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Morphisms Between Supersymmetric and Topological Quantum Field Theories *

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ABSTRACT

We find that topological invariants, isomorphic to Donaldson Polynomials, exist in chiral superfield theories. Twists between these invariants and those of the corresponding TQFT are given. In the topological sigma model, anti-commuting charges of integer spin are found which together with the BRST charge, fill out a D=2, N=2 supersymmetry algebra.

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Supersymmetric quantum mechanics has been used to study the topology of manifolds. For example, the calculation of the index $Tr(-)^F$ leads to proofs of the Atiyah-Singer index theorem [1] and the Morse inequalities [2]. This connection between topology and quantum field theory has been furthered through the introduction of a class of theories known as topological quantum field theories (TQFT's) [3]. Correlators of observables in these theories depend on the topological class of the field configuration under consideration (i.e. on the choice of in and out vacua). They are independent of the differentiable structure of the manifold. In chiral superfield theories (χST's) a similar situation arises. In Ref. [4] it has been shown that the correlators of the lowest components of chiral superfields are independent of the positions of the operators.

In this letter we will discuss how to make the connection between χST's and TQFT's precise. We will look at the examples of D=2, N=2 supersymmetric non-linear sigma model and D=4, N=2 supersymmetric Yang-Mills theory.

TQFT's can be constructed as the BRST gauge fixing of a local shift (topological) symmetry under which only the homotopy class of a field configuration is preserved [5,6]. Schematically, one has $\delta\mathcal{B}(x) = \mathcal{F}(x)$. In the BRST quantized theory, $\mathcal{F}$ carries opposite Grassmann statistics to $\mathcal{B}$. If $\mathcal{B}$ is a commuting boson then $\mathcal{F}$ is Grassmann odd. In one and two space-time dimensions where spin is of no physical consequence, the anti-commuting ghost $\leftrightarrow$ fermion identification becomes meaningful. Then the topological transformation is suggestive of a supersymmetry transformation.

The actions of TQFT's are of the form $S_{TOP} = \int_M [Q, K\}$. M is the space-time manifold. The quantity K depends on the metric on the manifold through the defintion of inner products of the fields. These fields constitute positive and negative ghost number multiplets under a BRST charge, Q. A BRST invariant vacuum is constructed by inserting ghost zero-modes into the naive vacuum as is done in string theory. By differentiating the partition function $\mathbf{Z}$ with respect to the metric one finds

$$\frac{\delta \mathbf{Z}}{\delta g^{\underline{ab}}} = \langle T_{\underline{ab}} \rangle = \langle [Q, \Lambda_{\underline{ab}}\} \rangle = 0 \ , \tag{1}$$

for some $\Lambda_{\underline{ab}}$. Correlation functions of the metric independent observables in the theory may be shown [3] to to be topological invariants. This proof employs the BRST invariance of the vacuum and the transformation laws of the various fields. It also assumes that the measure of $\mathbf{Z}$ is independent of the metric.

Let us now look at supersymmetric field theories. We quickly find an analogous result to eqn. (1). Consider, for example, D=4, N=1 supersymmetry with supercharge Q_α. It is well known that the energy-momentum tensor may be written as $T_{\underline{ab}} = -\frac{1}{16} tr(\gamma_{(\underline{a}|}[\bar{Q}, S_{|\underline{b})}\})$ so that for a super-translationally invariant vacuum, $\langle T_{\underline{ab}} \rangle = 0$ [7]. As in (1), one takes $\mathbf{Z}$ to be metric independent. Next consider a chiral superfield theory with $\bar{D}_{\dot{\alpha}} \Phi = 0$. Let $\Phi| = \phi$ and $D_\alpha \Phi| = \psi_\alpha$. Then for correlation functions of ϕ:

$$\partial_{\alpha\dot{\alpha}} \langle \phi(x_1) \cdots \phi(x_n) \rangle = \langle \phi(x_1) \cdots [\bar{Q}_{\dot{\alpha}}, \psi_\alpha\} \cdots \phi(x_n) \rangle = 0 \ . \tag{2}$$

This result holds for general chiral superfields [4]. As the metric on the manifold was not used, and the partition function is metric independent, such a correlation function is a topological invariant.

