This situation corresponds to non-connected Green's functions and leads to "collapsed" boundaries. Secondly, two of the hole vertices may be directly contracted, implying that boundaries can touch each other along an edge with no intervening worldsheet. Thirdly the surface may separate into many parts as a result of the dual diagram being composed of sub-diagrams connected only through hole vertices.

A rather convoluted interpretation of the triangulation may be dreamt up to redeem Kazakov's model. This involves considering the surface to be composed not just of the conventional tiles (triangles, squares etc), but also of thin "rods" carrying no area, but just boundary length. Then when a hole-vertex contracts with any other hole-vertex, including itself, we insert a rod in the tiling dual to the offending propagator. This means that two holes touching with no intervening worldsheet are really separated by a line of rods. Similarly a hole-vertex with q legs that has s self-contractions is to be interpreted as a bona fide hole of length $q - 2s$ in the worldsheet, with s rods protruding into its interior. The problem of surface fragmentation could be interpreted away by allowing tiles to be joined not only along edges, but also by being "hinged" together at vertices.

At this point we may be tempted to appeal to universality and claim that the continuum limit will be insensitive to the details of the tiling. In order to test this claim we would like to be able to solve a model that explicitly excludes the unwanted contractions. Comparison of this with Kazakov's model would then establish whether the universality classes are in fact the same.

Unfortunately, we do not have such a model at our disposal. However, we may check the universality in a more restricted context. Consider worldsheets with a single hole. As discussed before, these should properly be generated by the connected Green's functions G_q^c. If instead we used the full Green's functions G_q, then we would obtain contributions to the partition function that are analogous to the

unwanted graphs of (4.1). Hence comparing these two models will give an indication as to whether the presence of Kazakov's pathologies alters the universality class.

We thus consider a surface with just one boundary, whose length is allowed to vary dynamically. The partition function will be

$$Z = \sum_{\text{Surfaces}} (-g)^A t^q \tag{4.2}$$

where the sum is over surfaces with (varying) area A and perimeter q. We give g the usual interpretation in terms of the cosmological constant, and take $-\ln t$ to be the energy density of the boundary (corresponding to a mass at the end of an open string).

As explained in Chapter 2, the sum over triangulations with a well-behaved boundary is equivalent to the sum over connected Green's functions. We can thus write

$$Z = \sum_q G_q^c t^q \frac{1}{q} \tag{4.3}$$

where the symmetry factor $\frac{1}{q}$ corresponds to rotations of the surface with respect to the boundary. Now, as $q \to \infty$ we have $G_q^c \sim \frac{\alpha^q}{q^\beta}$, so the contribution to the partition function from large perimeters will be

$$\begin{aligned} Z &\sim \sum_q \frac{(\alpha t)^q}{q^{\beta+1}} \\ &= \sum_q \frac{(\alpha/\alpha_*)^q}{q^{\beta+1}} \end{aligned} \tag{4.4}$$

where α_* is the critical value of α, given by $\alpha_* = \frac{1}{t}$. Let g_* be the value of the cosmological constant corresponding to α_*. Then for a given value of t a phase transition will occur at g = g_*. We now investigate the behaviour of the surface in the vicinity of this point.

It is tempting to do this simply by inserting the asymptotic behaviour of G_q^c into (4.3). In doing this, though, care must be taken since as we alter g, the range of validity of the asymptotic series will change. Instead we notice that

$$\frac{\partial Z}{\partial t} = \frac{1}{t}\sum_{q=1}^{\infty} G_{2q}^c t^{2q} \qquad (4.5)$$

$$\lambda(t) - \frac{1}{\iota}$$

where as before X(z) $\equiv \frac{\psi(z)}{z}$, so the partition function is simply determined from the generating functional ψ.

To determine the behaviour of the random surface for g near g_* there are two cases to consider, these being $g_* > g_c$ and $g_* = g_c$ where g_c is the critical point for a surface with no holes.

