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ON THE STRUCTURE OF HIGH-ORDER TERMS IN CHIRAL PERTURBATION THEORY WITH EXTERNAL FIELDS*

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ABSTRACT

The external-field formalism of chiral perturbation theory, as developed by Gasser and Leutwyler, is extended to arbitrary orders in the chiral expansion. In view of the occurrence of the tensor $\varepsilon^{\mu\nu\rho\sigma}$ in the Wess-Zumino term and in invariant higher-order Lagrangeans, analytic operator regularization is adapted to chiral perturbation theory and shown to be consistent. Beyond one loop, a new class of invariant counterterms occurs, containing the fluctuation field besides the background and external fields. The relation between counterterm Lagrangeans with different powers of the fluctuation field depends in general on the renormalization conditions imposed on the fluctuation. The divergent part of the one-loop diagrams with one external fluctuation-field line is explicitly calculated and found to be derivable by expanding $\mathcal{L}_4(U)$.

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1. Introduction

Chiral perturbation theory (ChPT) succinctly summarizes the powerful constraints imposed on the low-energy behaviour of QCD by spontaneously broken chiral symmetry (ChS) and allows perturbative calculation of the light-quark mass effects. The most convenient representation of the algebra of chiral currents is by means of effective chiral Lagrangeans for the octet of the lowest-lying pseudoscalar bosons π, K and η.[1] A reasonably good description of the experimental data is thereby obtained.

On a more quantitative level, unitarity corrections and symmetry-breaking effects of higher order in the quark masses have to be taken into account. Even though the model is not renormalizable, quantum corrections can be calculated in the framework of a systematic momentum expansion of the effective action.[2] Divergences in loop diagrams can be absorbed by renormalizing higher-order chiral Lagrangeans. The newly introduced couplings have to be determined from some experimental data, but chiral symmetry still allows to predict the rates for a host of other processes.

At next-to-leading order (p^4), a particularly systematic and convenient framework has been set up by Gasser and Leutwyler[3] and used to determine the ten additional coupling constants arising at that level. This formalism has since been very successfully applied to a large variety of strong and electro-weak processes involving mesons and even baryons. It has recently been extended to incorporate quantum corrections to anomalous meson processes[4] and is ideally suited for proving the nonrenormalization of the chiral anomaly to all orders of ChPT.[5,6]

With one exception,[7] no attempts have so far been made to carry ChPT beyond one-loop order, essentially because the calculational effort and the number of required counterterms increase sharply. It is, however, of some theoretical interest to see whether the formalism can be consistently extended to all orders in the chiral expansion: Ref. 6 is an example of a general statement that can be made about chiral symmetry in ChPT; moreover, one can study in this model which properties of a theory are affected by its (perturbative) nonrenormalizability.

In this spirit, Charap[8] and Honerkamp[9] investigated some general properties of loop corrections in the four-dimensional nonlinear σ model. However, they did not consider any specific regularization scheme and its effects on chiral symmetry. One of the purposes of this work is to supply a discussion of this issue within the formalism of Ref. 3, extended to any finite order in the chiral expansion. Dimensional regularization (DR) has been used to great advantage at the one-loop level because it preserves chiral symmetry. However, both the anomalous Wess-Zumino term and many terms in the chirally invariant Lagrangeans of sixth order[6] and beyond contain the $\varepsilon^{\mu\nu\rho\sigma}$ tensor, so that DR cannot readily be applied. Neither a simple cutoff nor the Pauli–Villars method are useful in ChPT because the former violates chiral symmetry and the latter leads to difficulties with the external scalar and pseudoscalar fields. As a replacement, operator regularization (OR) appears ideally suited for the background-field formulation of ChPT: It preserves the symmetries, operates always in four dimensions and is based on analytic continuation. The original prescriptions by McKeon and Sherry[10] will be adapted somewhat (instead of the implicit subtractions performed by their procedure, the separation into divergent and finite parts will be made explicit).

Other topics arising in higher orders are the functional measure and its regularization, the use of the classical equations of motion and the subtraction procedure for subgraph divergences. The latter requires a new class of counterterms depending not only on the background and external fields, but on the fluctuations as well. Their coupling constants in general depend on the renormalization prescriptions imposed on the fluctuation field. In Sect. 4, explicit calculation of the $O(p^4)$ divergent terms linear in the fluctuation suggests a close relation between the new counterterms and those without fluctuation fields.

2. Operator Regularization

2.1. Regularization of the propagator

Operator regularization[10] is based on the observation[11] that loop diagrams converge if the propagator falls off as a sufficiently high power of p^2 at large momenta. Evaluating the diagrams with propagators $(p^2+m^2)^{-1-\varepsilon}$, one may separate the finite pieces from the divergent terms in the limit $\varepsilon \to 0$ after analytic continuation. In a problem with background fields, local symmetries are explicitly preserved if the free propagator is replaced by a suitable covariant differential operator $D^\star$, representing the propagator of the fluctuation field in the background. Such an operator D arises naturally in ChPT, see Ref. 3 or (3.16) and (3.17).