The presence of topological invariants stem from the fact that the energy-momentum tensors of these theories are given by the action of the super-charge on something. A general argument proceeds as follows. Consider an action for bosonic and fermionic fields with canonical kinetic terms. Let the action, S, be constructed to be invariant under the action of a charge Q: $[Q, S] = 0$. Also require that the energy-momentum tensor, $T_{\underline{ab}}$, may be written as $T_{\underline{ab}} = [Q, \Lambda_{\underline{ab}}\}$. This is immediate if $S = [Q, \Psi\}$ for some gauge fixing fermion Ψ. Applying the rules of canonical quantization we arrive at

$$P_{\underline{a}} = \int_Y T_{0\underline{a}} = \frac{1}{2}[Q, \bar{Q}_{\underline{a}}\} \ , \qquad \bar{Q}_{\underline{a}} \equiv 2\int_Y \Lambda_{0\underline{a}} \ , \qquad [P_{\underline{a}}, \Phi\} = -i\nabla_{\underline{a}}\Phi \ , \tag{3}$$

where $\tilde{M} \equiv Y \times R^1$, Φ is a generic field and $\nabla_{\underline{a}}$ is covariant with respect to some local symmetry which will not concern us. Construct an observable, $\mathcal{O}$, which is a singlet of the local symmetry and is independent of the metric on M. Furthermore, let it be a C^k function in any of the fields Φ for which $[Q, \Phi\} = 0$ and for k large enough. It then follows that

$$\partial_{\underline{a}} \langle \mathcal{O} \rangle \; = \; \langle [iP_{\underline{a}}, \mathcal{O}\} \rangle \; = \; \langle [i\frac{1}{2}Q, [\bar{Q}_{\underline{a}}, \mathcal{O}\}\} \rangle \; = \; 0 \; . \tag{4}$$

Thus the *vev* of $\mathcal{O}$ is independent of the differentiable structure on the manifold M. In general, it will depend on the topological class of M. Note that if $\Lambda_{0\underline{a}}$ is a conserved current, eqn. (3) yields a supersymmetry algebra, albeit unconventional. In supersymmetry we will see that the mechanism which allows for topological invariants (more general than (2)) is the existence of a propagating field which transforms chirally under the supersymmetry algebra.

Given that topological invariants exist in χST's, we are led to inquire if the latter are related to the observables of TQFT's. By relationship we will mean a morphism, termed a "twist" $\mathcal{T}$, which makes the diagram

$$\begin{array}{ccc} \mathcal{W}_{TOP} & \xrightarrow{\mathcal{T}} & \mathcal{W}_{SUP} \\ \uparrow & & \uparrow \\ TQFT & \xrightarrow{\mathcal{T}} & N=2\ \chi ST \end{array}$$

commutative. Here $\mathcal{W}_{TOP(SUP)}$ is the space of topological invariants constructed as correlation functions in the TQFT (χST). Henceforth, we will refer to this diagram as the Twist Diagram. We remark that $\mathcal{T}$ is purely a mathematical operation. Its physical interpretation, if any, has yet to be unveiled.

The physical states of a TQFT [8] comprise a BRST complex and are the ground states of the Hamiltonian. They will be in one to one correspondence with the ground states of the corresponding supersymmetric theory.

We will now illustrate these general remarks with examples in D=2 and D=4.

D = 2

At tree level, topological sigma models [3] (TSM's) can be formulated on an arbitrary almost complex manifold. In the case of Kähler manifolds, there is a relation to N=2 supersymmetric sigma models. Prior to our work, this was realized by twisting the two dimensional Lorentz algebra ($U(1)$) with the internal $U(1)$ of the $N=2$ algebra and discarding two of the four super-charges. This gave a construction of the world-sheet scalar BRST charge in terms of two of the remaining supersymmetry generators [3,9]. We find that the TSM is invariant under vector charges which form a full supersymmetry algebra with the BRST charge. Furthermore, certain correlators in the supersymmetric theory are shown to be topological and may be twisted into those of the TSM.