We treat first the case of $g_* > g_c$. For this the asymptotic behaviour of G_{2q}^c will be determined by a stationary point of F as described in Chapter 3. At g = g_* let the behaviour of F be given by

$$F(\lambda) \sim \frac{\iota}{\alpha_*} - A(\lambda - B)^{\prime\prime} \qquad \text{as } \lambda \to B \qquad (4.6)$$

where A and B are constants and we have used the fact that $F(B) = 1/\alpha_*$. At g = $g_* + \epsilon$ this will become

$$F(\lambda) \sim \frac{1}{\alpha_*} - A(\lambda - B)^n + C\epsilon(\lambda - B) + D\epsilon + O(E) \qquad \text{as } \lambda \to B \qquad (4.7)$$

where C and D are constants. To determine X(z) we solve the implicit equation z = $F(X)$. This yields from (4.7)

$$\lambda^{(r)}\left(\frac{1}{\alpha_*}\right) \sim \epsilon^{\frac{1}{n}-r} + \text{finite} \qquad \text{as } \epsilon \to 0 \qquad (4.8)$$

We have then from (4.5) that the divergent statistical moments of the length and

perimeter are given by

$$\langle q^r \rangle \sim \epsilon^{\frac{1}{n}-r}$$
$$\langle A' \rangle \sim \epsilon^{1+\frac{1}{n}-r} \tag{4.9}$$

where we have used the fact that since $\beta > 0$ for cases of interest, (4.4) implies that Z is finite at g = g_*. Surfaces are thus dominated by large perimeter length and small area.

Now we turn to the case of $g_* = g_c$. For this the asymptotic behaviour of G_q^c may be determined by either a stationary point of F or the behaviour of F at the edge of the Dyson sea, as explained in Chapter 3. The first possibility leads to the same types of behaviour as for $g_* > g_c$. We restrict attention in the following to the second case.

Suppose that the model is multicritical of order m as defined by (3.16). At $q = g_c$ we have

$$F(2a_c + \eta) \sim A + C\eta + O(\eta^{\frac{3}{2}}) \qquad \text{as } \eta \to 0 \tag{4.10}$$

where A and C are some constants. We begin by showing that at $q = g_c + \epsilon$ this becomes

$$F(2a + \eta) \sim A + B\epsilon^{\frac{1}{m}} + (C + D\epsilon^{\frac{1}{m}})\eta + E\epsilon^{1-\frac{1}{m}}\sqrt{\eta} + O(\eta^{\frac{3}{2}}) \tag{4.11}$$

where A, B, C, D, E are constants.

For this recall that as shown in [2], we may express the density of eigenvalues $u(\mu)$ as

$$u(\mu) = \frac{1}{\pi} \int_{\mu^2/4}^{a^2} \frac{W'(r)}{\sqrt{4r - \mu^2}} dr \tag{4.12}$$

with $W(r)$ given by

$$W(r) = r + g \sum_{p \geq 2} \xi_{2p} \frac{(2p)!}{p!(p-1)!} r^{p-1} \tag{4.13}$$

where the coefficients ξ_i are defined by (3.2). At multicriticality we have [6,7]

$$W(r) \sim 1 - K(a_c^2 - r)^m \qquad \text{as } r \to a_c^2 \tag{4.14}$$

where K is a constant. Hence from (4.13) we have at $g = g_c + \epsilon$

$$W(r) \sim 1 + \epsilon - K(1-\epsilon)(a_c^2 - r)^m \qquad \text{as } r \to a_c^2 \tag{4.15}$$

We can now determine a from the condition $W(a^2) = 1$, yielding

$$a^2 \sim a_c^2 - L\epsilon^{\frac{1}{m}} \qquad \text{as } \epsilon \to 0 \tag{4.16}$$

where L is a constant. We can then write, for $g = g_c + \epsilon$,

$$\begin{aligned} W(r) &\sim 1 + \epsilon - K(1-\epsilon)(a^2 + L\epsilon^{\frac{1}{m}} - r)^m \qquad \text{as } r \to a^2 \\ &= 1 + [(a^2 - r)^m + \epsilon^{\frac{1}{m}}(a^2 - r)^{m-1} + \ldots + \epsilon^{1-\frac{1}{m}}(a^2 - r)] \end{aligned} \tag{4.17}$$

where in the last line we have suppressed irrelevant constant coefficients. Now; from (4.12) we can see that a term of the form $(a^2 - r)^p$ in $W(r)$ leads to a term in $u(\mu)$ of the form $u(2a - \delta) \sim \delta^{p-\frac{1}{2}}$. We thus have from (4.17)

