SLAC–PUB–5233
April 1990
(M)

A REPLY TO "A COMMENT ON GENERALIZED ELECTROMAGNETISM AND DIRAC ALGEBRA"*

David Fryberger

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94309, USA

Received April 6, 1990.

Issues raised by W. A. Rodrigues, Jr. are discussed.

Key words: Dirac algebra, electromagnetism, magnetic monopoles.

Rodrigues[(1)] asserts that my claim that it is possible to derive by Hamilton's principle both the symmetrized dyality invariant Maxwell's equations and the dyality invariant equations of motion for both electrically and magnetically charged particles,[(2)] is wrong. In particular, Rodrigues takes issue with my derivation of the forces associated with the cross terms in the equations of motion, ***i.e.,*** the forces on electric charges due to the fields generated by magnetic charges, and vice versa. Unfortunately, Rodrigues did not follow my argument by which forces associated with the cross terms were obtained; the mathematical development that he presented in his "proof" is not equivalent to the development I employed. Hence, it is not surprising that he was unable to confirm my result. To dispel some of the confusion that Rodrigues has injected into this matter, I repeat my development here with amplifying discussions, detailing essential aspects of it that he overlooked. Reference (2) was evidently too brief on these points.

Submitted to Foundations ***of Physics Letters.***

⋆ **Work supported by Department of Energy contract** DE–AC03–76SF00515.

My development starts with the Lagrangian density

$$\mathcal{L} = \frac{1}{8\pi} < \mathcal{F}^* \mathcal{F} >_s - \frac{1}{c} < \mathcal{J} \mathcal{A} >_s \quad , \tag{76}$$

where $\mathcal{F}$, $\mathcal{J}$, and $\mathcal{A}$ are generalized electromagnetic quantities expressed as Clifford multivectors in Minkowski spacetime. Gaussian units are used and the notation and equation numbers are those of Ref. (2). (The * symbol is akin to complex conjugation in that it changes the sign of the γ_5 of a quantity. γ_5 is the Clifford unit pseudoscalar. The operation $< >_s$ means take the scalar part.)

It is shown in Ref. (2) that one can obtain from the $\mathcal{L}$ of Eq. (76) the symmetrized dyality invariant Maxwell's equations.' To develop my approach to obtain the (desired dyality invariant) generalized equations of motion for both electrically and magnetically charged particles, I first multiply out the generalized interaction term[2]

$$-\mathcal{J}\mathcal{A} = -(j - \tilde{k})(A - \tilde{M}) = -(jA + \tilde{k}\tilde{M} - j\tilde{M} - \tilde{k}A) \quad . \tag{77}$$

The tilde denotes the dual of the indicated quantity.[(2)] Thus, $\tilde{k} = \gamma_5 k$. The scalar part of $-\mathcal{J}\mathcal{A}$, which is equal to $- j \cdot A - k \cdot M$, comprises the standard interaction terms and leads to the proper equations of motion for electrically charged particles in an electromagnetic field[3] and for magnetically charged particles in a magnetoelectric field.[3] The cross terms in this Lagrangian, from which one wishes to obtain the cross interaction forces, comprise a sum of bivectors and pseudoscalars and do not survive the $< >_s$ operation. Hence, the analysis appears to be at an impasse.

This situation comes as no surprise to those who are familiar with the history of efforts trying to put the physics of magnetic monopoles on a more secure mathematical foundation, that is, efforts to derive the equations of motion from a variational principle.[4] It was shown over 20 years ago[(3,4)] that (given certain assumptions) one cannot derive from the same Lagrangian the dyality invariant Maxwell's equations and the equations of motion of electrically and magnetically charged particles. There are also more recent discussions.[(5–7)] In spite of these proofs, numerous efforts to construct suitable Lagrangians have been made. One approach is to diversify the functional form of the Lagrangian by augmenting the supply of vectors beyond the usual current and the vector potential. For example, in addition to these, the coordinate vector[3,8] and the unit vector [9] have been employed. However, all these approaches have encountered difficulties of one sort or another. There is also a more recent paper[10] on this topic, using Clifford algebra, but it too has difficulties.[2,11] In fact, even Rodrigues[1] now concedes that there are flaws in Ref. (10). Extensive reviews of these efforts and their difficulties have been published.[12]