A convenient exponential representation of $D^{-1-\varepsilon}$ is

$$(D^{-1})_{reg.} = \lim_{\varepsilon\to 0} \frac{\mu^{2\varepsilon}}{\Gamma(\varepsilon+1)} \int_0^\infty dt\, t^\varepsilon e^{-tD} \,. \tag{2.1}$$

In order to evaluate regularized amplitudes, one needs to expand $\exp(-tD)$ in powers of the background fields. Let $D = D_0 + D_I(\Phi)$ where D_0 is the free inverse propagator without background fields and D_I is the interaction piece. After applying formulae given by Schwinger[12] in a similar context,

$$\begin{aligned} e^{-t(D_0+D_I)} =& \sum_{n=0}^{\infty}(-t)^n \int_0^1 du_1 \ldots \int_0^1 du_n {u_n}^{n-1} e^{-u_1\cdots u_n t D_0} D_I \\ & \times e^{-(1-u_1)u_2\cdots u_n t D_0} D_I \cdots e^{-(1-u_{n-1})u_n t D_0} D_I e^{-(1-u_n)tD_0} \end{aligned} \tag{2.2}$$

$\star$ The implicit assumption here is that D is a self-adjoint operator. This is not the case for a chiral fermion interacting with gauge fields, and explicit calculation with OR indeed recovers the usual chiral anomaly: See the second paper in Ref. 10.

and

$$
\begin{aligned}
\mathrm{Tr}\left(e^{-t(D_0+D_I)}\right) = \mathrm{Tr}\Bigg(e^{-tD_0} &- tD_I e^{-tD_0} \\
&+ \sum_{n=1}^{\infty} \frac{(-t)^{n+1}}{n+1} \int_0^1 du_1 \ldots \int_0^1 du_n {u_n}^{n-1}\, D_I e^{-u_1\cdots u_n tD_0} D_I \\
&\times e^{-(1-u_1)u_2\cdots u_n tD_0} D_I \cdots e^{-(1-u_{n-1})u_n tD_0} D_I e^{-(1-u_n)tD_0}\Bigg)\,,
\end{aligned}
\tag{2.3}
$$

all integrations over vertex locations can be turned into standard Gaussian momentum integrals.

In applications of OR, momentum integrations, parameter integrals and the expansion of $\exp(-tD)$ are freely interchanged. In order to justify this procedure, one needs to note first that D in Euclidian space is strictly positive for weak external fields and nonvanishing quark masses. Thus the t-integration converges uniformly for all values of p. Conversely, the integration over the loop momentum associated with strings of propagators does not converge uniformly with respect to t as $t \to 0$, but for ε sufficiently large the pole at $t = 0$ is removed by the factor t^ε in (2.1). It follows that the t and momentum integrations may indeed be interchanged.

A sufficient criterion for uniform convergence of the expansions (2.2) and (2.3) is easily established if D_I is x-dependent but does not contain derivatives: By evaluating $\exp(-P(u)tD_0)$ in the momentum basis ($P(u)$ are polynomials of $u_1, \ldots, u_{n-1}$) and D_I in the x basis, one arrives at the condition $\int d^4x |D_I(x)|^n \leq K^n$ for all sufficiently large n and some arbitrary but fixed number K. In the case of interest in ChPT, where D_I also contains derivatives, the analogous condition involves higher and higher derivatives of the external fields and its form is not very illuminating. It appears, however, that there is a large class of sufficiently smooth and rapidly decaying external field configurations that satisfy the criterion. It will thus be assumed in the following that the external fields are such that (2.2) and (2.3) indeed represent valid expansions.

For any value of the loop momentum, the integration over the u-parameters converges uniformly, at a rate that depends only on the values of the momenta carried by external and background fields. Similarly, the convergence of the momentum integration is uniform for all values of the u_i. Thus, the momentum integration may be carried out first. For non-zero meson masses, the exponent of the exponential remaining after the momentum integration is strictly positive for sufficiently small external momenta (its zeroes correspond to physical thresholds) so that the u-integrations converge uniformly for all values of t; conversely, the t-integration also converges uniformly for all values of the u_i if it is carried out before the latter. Hence, the customary steps of interchanging integrations in the regularized theory are justified in OR.

Finally, note that OR does not distinguish between divergences of different orders with respect to a momentum cutoff; divergent loops uniformly lead to a factor $1/\varepsilon$. The method adds, for each loop, *all* terms of $O(\varepsilon^{-1})$ and $O(\varepsilon^{0})$ to obtain chirally invariant results. (See also Sect. 3.5 in this connection.)

2.2. Regularization of $\delta^{(4)}(0)$

As discussed in Sect. 3, the invariant functional measure leads to extra vertices derived from the determinant of the target space metric. The factor $\delta^{(4)}(0)$ occurring therein requires chirally invariant regularization. Its origin is the strict locality of chiral transformations; regularization essentially amounts to appropriate smearing of this δ function. The regularized expression must be an invariant function of the background and external fields since no other scale-setting entities are available. Thus the slightly paradoxical situation arises where $\ln\det\big(g_{ab}(\xi)\big)\delta^{(4)}(0)$ indeed depends on other fields besides the fluctuation ξ.

The most natural way to regularize $\delta^{(4)}(0)$ in the context of OR makes use of the equation

$$\delta_{ab}\delta^{(4)}(0) = \delta_{ab}\int\frac{d^4p}{(2\pi)^4} = \int\frac{d^4p}{(2\pi)^4}\lim_{\varepsilon\to 0} D_{ab}^{-\varepsilon}\ .$$

In the next step, the integration and the limit $\varepsilon \to 0$ are interchanged – even though the limit is not uniform with respect to p – to obtain the regularized form of the δ function:

$$(\delta^{(4)}(0))_{\rm reg.} = \lim_{\varepsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \frac{{\rm Tr}\,(D^{-\varepsilon})}{{\rm Tr}\,(1)} \,. \qquad (2.4)$$

Evaluating the integral one finds the somewhat counterintuitive result that this regulated form of $\delta^{(4)}(0)$ is finite in the limit $\varepsilon \to 0$. After a calculation equivalent to determining the divergent part of Tr (ln D) in DR[3]*, one obtains

$$\delta^{(4)}(0) = -\lim_{\varepsilon\to 0} \varepsilon \int d^4x \mathcal{L}_4(U_0) \,, \qquad (2.5)$$

i.e. the divergent piece of $\mathcal{L}_4$, multiplied by ε so that it becomes finite. Note that the regularizing operator need not be D; it can be chosen at will if it is chirally covariant and causes the integrals to converge. D stands out, however, as the simplest choice; in the momentum expansion, the order of Sp ln D exactly corresponds to the engineering dimension of the δ function.