$$u(2a - \delta) \sim 2\text{-M} + O(6) \tag{4.18}$$

From this we can calculate the Coulomb force just outside the sea for $g = g_c + \epsilon$, giving

$$C'(2a + \eta) \sim \epsilon^{1-\frac{1}{m}}\sqrt{\eta} + O(\eta) \tag{4.19}$$

We also have, from (4.16) that at $g = g_c + \epsilon$

$$V'(2a + \eta) \sim A + B\epsilon^{\frac{1}{m}} + (C + D\epsilon^{\frac{1}{m}})\eta + O(\eta^2) \tag{4.20}$$

Using (3.30) , the expression for F in terms of V' and C', we thus obtain (4.11) as required.

We now use (4.11) to find X(z) by solving the equation z = *F(X)*. This gives

$$\lambda^{(r)}\left(\frac{1}{\alpha_*}\right) \sim \epsilon^{1-\frac{2r+1}{2m}} + \text{ finite} \tag{4.21}$$

From this we obtain the divergent statistical moments of the length and perimeter as

$$\begin{aligned} \langle q^r \rangle &\sim \epsilon^{1+\frac{1}{2m}-\frac{r}{m}} \\ \langle A^r \rangle &\sim \epsilon^{1+\frac{1}{2m}-r} \end{aligned} \tag{4.22}$$

These show that for pure gravity, with m = 2, the area is the square of the perimeter, leading to "nice" surfaces. For higher multicritical models, the area is large and the perimeter is small, so the boundaries shrink to punctures.

We now compare these results to those obtained by using the full Green's functions G_q. For these we will have

$$\begin{aligned} t\frac{\partial Z}{\partial t} &= \sum_{q=0}^{\infty} G_{2q} t^{2q} \\ &= \phi(t) \end{aligned} \tag{4.23}$$

Using $\phi(j) \equiv \frac{1}{j}F(\frac{1}{j})$ and the expression (4.11) we can obtain for a multicritical point of order m that at $g = g_c + \epsilon$

$$\phi^{(r)}\left(\frac{1}{2a_c}\right) \sim \epsilon^{1-\frac{2r+1}{2m}} + \text{ finite} \tag{4.24}$$

The case of non-critical surfaces is obtained by using this with m = 1.

Now, in the case of unitary theories, the dominant stationary point of F away from criticality is always quadratic, so since (4.9) with n = 2 yields the same behaviour as (4.24) with m = 1, the behaviour of the surface is independent of whether it is constructed from full or connected Green's functions.

Further, for (multi) critical theories with a monotonic density of the Dyson sea the asymptotic behaviour of the G_q^c's is determined by the critical point of F at the edge of the sea, so again the behaviour of the surface is insensitive to the presence of the pathological contractions. Models with a non-monotonic Dyson sea, however, can give different behaviours depending on whether Kazakov's pathologies are present.

5. The Branched Polymer Phase for d=1

So far we have dealt exclusively with discretisations of a surface with no embedding dimension. As has been shown in [3] the method of matrix models may be extended to describe triangulations of surfaces embedded in one external dimension. To do this we write the partition function as

$$Z = \lim_{N\to\infty} \frac{1}{N^2} \ln \int \mathcal{D}\phi(t) e^{-\mathrm{Tr}[\dot{\phi}(t)^2 + NU(\phi(t)/\sqrt{N})]} \tag{5.1}$$

with

$$U(\phi) = \phi^2 + g(\xi_4\phi^4 + \xi_6\phi^6 + \ldots + \xi_{2r}\phi^{2r}) \tag{5.2}$$

where now the integral over the matrix ϕ has become a functional integral, with ϕ depending on the extrinsic coordinate t.

Note that the coefficients of $\dot{\phi}^2$ and ϕ^2 are chosen to be 1, not $\frac{1}{2}$ as in [1]. Our choice gives the propagator the desired normalisation, theirs leads to an extra factor of 2.