The reiteration of this problem, which was already pointed out in Ref. (2), is the essence of the Rodrigues comment. On the other hand, it seemed evident to me that, if one wished to proceed and utilize these cross terms, something new would be necessary. Rodrigues ignored or misconstrued the new aspects introduced in Ref. (2).

In order to find this something new, I explored the surface-integral form of the variation of the action associated with the standard interaction terms of S_{I_S}. (Since Rodrigues does not dispute the use of the surface-integral formulation for finding the variation of the action associated with an incremental variation in path, described by δx^{ν}, I do not repeat that argument here.[5]) Thus,

$$\delta_x S_{I_S} = -\frac{1}{c}\oint dx \cdot (eA + gM) \tag{78a}$$

$$= \frac{1}{c}\int (d\sigma \cdot \partial) \cdot (eA + gM) \tag{78b}$$

$$= \frac{1}{c}\int < d\sigma \partial (eA + gM) >_s \quad . \tag{78c}$$

Equation *(78a)* is the variation of S_{I_S} as the difference between the line integral along the displaced path and along the equilibrium path. This is the usual formulation, but written as the line integral around a closed loop. ($\delta x^{\nu} = 0$ at the ends of the path.) Equation *(78b)* is an equivalent surface integral form, found by using the boundary theorem.[13] Equation (78c) is an expression equivalent to Eq. *(78b).*

When the type of multiplication between the factors in the terms is specified, as in Eqs. (78a–78b), the resulting expression is already a Clifford scalar, which, of course, is suitable for determining the equations of motion. When the more general Clifford or geometric product is used in Eq. (78c), *e.g.*, *dad,* additional bivector and pseudoscalar terms arise, but such terms are not of interest for the derivation of the (usual) equations of motion and are discarded by employing the $<\ >_s$ operation. Thus, Eq. (78c) is still mathematically equivalent to the usual line integral formulation.

It is important to observe that Eq. (78b) incorporates a conceptual extension to the usual formulation of the variation of the action. The variation of the action as expressed in Eq. (78b) is no longer just the difference between the action along the displaced path and that along the equilibrium path; it involves the functional form of the potentials on an arbitrary (incremental) surface area spanning the loop defined by the two paths. That is, using this surface integral formulation, the values of the potentials in the entire incremental region in the vicinity of the equilibrium path simultaneously contribute to the variation of the action. Of course, the location of the displaced path as defined by δx^ν still precisely determines via Eq. (78b) the $\delta_x S_{I_S}$. While this step is not controversial because Eq. (78b) is mathematically equivalent to the usual line integral formulation, it leads one in a natural way to the next conceptual extension of $\delta_x S_I$.

It is the more general form of $\delta_x S_{I_S}$ given by Eq. (78c) that is key to the development shown in Ref.- (2). The crucial step is the assertion that it is appropriate to view Eq. (78c) as a valid expression in general for the variation of the action $\delta_x S_I$. Following this logic, then, one places the (appropriate form of the) generalized cross interaction terms of the Lagrangian $\mathcal{J}\mathcal{A}$ directly into the parenthesis of Eq. (78c). This step yields

$$\bar{\delta}_x S_I = \bar{\delta}_x(S_{I_S}+S_{I_C}) = \frac{1}{c}\int <\ d\sigma\partial(eA+gM-e\tilde{M}+g\tilde{A})\ >_s . \quad (81')$$

In deference to Rodrigues, in Eq. (81') I have changed the notation for the variation operator from δ_x to $\bar{\delta}_x$ (hence, the prime on the equation number) to indicate that there is something new here.