2.3. Regularization of ln D

One-loop graphs to $O(p^4)$ are calculated by evaluating $(i/2)$Tr (ln D). A regularized form of ln D is

$$(\ln D)_{\rm reg.} = -\lim_{\varepsilon\to 0} \frac{d}{d\varepsilon}\left(\mu^{2\varepsilon} D^{-\varepsilon}\right) = -\lim_{\varepsilon\to 0} \frac{d}{d\varepsilon}\left(\frac{\mu^{2\varepsilon}}{\Gamma(\varepsilon)} \int\limits_0^\infty dt\, t^{\varepsilon-1} e^{-tD}\right) \,. \qquad (2.6)$$

Note that this definition is well known from ζ-function regularization and reduces to the ordinary logarithm if D is a positive c-number. Only finite terms are obtained when it is applied to Sp (ln D).

⋆ In the following, the symbol Tr will denote the trace over flavour indices whereas Sp also includes the trace over the continuous label x.

At this point, one may check the consistency with the regularization proposed for $\delta^{(4)}(0)$ in Sect. 2.2. Formally, $\ln D$ may also be written as

$$\ln D = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(D^{-\varepsilon} - 1\right) . \tag{2.7}$$

The trace over the second term leads to $\delta^{(4)}(0)$, which is subsequently regularized with the help of eqn. (2.4):

$$\begin{aligned} \mathrm{Sp}\,(\ln D) &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\,(D^{-\varepsilon}) + \lim_{\varepsilon\to 0} \frac{\delta^{(4)}(0)}{\varepsilon}\,\mathrm{Tr}\,(1) \\ &= -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,(D^{-\varepsilon}) - \lim_{\alpha\to 0} \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,(D^{-\alpha}) \right) . \end{aligned} \tag{2.8}$$

This is exactly the same result as one obtains from (2.6).

In order to make close contact with the results obtained from the heat-kernel method combined with dimensional regularization, one may consider yet another definition of $(\ln D)_{\mathrm{reg.}}$: By simply dropping the second term in (2.7), one gets

$$(\ln D)_{\mathrm{reg.}} = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\, D^{-\varepsilon} . \tag{2.9}$$

For a c-number a, this evaluates to $\ln a - 1/\varepsilon$. This may appear artificial, but one may check that the results obtained for the trace of the logarithm of a second-order differential operator of the type of eqn. (3.17) are indeed equivalent to those obtained in DR, including the divergent pieces.

2.4. Subtraction of subgraph divergences

The subtraction of subgraph divergences in OR deserves separate discussion. In Ref. 13, two-loop graphs regulated according to Ref. 10 were found to be finite in the limit $\varepsilon \to 0$ even without introducing counterterms; implicit subtraction of overall divergences in the Laurent expansion of the regulated amplitude is built into the original method. However, agreement with other regularization methods could be achieved only if the analog of Bogoliubov–Parasiuk–Hepp–Zimmermann[14] subtractions was performed.

In the framework of a nonrenormalizable theory, it appears conceptually clearer to explicitly remove infinities by means of counterterms since these are part of a higher-order Lagrangean whose effects have to be included, even if there were no divergences. The prescriptions for a renormalizable theory can then immediately be transcribed to the present case by allowing for an increasing number of counterterms (see the second paper of Ref. 14, p. 322).

In general, one-loop subgraphs need to be treated with extra care in OR because they contain the fluctuation field in the external lines and thus are calculated from strings of propagators D^{-1} rather than from $\mathrm{Sp}\,(\ln D)$. OR regulates the two operators in somewhat different ways; additional finite subtractions may be needed to ensure consistent renormalization of all Green functions. Ref. 13 gives a prescription for accomplishing this without explicit counterterms by means of introducing an auxiliary scale. In ChPT with counterterms, the same effect can be obtained by adjusting the finite piece of the couplings in the higher-order Lagrangeans.

3. The Generating Functional beyond One-Loop Order

This section focuses on the structure of the generating functional of connected Green functions in chiral perturbation theory. For a detailed exposition of the external-field formalism in ChPT, the reader is referred to Ref. 3. The main point is that the perturbative expansion of the regulated generating functional can be obtained in terms of chirally covariant vertices and propagators. The invariant functional measure, however, induces a set of extra vertices whose role in the present formalism differs from that in earlier approaches.[15,16]

3.1. Chiral Lagrangeans with external fields

At low energies, the effective action for QCD coupled to external sources for the flavour currents contains only degrees of freedom related to the light quarks u, d and s. Assuming dynamical breaking of chiral symmetry at a scale $O(1\ \mathrm{GeV})$, the first term in its momentum expansion may be written as[3]

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{Tr}\left(\nabla_\mu U^\dagger \nabla^\mu U + U^\dagger X + X^\dagger U\right) + h\nabla_\mu\Theta\nabla^\mu\Theta\ . \tag{3.1}$$

The field U takes values in SU(3) and is parametrized as

$$U(x) = \exp(-\varphi(x)/f)\ , \qquad \varphi(x) = -i\lambda_a\varphi^a(x)\ , \tag{3.2}$$

where the $\varphi^a(x)$ represent the octet of pseudoscalar mesons π, K and η. The convention for the Gell-Mann matrices is $\mathrm{Tr}\,(\lambda_a\lambda_b) = 2\delta_{ab}$ for $a, b = 1, \ldots, 8$. To lowest order, the constant f is the pion decay constant, $f \approx 93\,\mathrm{MeV}$.