This partition function may be expanded in terms of Feynman diagrams as in the case of $d = 0$. The difference now is that to each vertex is associated a value of the embedding dimension, $t = t_i$, and for each pair of neighbouring vertices with $t = t_i$ and $t = t_j$ the partition function contains an additional factor of $e^{-|t_i - t_j|}$. The belief is that this propagator should yield the same continuum results as the Polyakov propagator $e^{-(t_i - t_j)^2}$, and so, by considering the tiling dual

to the Feynman diagram expansion, (5.1) should provide a discretisation of strings in one dimension.

The continuum theory of 2d gravity gives us hints that for embedding dimension $d > 1$ the intrinsic geometry of the worldsheet enters a "branched polymer phase" in which "spikes" in the internal metric are energetically favoured [12]. It is therefore of great interest to study the geometry of the surface at the critical value of the dimension $d = 1$, and in particular to investigate whether at this critical point there are any indications of the pathologies that lie beyond the branched polymer phase transition.

To do this we will first extend our previous discussion of loops on surfaces and describe how it may be used to test for the presence of a branched polymer phase. We first need to generalise the full Green's function $(\mathrm{Tr}\ \phi^q\rangle$ of $d = 0$ to the case of $d = 1$. The natural generalisation of this is the correlation function

$$\langle \mathrm{Tr}\ \phi(t_1)\phi(t_2)\ldots\phi(t_q)\rangle \tag{5.3}$$

which generates a surface with a (pathological) boundary of length q whose segments lie at positions $t = t_1, t = t_2, \ldots t = t_q$ in the embedding space. Unhappily, the calculation of this quantity requires an understanding of the angular variables [1] of the $d = 1$ theory, which is currently lacking.

For our purposes it will be sufficient to consider the specialised form of these correlation functions $(\mathrm{Tr}\ \phi(t_0)^q\rangle$. This corresponds to a surface with a pathological boundary of length q, where the entire boundary lies at the same value of the extrinsic coordinate $t = t_0$. We may use exactly the procedure described in Chapters 2 and 3 to convert these full equal time Green's functions to the connected equal time Green's functions $G_q^c(t_0)$. These correspond to surfaces with nice boundaries of length q lying at a single value of the extrinsic coordinate, $t = t_0$.

It should be noted that strictly speaking it is the centres of the tiles that carry embedding coordinates. The quantities computed here will correspond to bounded

surfaces in which the centres of the edgemost tiles lie close to the specified value of the embedding dimension.

As shown in Chapter 2, we can use $G_q^c(t_0)$ to calculate $\langle \#_{q-\text{cycles}}(t_0)\rangle$, the expected number of non-intersecting paths on the surface lying at $t = t_0$. This may be interpreted as giving the spectrum of intrinsic string lengths if the worldsheet is sliced at $t = t_0$. We hope that some indications of the intrinsic geometry of the surface may be obtained from this spectrum.

It is remarkable that the model (5.1) can be solved exactly. This is done by reformulating it as an N-body fermion problem as first illustrated by [1]. We may use the procedure of [13] to calculate any U(N) invariant quantity by reducing the integral over the matrix ϕ to an integral over its eigenvalues λ_i. This enables us to rewrite the partition function (5.1) as

$$Z = \lim_{N\to\infty} \frac{1}{N^2} \ln \int \prod_i \mathcal{D}\lambda_i(t)\Delta^2\{\lambda_i(0)\}\Delta^2\{\lambda_i(T)\}e^{-\int_0^T dt \sum_i \dot{\lambda}_i^2 + NU(\lambda_i/\sqrt{N})} \quad (5.4)$$

where A is a Vandermonde determinant

$$\Delta\{\lambda_i\} = \prod_{i<j}(\lambda_i - \lambda_j) \quad (5.5)$$

If these determinants were not present, the functional integral would describe the quantum mechanics of N bosons moving in a potential U. The determinants have the effect of giving the particles Fermi statistics. We can use this correspondence to calculate the equal time correlation functions in terms of fermion many body operators

$$\begin{aligned}\langle \mathrm{Tr}\, \phi(t_0)^q\rangle &= \frac{1}{Z}\int \prod_i \mathcal{D}\lambda_i(t)\Delta^2(0)\Delta^2(T)e^{-\int_0^T dt \sum_i \dot{\lambda}_i^2 + NU(\lambda_i/\sqrt{N})} \sum_i \lambda_i(t_0)^q \\ &= \langle \chi_0| \sum_i \hat{\lambda}_i^q |\chi_0\rangle\end{aligned} \quad (5.6)$$

where $|\chi_0\rangle$ is the N-fermion ground-state, and $\hat{\lambda}_i$ are the one body fermion operators. Note though that our choice of propagator means that the kinetic energy of

the fermions is $\dot{\lambda}^2$ rather than the usual $\frac{1}{2}\dot{\lambda}^2$.