The superspace action for an $N=2$ model for maps from the world-sheet to a Kähler manifold M is $S_{SUP} = \frac{1}{4}\int d^2\sigma d^4\theta K(\Phi,\bar{\Phi})$ where the Φ^I are chiral superfields and $K(\Phi,\bar{\Phi})$ is the Kähler potential on M. This action reduces to the component expression

$$\begin{aligned} S_{SUP} &= \int d^2\sigma[g_{I\bar{J}}\partial_z\phi^I\partial_{\bar{z}}\phi^{\bar{J}} + i\frac{1}{2}g_{I\bar{J}}\psi^{\bar{J}}\mathcal{D}_{\bar{z}}\psi^I + i\frac{1}{2}g_{I\bar{J}}\bar{\psi}^{\bar{J}}\mathcal{D}_z\bar{\psi}^I \\ &\quad + \frac{1}{4}R_{I\bar{J}K\bar{L}}\bar{\psi}^I\bar{\psi}^{\bar{J}}\psi^K\psi^{\bar{L}}] , \qquad (5) \\ \mathcal{D}_z\bar{\psi}^I &\equiv \partial_z\bar{\psi}^I + \partial_z\phi^K\Gamma^I_{JK}\bar{\psi}^J , \qquad \mathcal{D}_{\bar{z}}\psi^I \equiv \partial_{\bar{z}}\psi^I + \partial_{\bar{z}}\phi^K\Gamma^I_{JK}\psi^J . \end{aligned}$$

The N=2 supersymmetry algebra in two dimensions contains two left-handed super-charges $Q_\pm$, two right-handed super-charges $\bar{Q}_\pm$. We use the standard superconformal notation with the $\pm$ indices denoting R-symmetry weights and the barred/unbarred denoting handedness. In this way the charges satisfy the algebra

$$[Q_+, Q_-\} = i2\partial_z , \qquad [\bar{Q}_+, \bar{Q}_-\} = i2\partial_{\bar{z}} , \qquad (6)$$

where all other commutators are zero. S_{SUP} is invariant under the non-trivial N=2 supersymmetry transformations for the left-handed fields

$$\begin{aligned}
[Q_+,\phi^I\} &= i\psi^I \ , \qquad [Q_-,\phi^{\bar I}\} = i\psi^{\bar I} \ , \\
[Q_-,\psi^I\} &= 2\partial_z\phi^I \ , \qquad [Q_+,\psi^{\bar I}\} = 2\partial_z\phi^{\bar I} \ , \\
[\bar Q_+,\psi^I\} &= F^I \ , \qquad [\bar Q_-,\psi^{\bar I}\} = F^{\bar I} \ , \\
F^I &= i\Gamma^I_{JK}\psi^J\bar\psi^K \ .
\end{aligned} \tag{7}$$

The transformations for the right handed fields are obtained by interchanging barred and unbarred indices. The actions of the Lorentz $U(1)$ generator J and the $U(1)$ R-symmetry generators on the fields can be summarized in the form $\Phi(R,\bar R,J)$,

$$\begin{aligned}
&\phi(0,0,0)\ ;\quad \psi^I(1,0,\frac{1}{2})\ ;\quad \psi^{\bar I}(-1,0,\frac{1}{2})\ ;\quad \bar\psi^I(0,1,-\frac{1}{2})\ ;\quad \bar\psi^{\bar I}(0,-1,-\frac{1}{2})\ , \\
&Q_+(1,0,\frac{1}{2})\ ;\quad \bar Q_+(0,1,-\frac{1}{2})\ ;\qquad Q_-(-1,0,\frac{1}{2})\ ;\quad \bar Q_-(0,-1,-\frac{1}{2})\ .
\end{aligned} \tag{8}$$

The topological sigma model action for maps from the world sheet, Σ, to a Kähler manifold M [3] is given by

$$\begin{aligned}
S_{TOP} = \int d^2\sigma[g_{I\bar J}\partial_z\phi^I\partial_{\bar z}\phi^{\bar J} &- i\frac{1}{2}g_{I\bar J}\rho^I_z\mathcal{D}_{\bar z}\chi^{\bar J} - i\frac{1}{2}g_{I\bar J}\rho^{\bar J}_{\bar z}\mathcal{D}_z\chi^I \\
&- \frac{1}{4}R_{I\bar JK\bar L}\chi^I\chi^{\bar J}\rho^K_z\rho^{\bar L}_{\bar z}] \ ,
\end{aligned} \tag{9}$$

$$\mathcal{D}_{\bar z}\chi^{\bar J} \equiv \partial_{\bar z}\chi^{\bar J} + \partial_{\bar z}\phi^{\bar K}\Gamma^{\bar J}_{\bar K\bar L}\chi^{\bar L} \ , \qquad \mathcal{D}_z\chi^I \equiv \partial_z\chi^I + \partial_z\phi^K\Gamma^I_{KL}\chi^L \ .$$