Under chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ rotations, U transforms as

$$U'(x) = g_L^{-1}(x)\, U(x)\, g_R(x). \tag{3.3}$$

Chirally covariant derivatives are defined by

$$\begin{aligned} \nabla^\mu U &= \partial^\mu U + V_L^\mu U - U V_R^\mu\ , \\ \nabla^\mu U^\dagger &= \partial^\mu U^\dagger + V_R^\mu U^\dagger - U^\dagger V_L^\mu\ . \end{aligned} \tag{3.4}$$

The (anti-Hermitean) source fields for the left- and right-handed vector flavour currents are required to transform as connections:

$$V_{R,L}^\mu = -\frac{i}{2}\lambda_a V_{R,L}^{\mu,a} = V^\mu \pm A^\mu \rightarrow g_{R,L}^\dagger (V_{R,L}^\mu + \partial^\mu) g_{R,L}\ . \tag{3.5}$$

The (Hermitean) scalar and pseudoscalar flavour densities transform homogeneously:

$$X = \frac{v}{f^2}(S + iP) = \frac{v}{f^2}\frac{1}{2}\lambda_a(S^a + iP^a) \rightarrow g_L^\dagger X g_R\ ; \tag{3.6}$$

the constant v is related to the quark condensates, and the constant term in $S - iP\gamma_5$ contains the quark mass matrix.

The Lagrangean (3.1) is counted as $O(p^2)$: U is a dimensionless field, V^μ and A^μ are order p, while X, containing the square of the meson masses, is $O(p^2)$. Chirally invariant Lagrangeans $\mathcal{L}_{2n}$ of higher order in p may be constructed from covariant expressions like U, $\nabla_\mu U$, $\nabla_\mu \Theta$, X, $\nabla_\mu X$, $F^{\mu\nu}_{R,L}$, etc. At the level p^4, the WZ action should also be included in order to reproduce the triangle anomaly of QCD.

3.2. Saddle point expansion of the chiral Lagrangean

In order to calculate quantum corrections in a chiral-symmetry preserving way, one may expand $\mathcal{L}_2$ about the solution $U_0(x)$ of the classical equations of motion and integrate over the quantum fluctuations according to the number of loops.[3] As shown by Weinberg,[2] the divergences will be of successively higher orders in p^2 and have to be absorbed by the Lagrangeans $\mathcal{L}_4$, $\mathcal{L}_6$, etc. whose vertices in turn contribute at tree level as well as in loops. The full unitary theory will thus contain infinitely many independent couplings.

It is important to parametrize the quantum fluctuations in such a way that parity and chiral symmetry are manifest. A very convenient choice is[3]

$$U(x) = u(x)e^{-\xi(x)/f}u(x) , \qquad u(x) := U_0^{1/2}(x) . \tag{3.7}$$

The fluctuation fields ξ transform non-linearly under the chiral group and linearly under the vectorial subgroup:

$$\xi'(x) = \tilde{h}^\dagger(g_L(x), g_R(x), u(x))\, \xi(x)\, \tilde{h}(g_L(x), g_R(x), u(x)) , \tag{3.8}$$

where

$$\tilde{h}(g_L(x), g_R(x), \varphi_0(x)) = u(x)\, g_R(x)\, u'^\dagger(x) = u^\dagger(x)\, g_L(x)\, u'(x) . \tag{3.9}$$

It is easy to see that $\xi(x)$ is a pseudo-scalar field. For the purpose of generating graphs in a perturbative expansion, one may introduce a source $J(x)$ for $\xi(x)$ with the transformation law $J' = \tilde{h}^\dagger J \tilde{h}$ so that $\mathrm{Tr}\,[J(x)\xi(x)]$ is invariant under chiral transformations.

In order to make chiral invariance manifest for each term in the action, one defines the covariant derivative of ξ through the formula[17,3]

$$\begin{aligned} \nabla_\mu U &= u\big(d_\mu e^{-\xi} + \{\Delta_\mu,\, e^{-\xi}\}\big)u, \\ \nabla_\mu U^\dagger &= u^\dagger\big(d_\mu e^{\xi} - \{\Delta_\mu,\, e^{\xi}\}\big)u^\dagger\ , \end{aligned} \tag{3.10}$$

where

$$d_\mu := \partial_\mu + [\Gamma_\mu,\ \ ]\ . \tag{3.11}$$

The anti-Hermitean fields Γ_μ and Δ_μ are vector and axial-vector fields, respectively:

$$\Gamma^\mu = \tfrac{1}{2}\,[u^\dagger, \partial^\mu u] + \tfrac{1}{2}\,uV_R^\mu u^\dagger + \tfrac{1}{2}\,u^\dagger V_L^\mu u\ , \tag{3.12}$$

$$\begin{aligned} \Delta^\mu &= \tfrac{1}{2}\,\{u^\dagger, \partial^\mu u\} - \tfrac{1}{2}\,uV_R^\mu u^\dagger + \tfrac{1}{2}\,u^\dagger V_L^\mu u \\ &= \tfrac{1}{2}\,u^\dagger \nabla^\mu U_0 u^\dagger = -\tfrac{1}{2}\,u\nabla^\mu U_0^\dagger u\ . \end{aligned} \tag{3.13}$$

While Δ_μ transforms homogeneously under the chiral group, Γ_μ is a connection, as anticipated by (3.11):

$$\Gamma'_\mu = \tilde{h}^\dagger(\Gamma_\mu + \partial_\mu)\tilde{h} \qquad \text{and} \qquad \Delta'_\mu = \tilde{h}^\dagger \Delta_\mu \tilde{h}\ . \tag{3.14}$$

In addition, one defines

$$\sigma_\pm = \frac{1}{2}(u^\dagger X u^\dagger \pm uX^\dagger u)\ . \tag{3.15}$$

It is now easily seen that each term in the expansion of a chiral Lagrangean $\mathcal{L}_{2k}$ is a chiral invariant by itself.