For surfaces of spherical topology we take the large N limit of the Green's functions. This can be done by using the WKB approximation [1,3], yielding

$$G_q(t_0) = \frac{1}{2\pi} \int_{-\lambda_{\epsilon_F}}^{\lambda_{\epsilon_F}} d\lambda \; \lambda^q \sqrt{\epsilon_F - U(\lambda)} \tag{5.7}$$

where ϵ_F is the Fermi-level and λ_{ϵ_F} is the position of the edge of the Fermi sea, so $U(\lambda_{\epsilon_F}) = \epsilon_F$. The Fermi-level ϵ_F can be obtained in terms of g from the normalisation condition

$$1 = \frac{1}{2\pi} \int_{-\lambda_{\epsilon_F}}^{\lambda_{\epsilon_F}} d\lambda \sqrt{\epsilon_F - U(\lambda)} \tag{5.8}$$

We may check (5.7) by substituting g = 0. This will correspond to the external legs being joined just by propagators. The absence of vertices means that the Green's functions are insensitive to the number of embedding dimensions: Comparison with the solution of ϕ^4 theory in d = 0 given by [1] shows that this is indeed the case.

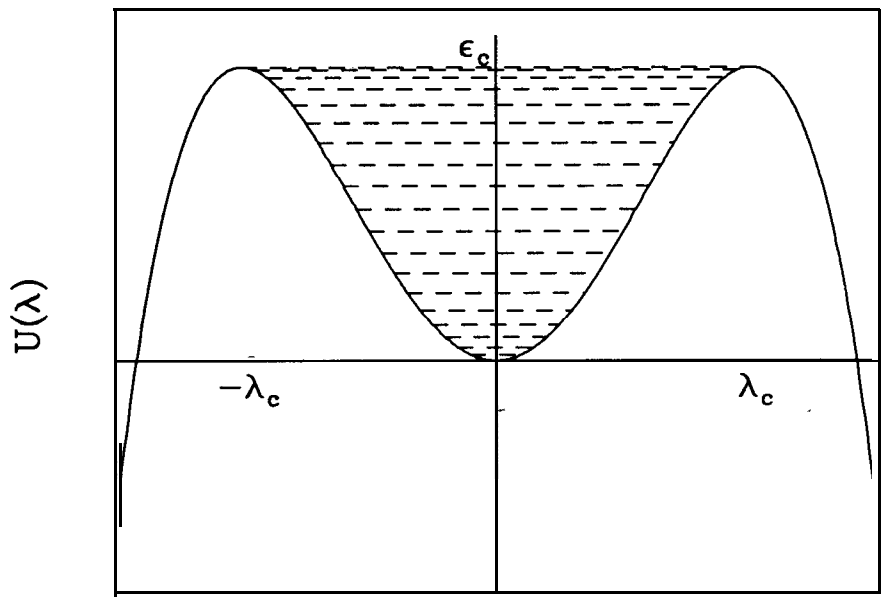
To calculate the properties of loops on the surface we are really interested in the connected Green's functions $G_q^c(t_0)$. We find the asymptotic behaviour of these as was done for the case of d = 0 by defining

$$\begin{aligned} F(\mu) &= \left\langle \mathrm{Tr}\left(\frac{1}{\mu - \phi(t_0)}\right)\right\rangle \\ &= \frac{1}{2\pi} \int_{-\lambda_{\epsilon_F}}^{\lambda_{\epsilon_F}} d\lambda \frac{\sqrt{\epsilon_F - U(\lambda)}}{\mu - \lambda} \end{aligned} \tag{5.9}$$

We then search for the stationary points and singularities of F as described before. For simplicity we will restrict attention to the critical point, at which the Fermi sea almost spills over the potential and the characteristic area of the surface becomes infinite. At this point we will write $\epsilon_F \to \epsilon_c$ and $\lambda_{\epsilon_F} \to \lambda_c$.