The fields χ and ρ are anti-commuting world-sheet scalars and vectors, respectively. The TSM action is invariant with respect to the following non-trivial BRST transformations

$$\begin{aligned}
[Q_l,\phi^{\bar I}\} &= i\chi^{\bar I} \ , \qquad [Q_r,\phi^I\} = i\chi^I \ , \\
[Q_l,\rho^I_z\} &= 2\partial_z\phi^I \ , \qquad [Q_r,\rho^I_z\} = -i\Gamma^I_{JK}\chi^J\rho^K_z \ , \\
[Q_l,\rho^{\bar I}_{\bar z}\} &= -i\Gamma^{\bar I}_{\bar J\bar K}\chi^{\bar J}\rho^{\bar K}_{\bar z} \ , \qquad [Q_r,\rho^{\bar I}_{\bar z}\} = 2\partial_{\bar z}\phi^{\bar I} \ .
\end{aligned} \tag{10}$$

We have found that the action is also invariant under the new vector charges Q_z and $Q_{\bar z}$ with

$$\begin{aligned}
[Q_{\bar z},\phi^{\bar I}\} &= i\rho^{\bar I}_{\bar z} \ , \qquad [Q_z,\phi^I\} = i\rho^I_z \ , \\
[Q_{\bar z},\chi^I\} &= 2\partial_{\bar z}\phi^I \ , \qquad [Q_z,\chi^I\} = i\Gamma^I_{JK}\chi^J\rho^K_z \ , \\
[Q_{\bar z},\chi^{\bar I}\} &= i\Gamma^{\bar I}_{\bar J\bar K}\chi^{\bar J}\rho^{\bar K}_{\bar z} \ , \qquad [Q_z,\chi^{\bar I}\} = 2\partial_z\phi^{\bar I} \ .
\end{aligned} \tag{11}$$

The BRST and vector charges satisfy the supersymmetry algebra

$$[Q_l, Q_z\} \;=\; i2\partial_z \;\;, \qquad [Q_r, Q_{\bar z}\} \;=\; i2\partial_{\bar z} \;\;, \tag{12}$$

with all other anti-commutators being zero. In particular each Q is nilpotent. This is an on-shell closure of the algebra. Presumably, with the introduction of the BRST auxiliary field, this algebra will close off-shell much as a supersymmetry algebra closes off-shell with the appropriate introduction of auxiliary fields. The left and right ghost number ($\mathcal{G}_l$, $\mathcal{G}_r$) and Lorentz (J) weights of the various fields and operators are as follows with the notation $\Phi(\mathcal{G}_l, \mathcal{G}_r, J)$

$$\begin{aligned} &\phi(0,0,0) \;\;;\;\; \rho_z^I(-1,0,1) \;\;;\;\; \rho_{\bar z}^{\bar I}(0,-1,-1) \;\;;\;\; \chi^I(0,1,0) \;\;;\;\; \chi^{\bar I}(1,0,0) \;\;, \\ &Q_l(1,0,0) \;\;;\;\; Q_r(0,1,0) \;\;;\;\; Q_{\bar z}(0,-1,-1) \;\;;\;\; Q_z(-1,0,1) \;\;. \end{aligned} \tag{13}$$

Now re-define the Lorentz generator to be

$$J' \;\equiv\; J \;+\; \frac{1}{2}(\mathcal{G}_l \;-\; \mathcal{G}_r) \;\;, \tag{14}$$

and identify $R \equiv -\mathcal{G}_l$ and $\bar R \equiv \mathcal{G}_r$. Then eqns. (12) and (13) become identical to eqns. (6) and (8), respectively, with the renamings

$$\begin{aligned} &\rho_z^I \equiv -i\psi^I \;\;, \qquad \rho_{\bar z}^{\bar I} \equiv -i\bar\psi^{\bar I} \;\;, \qquad \chi^I \equiv -i\bar\psi^I \;\;, \qquad \chi^{\bar I} \equiv -i\psi^{\bar I} \;\;, \\ &Q_l \equiv Q_- \;\;, \qquad Q_r \equiv \bar Q_+ \;\;, \qquad Q_{\bar z} \equiv \bar Q_- \;\;, \qquad Q_z \equiv Q_+ \;\;. \end{aligned} \tag{15}$$

One also verifies that $S_{TOP} \rightarrow S_{SUP}$.