It is straightforward, as was demonstrated in Ref. (2), to show that the usual condition for an extremum, $\bar{\delta}_x(S_I + S_P) = 0$, where S_P is the action for the particle, indeed leads to the desired generalized dyality invariant equations of motion for electrically and magnetically charged particles in any combination of electromagnetic and magnetoelectric fields.

In order to understand how this new result is achieved, in spite of the proofs to the contrary, I first note that the scope of the Lagrangian builder's license has been extended to include $\delta_x S_I$ as well as S_I; in going from Eq. (76) to Eq. (81') the scalar assumption for the Lagrangian has been tacitly shifted from S_I to $\delta_x S_I$. While this is a weaker form of the scalar assumption, we see that no damage has been done to the final result because although all of the relevant terms of S_I are not scalars, S_I is written in covariant form and the resulting equations of motion derived from $\bar{\delta}_x S = 0$ are Lorentz invariant, which was the main motivation for assuming a scalar S in the first place. The next observation is to note that Clifford algebra pointed to a new way of combining the vectors we already have, obviating the need to look for new vectors. In Eq. (81') we are effectively combining into one equation Eq. (78b) for $\delta_x S_{I_S}$ and its analog

$$\bar{\delta}_x S_{I_C} = \frac{1}{c}\int (\mathrm{da} \wedge \partial) \cdot (-e\tilde{M} + g\tilde{A})$$

for the cross interaction. It is of course reasonable for $(\mathrm{da} \cdot \partial) \cdot (eA + gM)$ of Eq. (78b) to represent the standard interaction terms; this expression is mathematically equivalent to the usual line integral formulation for the vector . vector interaction $(j \cdot A + k \cdot M)$. Examination of Eq. (81') reveals that for the cross terms, it is the bilinear combinations $j \cdot \tilde{M} + \tilde{k} \cdot A$ that contribute the scalar terms to $\bar{\delta}_x S_I$, and hence the force terms describing the cross interactions. These terms are of the form vector . pseudovector,[6] which in space-time algebra yields bivectors, not scalars. But these terms yield suitable scalars when one forms $\bar{\delta}_x S_I$ using the surface integral formulation in the context of the weaker form of the scalar assumption. The vectors that enable this result are the ∂ and the δx (which is subsumed in the do). From a geometric point of view a new Lagrangian representation of the relationship between a current and a potential is now possible.

This development has the merit that the appropriate scalar terms associated with the cross interaction in the variation of the action as given by Eq. (81') appear in a natural way. It is a further benefit that Clifford algebra furnishes a mathematically uniform way of treating the standard terms and cross terms. Equation (81') prescribes that first the indicated geometric products are taken and then only the scalar parts are kept. In effect, Eq. (81') becomes the basis for a more general definition for the operation of finding the variation of the action for generalized electromagnetism. Being more general, this new definition is ***not*** equivalent to the usual line integral formulation, in particular for the cross terms. Thus, the something new alluded to in Ref. (2) can be considered to be a newly defined variation operator $\bar{\delta}_x$, or the view that Hamilton's principle in this case is appropriately defined directly in terms of the variation of the action rather than the action itself.

In conclusion, Nature is the ultimate authority in all efforts to develop physical theories. After all, Lagrangian building can be viewed(15) as an experimental science.[7] The criterion for the usefulness of a given Lagrangian, or action, is the suitability of the equations of motion that are derived from it. Application of this criterion to $\bar{\delta}_x S_I$ leads one to the conclusion that the development in Ref. (2) evidently is an appropriate approach. If, when magnetic monopoles are discovered, it turns out that these dyality invariant equations of motion correctly describe monopole physics, then the fact that they are derived by this strategem substantiates the assertion that the use of the geometric product in the surface integral formulation enables one to describe more comprehensively interactions between electric and magnetic charge currents and their vector potentials. This would also be an important impetus to seriously investigate further the physics implications of this mathematical description of monopoles. On the other hand, if monopoles are found and they do not exhibit any cross interactions, as contemplated in the model of Comay,(18) then this approach must be abandoned. Until monopoles are found the question is moot.