Consider first the case of unitary theories. For these, as described below (3.34) the only stationary points that can determine the asymptotic behaviour of $G_q^c(t_0)$ are the ones on the real line at which F takes real values.

At criticality, a unitary theory will have a potential U(X) of the form shown in Figure 7. Now, since $F(p) \to 0$ or ∞ as $\mu \to \infty$ depending on the branch, there can be no nonzero finite values of z at which $\psi(z) \sim \infty$, exactly as we found in the case of $d = 0$.



A

Figure 7. *U(X)* and the Fermi sea at criticality.

We consider now the stationary points of F that may contribute to α. Observe that $F(p)$ has a domain consisting of infinitely many sheets sewn together along the interval $[-\lambda_c, \lambda_c]$. As we move μ around a closed path in the complex plane, with the path crossing $[-\lambda_c, \lambda_c]$ in the direction of decreasing imaginary part a total of n times, we can easily see from (5.9) that

$$F(\mu) \to F(\mu) + in\sqrt{\epsilon_c - U(\mu)} \tag{5.10}$$

The important observation that distinguishes this from the case of $d = 0$ is that for $\mu \in$ R, $\sqrt{\epsilon_c - U(\mu)} \in$ R. Hence $F(p)$ is only real on the real axis for the first branch. But on this branch F is just the Coulomb force due to a charged sea of density $\sqrt{\epsilon_c - U(\mu)}$. This force clearly has no stationary points away from the origin.

The only points that can contribute to α are thus the critical points $\mu = \pm\lambda_c$, and these give $\alpha = 1/F(\lambda_c)$. Since

$$F(X) \sim (\lambda - \lambda_c)\ln(\lambda - \lambda_c) \qquad \text{as } \lambda \to \lambda_c \tag{5.11}$$

we have from (3.21) that $2 < \beta < 3$.

We now establish a lower bound of unity on α. This is done by the same argument as in the $d = 0$ case. The limiting critical unitary potential giving the greatest value of $F(\lambda_c)$ will be the one behaving as ϕ^2 for $|\phi| < \lambda_c$ and dropping off steeply outside this region. The normalisation condition (5.8) shows that this has $\epsilon_c = 4$, and then from (5.9) we have $F(\lambda_c)=1$. Any actual critical unitary potential will then have $F(\lambda_c) < 1$ and hence $\alpha > 1$.

This shows that any unitary $d = 1$ matrix model will have a long loop phase in which $\langle \#_{q-\text{cycles}}(t_0)\rangle$ grows exponentially with q. This was not a priori clear from the $d = 0$ result, since one might imagine that the restriction that loops lie on a given slice of the embedding dimension would significantly decrease their number.

We may imagine that if the worldsheet lay in a branched polymer phase for this $d = 1$ case, then the characteristic size of non-intersecting loops at a single value of t would be small. More precisely there would be many loops of length on the order of the circumference of the polymers, but very few of greater length. Since we have established the existence of a long loop phase, this would seem to imply that at $d = 1$ there is no branched polymer phase.

6. Conclusions

We have pointed out a flaw in the usual boundary operators of the matrix models of 2d gravity, and shown how to correct this defect. This has led to a means of calculating the statistical distributions of lengths of non-intersecting paths on a triangulation. By investigating the point at which the expected length of these paths diverges, we have established the existence of a new "long loop" phase in which the characteristic loop length is infinite, but the characteristic area is still finite.

This analysis has been carried out not only for unitary models, but also for Kazakov's multicritical models [10]. We find that when we consider perturbations of the potential in non-unitary directions, the phase diagram becomes substantially richer. If we restrict attention to potentials yielding a Dyson sea whose density decreases monotonically from the centre to the edge, then the asymptotic behaviour of the full Green's functions is qualitatively similar to that of the connected Green's functions.

Our objections to the conventional boundary operators also hold for Kazakov's proposed solution to non-critical open string theory, which we believe leads to pathological behaviour of the worldsheet boundaries. Though we have been unable to solve the correct version of this theory in its full glory, we have considered the simplest case in which the worldsheet has just one boundary. The presence of Kazakov's pathologies is shown to change the continuum behaviour for general potentials, though not for unitary models.