3.3. THE STRUCTURE OF THE GENERATING FUNCTIONAL

The chiral perturbation series is generated from the path integral in the usual way. Let Φ collectively denote all the external fields except J introduced after (3.9), and designate the chirally invariant Lagrangean of $O(p^{2k})$ with n powers of the fluctuation field ξ by $\mathcal{L}_{2k,n}(U_0, \xi)$. The generating functional of connected Green functions can then be represented as

$$\begin{aligned} e^{iW[\Phi]} &= e^{i\sum_{k=1}^{\infty} S_{2k}[U_0,\Phi]} \cdot e^{i\sum_{k=1}^{\infty} \sum_{n=1}^{\prime\infty} S_{2k,n}[U_0,\Phi,i\delta/\delta J]} \\ &\quad \times \int d\mu(\xi) e^{i\int d^4x \left(\frac{1}{2}\xi^a D_{ab}[U_0,\Phi]\xi^b - \xi^a J^a\right)} \Big|_{J=0} \\ &= e^{i\sum_{k=1}^{\infty} S_{2k}[U_0,\Phi]} \cdot e^{\frac{i}{2}\mathrm{Tr}\,(\ln D_{ab})} \cdot e^{i\sum_{k=1}^{\infty}\sum_{n=3}^{\prime\infty} S_{2k,n}[U_0,\Phi,i\delta/\delta J]} \\ &\quad \times e^{\frac{1}{2}\delta^{(4)}(0)\int d^4x \ln\det[g_{ab}(i\delta/\delta J(x))]} \cdot e^{-\frac{i}{2}\int d^4x d^4y\, J^a(x) D^{-1}_{ab}(x,y|U_0,\Phi) J^b(y)} \Big|_{J=0} , \end{aligned} \tag{3.16}$$

where $\sum'$ means that the terms with $n = 1, 2$ should be dropped for $k = 1$. The inverse propagator of ξ in the presence of the external fields Φ and the background U_0 is given by

$$D_{ab}(x|U_0, \Phi) = \frac{\delta^2}{\delta\xi^a(x)\delta\xi^b(x)} S_{2,2}[U_0, \Phi, \xi] = (d_\mu d^\mu)_{ab} + \hat{\sigma}_{ab} . \tag{3.17}$$

It is covariant under the full chiral group, notwithstanding the non-linear realization of the symmetry (note that the transformation matrices $\tilde{h}$ depend on g_L, g_R and the background field u, but not on ξ itself). In the representation used above, the covariant derivative is

$$(d^\mu)_{ab} = \delta_{ab}\partial^\mu + \hat{\Gamma}^\mu_{ab} , \qquad \hat{\Gamma}^\mu_{ab} = -\tfrac{1}{2}\,\mathrm{Tr}\left([\lambda_a, \lambda_b]\Gamma^\mu\right) \tag{3.18}$$

and the matrix $\hat{\sigma}$ is defined as in Ref. 3:

$$\hat{\sigma}_{ab} = \tfrac{1}{2}\,\mathrm{Tr}\left([\lambda_a, \Delta_\mu][\lambda_b, \Delta^\mu]\right) + \tfrac{1}{4}\,\mathrm{Tr}\left(\{\lambda_a, \lambda_b\}\sigma_+\right) . \tag{3.19}$$

The functional measure $d\mu(\xi)$ will be seen below to respect chiral symmetry, so the whole expression (3.16) is formally a chiral invariant. (The Wess-Zumino term has been omitted for simplicity; its effect is studied in detail in Ref. 6.) Since Sect. 2 has established OR as a valid symmetry-preserving regularization procedure, the regularized and renormalized generating functional W is indeed chirally invariant to any given order of ChPT.

As in the more familiar renormalizable field theories, renormalization proceeds in steps: In a divergent graph with more than one loop, one must first subtract all its subgraph divergences, starting with the innnermost divergent subgraphs. Whereas all external lines in physical Green functions correspond to external fields Φ or meson background fields U_0, subgraphs do contain the fluctuation field ξ in the external lines.

Two different questions arise in this context: (i) Are the divergent pieces in $\mathcal{L}_{2k,n}$ simply derivable from those in $\mathcal{L}_{2k}$ (e.g. by expanding $\mathcal{L}_{2k}(U)$ to n-th order in ξ)? (ii) How are the finite (renormalized) higher-order couplings related? With respect to (i), note that all $\mathcal{L}_{2k,n}$ are chiral invariants; chiral symmetry alone cannot relate them to one another. While this paper cannot definitively answer the question, Sect. 4 will begin to explore it by means of calculating all divergent pieces in $\mathcal{L}_{4,1}$. Contrary to the situation in (i), fine points of the regularization scheme, such as the difference in the treatment of $\ln D$ and D^{-1} in OR (see Sect. 2.4) matter for the finite pieces. Also, renormalization conditions for the "unphysical" Green functions (with fluctuation fields in the external legs) have to be chosen. The most natural choice is to impose the same conditions as on the "physical" Green functions, but other procedures might also be consistent and would lead to different relations.