REFERENCES

1. W. A. Rodrigues, Jr., ***Found. Phys. Lett. 3, 95*** (1990).
2. D. Fryberger, ***Found. Phys.*** 19, 125 (1989).
3. F. Rohrlich, ***Phys.*** Rev. 150, 1104 (1966).
4. D. Rosenbaum, ***Phys.*** Rev. 147, 891 (1966).
5. J. Godfrey, ***Nuovo Cimento* 71A,** 134 (1982).
6. D. G. B. Edelen, ***Applied Exterior*** Calculus (Wiley, New York, 1985).
7. E. Comay, ***Nuovo Cimento* 80B,** 159 (1984); ***ibid.,* 89B,** 224 (1985).

8. D. Villarroel, *Nuovo Cimento Lett.* **15, 507 (1976).**
9. D. Zwanziger, *Phys.* Rev. **D3, 880** (1971).
10. M. A. de Faria-Rosa, E. Recami, and W. A. Rodrigues, Jr., ***Phys. Lett.*** **B173, 233** (1986); ***ibid., 188,*** 511 (1987).
11. E. Comay, ***Phys. Lett.*** **B207, 56** (1988).
12. V. I. Strazhev and L. M. Tomil'chik, *Fiz.* ***Elem. Chast. At. Yud.*** 4,187 (1973), [S*ov.* ***J. Part. and*** *Nucl.* 4, 78 (1973)]; Yu. D. Usachev, ***Fiz.*** *Elem.* ***Chast. At. Yad. 4, 225*** (1973) ***[Sov. J. Part. and*** *Nucl.* 4, 94 (1973)]; E. Ferrari, "Formulations of electrodynamics with magnetic monopoles," in ***Tachyons, Monopoles, and Related Topics,*** edited by E. Recami (North-Holland, Amsterdam, 1978), pp. 203-225.
13. D. Hestenes, ***Space-Time Algebra*** (Gordon & Breach, New York, 1966).
14. E. M. Purcell and N. F. Ramsey, ***Phys. Rev. 78,*** 807(L) (1950); N. F. Ramsey, ***Phys. Rev.*** **109, 225(L)** (1958).
15. H. Goldstein, ***Classical Mechanics*** (Addison-Wesley Publishing, Reading, 1950), p. 365; F. Rohrlich, ***Classical Charged Particles*** (Addison-Wesley, Reading, 1965), p. 86.
16. M. Kline, ***Mathematics, The Loss of Certainty*** (Oxford University Press, Oxford, 1980), Chap. 15.
17. M. Sachs, ***Found. Phys.*** **19, 1525** (1989).
18. E. Comay, *Nuovo* Cimento ***Lett.*** 43, 150 (1985).

NOTES

1. This is not a new result; see, e.g., Rohrlich.[3]
2. A typographical error in Eq. (77) is corrected here: The product jA in the right-hand parentheses was erroneously transcribed in Ref. (2) as $\mathcal{J}\mathcal{A}$.
3. I define electromagnetic field $F = \partial A$ to be that generated by electric charges and the magnetoelectric field $G = \partial M$ to be that generated by magnetic monopoles: $\mathcal{F} \equiv F + \gamma_5 G$.
4. Rodrigues, on the other hand, takes the position that the importance of the Lagrangian formulation should be downgraded if not discarded altogether: ". . . it is redundant to look for Lagrangians."[1]

5. In fact, he reformulates it using the language of differential forms.
6. It is interesting to observe that this bilinear form has the additional virtue of being appropriate for dealing with the monopole-charge parity question, which was pointed out long ago.[14]
7. In fact, even mathematics looks to Nature for its authority.[16] There is evidence that Rodrigues does not understand this concept.[17]