We have defined the map in the lower branch of the Twist Diagram. Let us now find the topological invariants of the N=2 supersymmetric sigma model. One can show that, for example, that given an element $A_{I\bar J}$ of $H^{(1,1)}(M)$, the operator

$$\mathcal{O}^{(-1,1)}_{(0)} \;=\; A_{I\bar J}\bar\psi^I\psi^{\bar J} \;\;, \tag{16}$$

is an element of a $\bar Q_+ \oplus Q_-$ cohomolgy. (This means it is a twisted chiral operator in the language of ref. [10].) Our notation is $\mathcal{O}^{(R,\bar R)}_{(k)}$ where k denotes the real degree of $\mathcal{O}$ on Σ

with exterior derviative d. Furthermore, the ascent equations

$$
\begin{aligned}
d\mathcal{O}^{(-1,1)}_{(0)} &= i[Q_-, \mathcal{O}^{(0,1)}_{(1)}\} - i[\bar{Q}_+, \mathcal{O}^{(-1,0)}_{(1)}\} \ , \\
\mathcal{O}^{(0,1)}_{(1)} &\equiv A_{I\bar{J}}\bar{\psi}^I d\phi^{\bar{J}} \ , \qquad \mathcal{O}^{(-1,1)}_{(1)} \equiv A_{I\bar{J}}\psi^{\bar{J}} d\phi^I \ , \\
d\mathcal{O}^{(0,1)}_{(1)} &= -i[\bar{Q}_+, \mathcal{O}^{(0,0)}_{(2)}\} \ , \qquad d\mathcal{O}^{(-1,0)}_{(1)} = -i[\bar{Q}_+, \mathcal{O}^{(0,0)}_{(2)}\} \ , \\
\mathcal{O}^{(0,0)}_{(2)} &\equiv A_{I\bar{J}} d\phi^I \wedge d\phi^{\bar{J}} \ ,
\end{aligned}
\tag{17}
$$

imply that the correlation functions of k-cycle integrals of the $\mathcal{O}$ operators are independent of the points on the manifolds. Equations (16-17) may be generalized to elements of $H^{(p,q)}(M)$. In fact given such a form, there are two complexes of operators $\mathcal{O}$ which may be formed. For example, in addition to $\mathcal{O}^{(-1,1)}_{(0)}$, there is also $\tilde{\mathcal{O}}^{(1,-1)}_{(0)} \equiv A_{I\bar{J}}\psi^I\bar{\psi}^{\bar{J}}$ which is an element of a $Q_+ \oplus \bar{Q}_-$ cohomology. Under the twisting, the set of operators $\mathcal{O}(\tilde{\mathcal{O}})$ maps into the BRST (anti-BRST) observables of ref. [3] with positive (negative) ghost number. Thus at least at tree level, the Twist Diagram commutes. Also note that eqn. (16) is the zero-momentum limit of the vertex operator for the axion-like field in the $M^4 \times K$ compactification of the superstring [11].

Having discussed the two-dimensional theories, we now make the following remarks about the D=1 topological sigma model [12] and its relationship to N=2 supersymmetric quantum mechanics. The lower branch of the Twist Diagram between topological quantum mechanics and N=2 supersymmetric quantum mechanics was discussed at length in ref. [8]. The super-charges were realized as linear combinations of the BRST and anti-BRST charges. Based on this or by dimensional reduction of our D=2 results, it is trivial to see that there are topological invariants of the form $\mathcal{O}_n = A_{a_1\cdots a_n}(x)\prod_{i=1}^n \psi^{a_i}$ in both theories. The a_i are vector indices on M and $A_k \in H^k(M)$. The fields ψ^{a_i} are ghosts for the TQFT and fermions for the supersymmetric theory and are interpreted as sections of the pull-back $\phi^*(TM)$ of the tangent bundle on M ($x \in \phi : R \to M$). The twist of ref. [8] relates them and allows us to conclude that the Twist Diagram is commutative.

D = 4

Gaugino condensates of D=4, N=1 SYM are known to form topological invariants [4]. However, these invariants are unrelated to the Donaldson Polynomials of TYM. The relationship is between D=4, N=2 SYM and TYM. The twisting, at the lagrangian level, between these theories has been discussed in refs. [3,13]. We will show that topological invariants exist in the *untwisted* supersymmetric theory. If one chooses, these may be twisted to the Donaldson Polynomials.