Because non-critical strings with $d = 1$ lie at the boundary of the branched polymer phase, it is of particular interest to investigate the intrinsic geometry of the worldsheet for these models. This is done by extending the above discussions to a model with an infinite chain of matrices. We have thus been able to calculate the properties of loops on the surface of a string in one embedding dimension. Because of a lack of understanding of the angular variables, we needed to restrict these loops to lie at a single value of the external coordinate. The presence of the

long loop phase even for paths restricted in this way hints that these $d = 1$ models are probably not in a branched polymer phase.

We may imagine generalising these constructions in various ways. Firstly it should be relatively straightforward to find the properties of loops on a random surface coupled to an $O(n)$ model or a Q-state Potts model [14]. Similarly the (p, q) matrix chain models of Douglas [15] should lend themselves to this analysis. In these cases there are two main possibilities for the boundary conditions. Either we may consider boundaries consisting of just one type of spin, or we may sum over their spin structures. In the case of the O(n) model this translates into the choice of whether to allow the self-avoiding-walks to cross the boundary [14]. For matrix chains it corresponds to the choice of allowing one or many types of matrix to reside on the boundary.

We have only applied our methods to the case of genus zero surfaces with a single boundary. To extend them to higher genera would require a generalisation of (2.6). Extension to higher numbers of holes requires subtractions from the dual Feynman diagrams corresponding to boundaries touching, as described in Chapter 4. With some effort both of these extensions should be possible, at least in the cases in which the genus and number of boundaries is fairly small.

Finally, it would be of great interest to reproduce these results from a continuum model. For the case of unitary models this could be attempted directly from the Liouville theory of David, Distler and Kawai [16]. Recently, Witten has connected 2d gravities at multicritical points with topological gravity [17]. If these models can be generalised to allow the presence of a boundary, their results should be directly comparable to the ones presented here.

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APPENDIX

In this appendix we calculate the number of zeros of $F'(X)$ in the complex plane for $d = 0$. Suppose that we start with an even potential of order $2r$

$$V(\phi) = \frac{1}{2}\phi^2 + \ldots + k\phi^{2r} \tag{A.1}$$

Then $F(X)$ is found from the Schwinger-Dyson equations as described in the text and takes the form

$$F(\lambda) = \frac{1}{2}\left\{V'_{(2r-1)}(\lambda) - P_{(2r-2)}(\lambda)\sqrt{\lambda^2 - 4a^2}\right\} \tag{A.2}$$

where P is some polynomial in λ and we use bracketed subscripts to denote the order of polynomials. From this we get

$$F'(X) = \frac{1}{2\sqrt{\lambda^2 - 4a^2}}\left\{V''_{(2r-2)}(\lambda)\sqrt{\lambda^2 - 4a^2} - Q_{(2r-1)}(\lambda)\right\} \tag{A.3}$$

where Q is a polynomial given by Q(X) = $P'(\lambda)[\lambda^2 - 4a^2] - P\lambda$. Now recall that if we take the positive branch in (A.2) then- $F(X) \sim O(1/\lambda)$ as $\lambda \to \infty$. Hence $F'(X) \sim O(1/\lambda^2)$ as $\lambda \to$ co. Substituting this requirement into (A.3) we have

$$V''_{(2r)}(\lambda)\sqrt{\lambda^2 - 4a^2} \sim Q_{(2r-1)}(\lambda) + O(\frac{1}{\lambda}) \qquad \text{as } \lambda \to \infty \tag{A.4}$$

and squaring this,

$$(V''(\lambda))^2_{(4r-4)}(\lambda^2 - 4a^2) = (Q(\lambda))^2_{(4r-2)} + R_{(2r-2)}(\lambda) \tag{A.5}$$

where R is another polynomial in λ.

From this it is clear that solutions of $F'(X) = 0$ are given by roots of $R_{(2r-2)}(\lambda)$. By the fundamental theorem of algebra, there are generically $2r - 2$ of these roots in the complex plane. Since we are allowing both branches of F, all of these will be realised as solutions of $F' = 0$.

So in general, $F(X)$ has $2r - 2$ stationary points in the complex plane, and these come in pairs of opposite sign, since the potential $V(\phi)$ was chosen to be even.

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