3.4. The equations of motion

To order p^4, the Lagrangean $\mathcal{L}_4$ has to be evaluated only for the configuration $U_0(x)$ that solves the classical equations of motion of $\mathcal{L}_2$;[3] this circumstance simplifies $\mathcal{L}_4$ somewhat. If one wishes to go beyond $O(p^4)$, one needs to expand $\mathcal{L}_4$ in powers of the fluctuation field ξ and hence has to use a more general expression:

$$\mathcal{L}_4 = \sum_{j=1}^{12} L_j F_j(U,\Phi) + \sum_{k=1}^{2} H_k G_k(\Phi) \,, \tag{3.20}$$

where the terms with $j = 1, \ldots, 10$ and $k = 1, 2$ are those given by Gasser and Leutwyler. In SU(3), there are two additional terms:

$$\begin{aligned} F_{11}(U) =& \mathrm{Tr}\left[(U^\dagger X - X^\dagger U)(U^\dagger \nabla^2 U - \nabla^2 U^\dagger U)\right] \,, \\ F_{12}(U) =& \mathrm{Tr}\left[(U^\dagger \nabla^2 U - \nabla^2 U^\dagger U)^2\right] \,. \end{aligned} \tag{3.21}$$

Generally speaking, the equations of motion derived from $\mathcal{L}_2$ may be used in $\mathcal{L}_{2k}$ once all the necessary expansions in powers of ξ have been performed. E.g., if one wishes to compute W up to and including $O(p^6)$, $\mathcal{L}_{4,1}$ and $\mathcal{L}_{4,2}$ are required and have to be derived from the complete expression for $\mathcal{L}_4$; where possible, the resulting expressions in terms of U_0 may be simplified by means of the equations of motion. In contrast, $\mathcal{L}_6$ may be immediately evaluated at $U(x) = U_0(x)$ in this context.

One might also consider solving the equations of motion of $\mathcal{L}_2 + \mathcal{L}_4 + \ldots + \mathcal{L}_{2k}$ if one wishes to calculate up to order p^{2k} (let this solution be denoted by U_{k-1}). In this way, no one-particle reducible loop graphs need to be calculated since $\sum_{l=1}^{k} \mathcal{L}_{2l,1}(U_{k-1}) = 0$. This is indeed a valid procedure, provided one truncates the p^2 expansion of that sum at $O(p^{2k})$: From an iterative solution of the equations of motion and the power counting rules, one readily sees that $U_{k-1}(x)$ contains components of arbitrarily high order in p if $k \geq 2$, whereas U_0 is strictly of $O(p^0)$.

3.5. The functional measure

In eqn. (3.16), the functional measure $d\mu(\xi)$ was taken to be the invariant Haar measure

$$d\mu(\xi) = \left(\prod_{x,a} d\xi^a(x)\right) \cdot \det{}^{1/2}\big(g_{ab}(\xi)\big); \tag{3.22}$$

$g_{ab}(\xi)$ is the metric on the target manifold in the parametrization of the ξ. It is chirally invariant because the chiral transformations by construction are precisely the isometries of the metric.[18] Since the coordinates φ and ξ, defined by $U = \exp(-\varphi) = u\exp(-\xi)u$, are related by the field-dependent chiral transformation $g_L = g_R^\dagger = u$, it follows that

$$g'_{ab}(\xi) = g_{cd}(\varphi(\xi))\,\frac{\partial\varphi^c}{\partial\xi^a}\,\frac{\partial\varphi^d}{\partial\xi^b} = g_{ab}(\xi). \tag{3.23}$$

This also shows that $g_{ab}(\xi)$ is independent of the background field u.

Standard evaluation[19] of the determinant gives a highly singular extra contribution to the action:

$$\prod_{x,a} d\xi^a \ \det{}^{1/2}\left[g_{ab}(\xi(x))\right] = \left(\prod_{x,a} d\xi^a\right)\, e^{\frac{1}{2}\delta^{(4)}(0)\int d^4x\, \ln\det g(\xi(x))}. \tag{3.24}$$

Either by restituting appropriate powers of $\hbar$ or by counting powers of momentum [where $\delta^{(4)}(0) = O(p^4)$] one finds that this term is $O(p^4)$. All its vertices are at least $O(\xi^2)$, so it contributes to "physical" Green functions only above order p^4.

In his original investigation of higher orders in "naïve" *massless* pion perturbation theory,[8] Charap found chiral symmetry to be violated by quartically or worse divergent loop diagrams unless a specific parametrization was chosen for the pion field. It was subsequently realized[20,21] that the extra graphs with vertices from the determinant eqn. (3.24) have the same type of divergence (in some implicit cutoff regularization) as the offending loop graphs and in fact precisely cancel them.

The role of the determinant strongly depends on the regularization scheme: E.g., in dimensional regularization one may consistently set to zero both $\delta^{(4)}(0) = (2\pi)^{-4}\int d^4p$ and the tadpole graphs of massless particles[22]. This is not possible in operator regularization, but both the regularized determinant and the tadpole graphs are chirally invariant; while the former is finite, the latter in general contain divergences. The determinant thus has lost its chiral-symmetry restoring function.

The extra vertices due to the target space metric lead to finite renormalizations of the chiral Lagrangeans $\mathcal{L}_{4,2n}$, $n \geq 1$ [$\ln\det g(\xi)$ contains even powers of ξ, starting at the second order]. Their contributions have a direct impact on the finite part of the "unphysical" counterterm Lagrangeans (i.e., those depending on fluctuation fields). It remains to be seen how these terms affect the relation between the "physical" and "unphysical" Lagrangeans.

4. Renormalization of $\mathcal{L}_{4,1}$

This section presents the calculation of the divergent terms of all one-loop graphs with one external ξ-line. It serves as an illustration of OR, an exploration of the structure of counterterms with external fluctuation fields, and also as a first step towards the calculation of W_6 (the generating functional to order p^6).