The D=4, N=2 SYM multiplet [14,15] contains one spin-1 field, $A_{\underline{a}}$, two spin-0 fields, C and C^*, two spin-$\frac{1}{2}$ fields, $\lambda_{\underline{\alpha}}$ and an auxiliary field, B_{ab}. Apart from the gauge group, the bosons are singlets under a rigid, internal $SU(2)_I$ group for which the gaugino is a doublet (labelled by the "a" index, $\underline{\alpha} \equiv \alpha a$) and B is a triplet. Additionally, the theory is invariant under a rigid $U(1)$ group for which the charges of the set of fields $(A_a, C, \lambda_{\underline{\alpha}})$ are $(0,2,1)$. The supersymmetry transformation laws which will be relevant to our discussion are

$$[\bar{Q}_{\underline{\dot{\alpha}}}, F_{\underline{ab}}\} = i\sigma_{[\underline{a}|\alpha\dot{\alpha}}\mathcal{D}_{|\underline{b}]}\lambda^{\underline{\alpha}} \ , \qquad [\bar{Q}_{\underline{\dot{\alpha}}}, \lambda_{\underline{\beta}}\} = i\sigma^{\underline{a}}{}_{\beta\dot{\alpha}}\mathcal{D}_{\underline{a}}C\delta_b{}^a \ , \qquad [\bar{Q}_{\underline{\dot{\alpha}}}, C\} = 0 \ . \quad (18)$$

Let us, for simplicity, take the Yang-Mills gauge group to be $SU(2)$. Take the object $\Omega_{(0)} \equiv Tr(C^2)$. From eqn. (18) one immediately sees that $[\bar{Q}^{\underline{\dot{\alpha}}}, \Omega_{(0)}\} = 0$. Given this, the following ascent equations may be derived:

$$\begin{aligned}
d\Omega_{(0)} &= [\bar{Q}^{\underline{\dot{\alpha}}}, \Omega_{(1)\underline{\dot{\alpha}}}\} \ , \qquad d\Omega_{(1)\underline{\dot{\alpha}}} = [\bar{Q}^{\underline{\dot{\beta}}}, \Omega_{(2)\underline{\dot{\beta}\dot{\alpha}}}\} \ , \\
d\hat{\Omega}_{(2)} &= [\bar{Q}^{\underline{\dot{\alpha}}}, \hat{\Omega}_{(3)\underline{\dot{\alpha}}}\} \ , \qquad d\tilde{\Omega}_{(2)\underline{\dot{\alpha}\dot{\beta}}} = [\bar{Q}^{\underline{\dot{\gamma}}}, \tilde{\Omega}_{(3)\underline{\dot{\gamma}\dot{\alpha}\dot{\beta}}}\} \ , \\
d\hat{\Omega}_{(3)\underline{\dot{\alpha}}} &= [\bar{Q}_{\underline{\dot{\alpha}}}, \hat{\Omega}_{(4)}\} \ , \qquad d\tilde{\Omega}_{(3)\underline{\dot{\gamma}\dot{\alpha}\dot{\beta}}} = [\bar{Q}_{(\underline{\dot{\alpha}}|}, \tilde{\Omega}_{(4)|\underline{\dot{\beta}})\underline{\dot{\gamma}}}\} \ , \\
d\hat{\Omega}_{(4)} &= 0 \ , \qquad d\tilde{\Omega}_{(4)\underline{\dot{\alpha}\dot{\beta}}} = 0 \ ,
\end{aligned} \qquad (19)$$