By expanding $\mathcal{L}_2$ to third order in ξ, one easily obtains the vertex:

$$\mathcal{L}_{2,3} = \frac{1}{6f}\mathrm{Tr}\left([\xi,[\xi,d_\mu\xi]]\Delta^\mu\right) + \frac{1}{12f}\mathrm{Tr}\left(\xi^3\sigma_-\right) . \tag{4.1}$$

Replacing two of the ξ fields with $-i\lambda_a \cdot i\delta/\delta J^a$ and letting it act on

$$-\frac{i}{2}\int d^4y d^4z J^b(y) D^{-1}_{b,c}(y,z) J^c(z)$$

[compare with (3.16)], one obtains the following form:

$$\begin{aligned} W^{\mathrm{loop}}_{4,1}[U_0,\Phi,\xi] = \int d^4x \lim_{y\to x} \Big[&\mathcal{V}^{ba}_1(x) + \mathcal{V}^{ba}_{2,\mu}(x)(d^\mu_x - d^\mu_y) \\ &+ \mathcal{V}^{ba}_{3,\mu}(x)d^\mu_x + \mathcal{V}^{ab}_{3,\mu}(y)d^\mu_y\Big] D^{-1}_{ab}(x,y) . \end{aligned} \tag{4.2}$$

The three different vertex structures are given by

$$
\begin{aligned}
\mathcal{V}_1^{ba}(x) =& \frac{(-i)^2}{12f}\,\mathrm{Tr}\,\Big(\{\lambda_a,\lambda_b\}\{\xi,\sigma_-\} + \lambda_a\xi\lambda_b\sigma_- + \lambda_a\sigma_-\lambda_b\xi\Big) \\
& - \frac{(-i)^2}{6f}\,\mathrm{Tr}\,\Big([\lambda_a, d_\mu\xi][\lambda_b,\Delta^\mu] + (a\leftrightarrow b)\Big)\ , \\
\mathcal{V}_{2,\mu}^{ba}(x) =& \frac{(-i)^2}{6f}\,\mathrm{Tr}\,\Big([\lambda_b,\lambda_a][\Delta_\mu,\xi]\Big)\ , \\
\mathcal{V}_{3,\mu}^{ba}(x) =& \frac{(-i)^2}{6f}\,\mathrm{Tr}\,\Big([\lambda_b,\xi][\lambda_a,\Delta_\mu]\Big)\ .
\end{aligned}
\tag{4.3}
$$

In view of future calculations, it is most economical to first obtain the divergent pieces of the quantities $(D_{ab}^{-1})_{\mathrm{reg.}}(x,x)$, $\lim_{y\to x}(d_x^\mu - d_y^\mu)(D_{ab}^{-1})_{\mathrm{reg.}}(x,y)$ and $\lim_{y\to x}\left[d_x^\mu(D_{ab}^{-1})_{\mathrm{reg.}}(x,y) + d_y^\mu(D_{ba}^{-1})_{\mathrm{reg.}}(x,y)\right]$ that are required by the vertex structures exhibited above. After straightforward application of the steps discussed in Sect. 2 and many cancellations of chirally variant expressions among the first three terms in the expansion (2.2), one arrives at the simple and chirally covariant result for the divergent part of the propagator at coincident points:

$$
D_{ab}^{-1}(x,x) = \frac{i}{16\pi^2}\cdot\frac{1}{\varepsilon}\cdot\hat{\sigma}_{ab}(x) + O(1)\ . \tag{4.4}
$$

The covariant derivatives of D^{-1} are obtained by separately calculating the ordinary derivatives of D^{-1} and the terms involving $\hat{\Gamma}$; combining the results, covariant expressions must result:

$$
\lim_{y\to x}(d_x^\mu - d_y^\mu)(D_{ab}^{-1})_{\mathrm{reg.}}(x,y) = \frac{i}{16\pi^2}\cdot\frac{1}{\varepsilon}\cdot\frac{-1}{3}\,d_\nu\hat{\Gamma}_{ab}^{\mu\nu}(x) + O(1) \tag{4.5}
$$

and

$$
\begin{aligned}
&\lim_{y\to x}\Big(d_x^\mu(D_{ab}^{-1})_{\mathrm{reg.}}(x,y) + d_y^\mu(D_{ba}^{-1})_{\mathrm{reg.}}(x,y)\Big) \\
&= \frac{i}{16\pi^2}\cdot\frac{1}{\varepsilon}\cdot\Big(\frac{1}{6}\,d_\nu\hat{\Gamma}_{ab}^{\mu\nu}(x) + \frac{1}{2}\,d^\mu\hat{\sigma}_{ab}(x)\Big) + O(1)\ .
\end{aligned}
\tag{4.6}
$$

The field strength related to Γ_μ has been defined as[3]

$$\Gamma^{\mu\nu} = d^{[\mu}\Gamma^{\nu]} + \Gamma^{[\mu}\Gamma^{\nu]} \,, \qquad \hat{\Gamma}^{\mu\nu}_{ab} = -\tfrac{1}{2}\,\mathrm{Tr}\left([\lambda_a, \lambda_b]\Gamma^{\mu\nu}\right) . \tag{4.7}$$

The chiral invariance of (4.5) and (4.6) is thus manifest.

In the next step, the appropriate propagator terms are multiplied into the vertices; the flavour traces are evaluated with the help of the formula

$$\sum_{a=1}^{N^2-1} (\lambda_a)_{ij}(\lambda_a)_{kl} = 2\delta_{il}\delta_{jk} - (2/N)\delta_{ij}\delta_{kl} \,, \tag{4.8}$$

which is valid for SU(N). From (4.3) it is clear that all the terms so obtained are chiral invariants.