where

$$
\begin{aligned}
\Omega_{(0)} &= Tr(C^2) \ , \\
\Omega_{(1)\underline{\dot\alpha}} &= -i\frac{1}{2}Tr(\sigma_{\underline{a}\alpha\dot\alpha}\lambda^{\underline{\alpha}}C)dx^{\underline{a}} \ , \\
\Omega_{(2)\underline{\dot\alpha\dot\beta}} &\equiv \hat\Omega_{(2)}C_{\underline{\dot\alpha\dot\beta}} \ + \ \tilde\Omega_{(2)\underline{\dot\alpha\dot\beta}} \ , \\
\hat\Omega_{(2)} &= \frac{1}{4}Tr(F_{\underline{ab}}C \ + \ \frac{1}{2}\lambda_c\sigma_{\underline{ab}}\lambda^c)dx^{\underline{a}}\wedge dx^{\underline{b}} \ , \\
\tilde\Omega_{(2)\underline{\dot\alpha\dot\beta}} &= \frac{1}{8}Tr(\sigma_{\underline{ab}\dot\alpha\dot\beta}\lambda^{\gamma a}\lambda_\gamma{}^b)dx^{\underline{a}}\wedge dx^{\underline{b}} \ , \\
\hat\Omega_{(3)\underline{\dot\alpha}} &= i\frac{1}{16}Tr(F_{\underline{ab}}\sigma_{\underline{c}\alpha\dot\alpha}\lambda^{\underline{\alpha}})dx^{\underline{a}}\wedge dx^{\underline{b}}\wedge dx^{\underline{c}} \ , \\
\tilde\Omega_{(3)\underline{\dot\gamma\dot\alpha\dot\beta}} &= i\frac{1}{32}Tr(F_{\underline{ab}}C_{\dot\gamma(\dot\alpha|}\sigma_{\underline{c}\gamma|\dot\beta)}\lambda^{\gamma(b}C^{a)c})dx^{\underline{a}}\wedge dx^{\underline{b}}\wedge dx^{\underline{c}} \ , \\
\hat\Omega_{(4)} &= \frac{1}{64}Tr(F_{\underline{ab}}F_{\underline{cd}})dx^{\underline{a}}\wedge dx^{\underline{b}}\wedge dx^{\underline{c}}\wedge dx^{\underline{d}} \ , \\
\tilde\Omega_{(4)\underline{\dot\alpha\dot\beta}} &= \frac{1}{2}\hat\Omega_{(4)}C_{\underline{\dot\alpha\dot\beta}} \ .
\end{aligned}
\tag{20}
$$

Here $C_{\underline{\dot\alpha\dot\beta}}$ is the complex conjugate of the anti-symmetric symbol of $SL(4,C) \supset SU(2)\otimes SL(2,C)$ defined by $C_{\underline{\dot\alpha\dot\beta}} \equiv C_{\dot\alpha\dot\beta}C^{ab}$. $\sigma_{\underline{ab}}$ is defined as $\sigma_{\underline{ab}\alpha}{}^\beta \equiv \frac{1}{4}\sigma_{[\underline{a}|\alpha\dot\alpha}\bar\sigma_{\underline{b}]}{}^{\dot\alpha\beta}$ with $\{\sigma_{\underline{a}},\sigma_{\underline{b}}\} = -2\eta_{\underline{ab}}\delta_\alpha{}^\beta$. These expressions were derivable, in part, because C is chiral. It is the lowest component of a chiral superfield (in N=2 superspace), W [15] : $\bar\nabla_{\underline{\dot\alpha}}W = 0$. The entire geometry of D=4, N=2 SYM superspace may be specified in terms of this superfield and its hermitian conjugate. This leads us to conjecture that eqns. (19-20) (which carry topological information only) may be obtained from $Tr(W^2)$ by the standard superspace projection techniques in the spirit of the construction of the Donaldson Polynomials for TYM given in ref. [8]. Furthermore, although we have not checked it, we do expect that the $\Omega_{(0)}$ may be generalized to $Tr(C^{2n})$ for a rank n group. There will be a similar generalization for the remaining $\Omega_{(k)}$'s. The $U(1)$ charge of each $\Omega_{(k)}$ is $(4-k)$. The space, $\mathcal{W}_{SUP}$, of topological invariants for D=4, N=2 SYM is composed of various Lorentz scalar products of homology k-cycles of the $\Omega_{(k)}$'s in a analgous manner to the construction of the Donaldson maps given in ref. [3]. It would be interesting to understand the connection between these invariants, moduli space, supersymmetry breaking, *etc.*