The result may be simplified by use of the equations of motion, which are most conveniently obtained by demanding that the linear term in the ξ-expansion of $\mathcal{L}_2(U)$ vanish: $\mathrm{Tr}\left(\xi(-d\cdot\Delta + \tfrac{1}{2}\sigma_-)\right) = 0$. Note that ξ is a pure octet field whence follows that the singlet part of $d\cdot\Delta$ is *not* determined by the (generally nonvanishing) singlet piece in σ_-. In fact, the definitions of Γ^μ and Δ^μ imply that they as well as $d\cdot\Delta$ are pure octet fields if a parameter related to the QCD vacuum angle[3] is set to zero. With this circumstance taken into account, the equations of motion are

$$-d\cdot\Delta + \tfrac{1}{2}\sigma_- - \tfrac{1}{6}\,\mathrm{Tr}\,(\sigma_-) = 0 \,. \tag{4.9}$$

Other simplifications are obtained by partial integration and by use of the identities

$$\Gamma_{\mu\nu} = \tfrac{1}{2}\,\mathcal{F}^+_{\mu\nu} - [\Delta_\mu, \Delta_\nu] \tag{4.10}$$

and

$$d_\mu\Delta_\nu - d_\nu\Delta_\mu = -\tfrac{1}{2}\,\mathcal{F}^-_{\mu\nu} \,, \tag{4.11}$$

where the notation

$$\mathcal{F}^{\pm}_{\mu\nu} = uF^R_{\mu\nu}u^\dagger \pm u^\dagger F^L_{\mu\nu}u \ ; \tag{4.12}$$

has been used. In the case of phenomenological interest where the symmetry group is SU(3), an additional identity for traceless 3×3 (anti-)Hermitean matrices may be used to combine certain terms[3]:

$$\mathrm{Tr}\,(ABAB) = -2\,\mathrm{Tr}\,(A^2B^2) + \tfrac{1}{2}\,\mathrm{Tr}\,(A^2)\,\mathrm{Tr}\,(B^2) + \left[\mathrm{Tr}\,(AB)\right]^2 . \tag{4.13}$$

Written in terms of ξ, Δ^μ, $\sigma_\pm$ and $\mathcal{F}^{\pm}_{\mu\nu}$, the divergent part of the counterterm Lagrangean $\mathcal{L}_{4,1}$ for $N = 3$ finally emerges as

$$\begin{aligned} S^{\text{div.}}_{4,1} = \frac{i}{\varepsilon} \cdot \frac{1}{16\pi^2} \cdot \frac{1}{12f} \int d^4x \\ \times\Big\{ & 3\,\mathrm{Tr}\,(\Delta^2)\,\mathrm{Tr}\,(\Delta^\mu d_\mu \xi) + 6\,\mathrm{Tr}\,(\Delta^\mu\Delta^\nu)\,\mathrm{Tr}\,(\Delta_\mu d_\nu \xi) \\ & + \mathrm{Tr}\,(\Delta^2)\,\mathrm{Tr}\,(\xi\sigma_-) - \mathrm{Tr}\,(\sigma_+)\,\mathrm{Tr}\,(\Delta^\mu d_\mu \xi) \\ & + \tfrac{3}{2}\,\mathrm{Tr}\left(\{\Delta^2,\sigma_-\}\xi\right) - \tfrac{3}{2}\,\mathrm{Tr}\left(\{\sigma_+,\Delta^\mu\} d_\mu \xi\right) \\ & - \tfrac{11}{18}\,\mathrm{Tr}\,(\sigma_+)\,\mathrm{Tr}\,(\xi\sigma_-) - \tfrac{5}{12}\,\mathrm{Tr}\left(\{\sigma_+,\sigma_-\}\xi\right) \\ & + \tfrac{1}{2}\,\mathrm{Tr}\left([\Delta^\mu\Delta^\nu,\mathcal{F}^-_{\mu\nu}]\,\xi\right) - \tfrac{1}{2}\,\mathrm{Tr}\left([\Delta^\nu,\mathcal{F}^+_{\mu\nu}]\,d^\mu\xi\right) \\ & + \tfrac{1}{8}\,\mathrm{Tr}\left([\mathcal{F}^+_{\mu\nu},\mathcal{F}^{-\,\mu\nu}]\,\xi\right) \Big\} . \end{aligned} \tag{4.14}$$

On expanding the divergent part of the general chiral Lagrangean $\mathcal{L}_4$, given by Gasser and Leutwyler, to first order in ξ, one verifies that it is identical to (4.14). Neither any of the additional terms listed in Sect. 3.4 nor those proportional to L_3 and L_7 are required for divergence cancellation.

In view of the discussion of the counterterm structure given in Sect. 3, this result is a hint that at least the divergent parts of the "unphysical" counterterm Lagrangeans are simply related to the "physical" ones. It remains to be seen if this situation also prevails in $\mathcal{L}_{4,2}$ and beyond.

5. Conclusions and Outlook

The main result of this investigation is that the external-field formalism of ChPT is a consistent perturbative framework for imposing the constraints of chiral symmetry beyond the one-loop order. The ambiguities in applying dimensional regularization to $\varepsilon^{\mu\nu\rho\sigma}$ are circumvented in operator regularization; this method explicitly preserves the chiral symmetry while working in four dimensions. The original prescriptions by McKeon and Sherry have been adapted in order to explicitly exhibit the divergences and to facilitate the subtraction of subgraph divergences.

One question remains open in this paper: How are the counterterms of a given order in the momentum expansion, but with different numbers of fluctuation fields, related to each other? On general grounds, one has to admit the possibility that they may be unrelated. It is as yet unclear exactly which renormalization conditions have to be imposed on the proper vertex functions of the fluctuation fields; this problem did not occur in the one-loop calculations carried out thus far. It is conceivable that physically reasonable conditions provide enough constraints to link all $\mathcal{L}_{2k,n}$ unambiguously to $\mathcal{L}_{2k}$. The calculation described in the previous section likewise hints at the existence of strong relations among the counterterms of a given order in p.

Multi-loop calculations in ChPT will not become commonplace in view of the considerable computational effort required and of the multitude of additional couplings appearing at each successive order. Nevertheless, it is reassuring to know that ChPT is theoretically sound and that powerful all-order statements such as the nonrenormalization of the chiral anomaly can be deduced on this foundation.

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