The twisting to D=4 TYM is achieved by first identifying the $SU(2)_I$ index a with the dotted $SU(2)_R$ index $\dot{\alpha}$. That is, with $SO(4) \sim SU(2)_L \otimes SU(2)_R$ as the Lorentz group on the four-manifold (with Euclidean signature), the symmetry group of D=4, N=2 supersymmetry $SO(4) \otimes SU(2)_I \otimes U(1)$ becomes $SU(2)_L \otimes SU(2)_D \otimes U(1)$ [3]. Here $SU(2)_D$ is the diagonal part of $SU(2)_R \otimes SU(2)_I$. This means that the super-charge becomes $\bar{Q}_{\underline{\dot{\alpha}}} = \bar{Q}_{\dot{\alpha}}{}^{a} \rightarrow \bar{Q}_{\dot{\alpha}}{}^{\dot{\alpha}}$. The latter is then identified as the anti-commuting, scalar BRST charge. Similarly, the spin-$\frac{1}{2}$ gaugino field is identified as an anti-commuting vector field, $\psi_{\underline{a}}$, through the twisting $\sigma_{\underline{a}\alpha\dot{\alpha}}\lambda^{\underline{\alpha}} = \sigma_{\underline{a}\alpha\dot{\alpha}}\lambda^{\alpha a} \rightarrow \sigma_{\underline{a}\alpha\dot{\alpha}}\lambda^{\alpha\dot{\alpha}} \rightarrow \psi_{\underline{a}}$. Accordingly, the supersymmetry transformations (18) become

$$[Q, F_{\underline{ab}}\} = i\mathcal{D}_{[\underline{a}}\psi_{\underline{b}]} \ , \qquad [Q, \psi_{\underline{a}}\} = -\mathcal{D}_{\underline{a}}\phi \ , \qquad [Q, \phi\} = 0 \ , \tag{21}$$

where we have identified the spin-0 field C as the commuting, scalar field ϕ of the TYM ghost multiplet. The ghost numbers of the TYM fields are the same as the $U(1)$ charges of the corresponding fields in the SYM multiplet. As the construction of the Donalsdon Polynomials is based solely on eqn. (21), it is clear that the twisting leads to them. One may also explicitly check that under this twisting, eqn. (20) leads to the Donaldson polynomials of the TYM theory. For example, the most complicated expression in (20), namely $\Omega_{(3)\underline{\dot{\gamma}\dot{\alpha}\dot{\beta}}}$ evaluates to (up to a normalization) $Tr(i\psi \wedge F)$ which is $W_{(3)}$ in ref. [3]. Thus we have found the twisting $\mathcal{T} : \mathcal{W}_{SUP} \rightarrow \mathcal{W}_{TQFT}$ from the space of topological invariants of D=4, N=2 SYM to the corresponding space in TYM. Given the construction, in refs. [13], of the D=4 TYM action from that of D=4, N=2 SYM, we have found strong evidence that the Twist Diagram is commutative.

In taking only the diagonal sum of the $SU(2)$'s, we have disposed of the extra charges. However, these may be recycled in the following way. Generally, with the identification of the $SU(2)_I$ index as a $SU(2)_R$ index, we will have $\bar{Q}_{\underline{\dot{\alpha}}} \rightarrow Q \oplus Q_{\underline{ab}}$ and $Q_{\underline{\alpha}} \rightarrow Q_{\underline{a}}$ where $Q_{\underline{ab}}$ is anti-symmetric and self-dual. Based on our experience with the vector charges of the

TSM (sect. III), we expect that these will also generate a symmetry of the TYM theory and form a supersymmetry algebra.

Our discussion of the twist diagram has been at the tree level. The TSM is conformally invariant at the one-loop level only if the target manifold is Ricci scalar flat [16]. Ricci flat Kähler manifolds meet this criterion. This is the condition for the vanishing of the one-loop β-function of the N=2 supersymmetric sigma model [17]. This constraint on the geometry of the target manifold of the TSM, a theory which is supposed to be topologically invariant, is surprising. It arises from the metric dependence of the measure of the partition function. Nevertheless, it suggests that the Twist Diagram is commutative also at one-loop order for Ricci flat Kähler manifolds. For the D=4 TYM theory, the one-loop β-function for the gauge coupling is $\beta(g) = -\frac{2g^3}{(4\pi)^2} C_2^{\text{adj.}}(G)$ [6]. This is exactly the one-loop β-function of D=4, N=2 pure SYM (no hypermultiplets) [18] and the result also suggests that the Yang-Mills Twist Diagram is commutative at one-loop order.

In conclusion, we would like to stress the following points. There are general classes of topological invariants in D=2, N=2 non-linear sigma model and pure D=4, N=2 SYM. These invariants are isomorphic to the generalized Donaldson Polynomials of TQFT's. Twists between the respective χSFT's and TQFT's have been defined. Ground states of χSFT's are in one to one correspondece with the physical states of the corresponding TQFT's. It is an interesting question as to how the twisting procedure implements the restrictions on the physical Hilbert spaces.

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