SLAC-PUB-5197
February 1990
(T)

Charged Vortex and Duality in Three-Dimensional Abelian Gauge Theory with a Chern-Simons Term*

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Abstract

The duality property of the three-dimensional Abelian Higgs model with a Chern-Simons term is discussed. It is argued that in a certain limit the charged vortices in the original theory can be described by a dual gauge theory with also a Chern-Simons term. The duality relations between the couplings of the two theories are thus obtained. The relevance of this duality to the fractional quantized Hall effect, in particular, the Laughlin-Haldane-Halperin hierarchy, is discussed.

Submitted to Nuclear Physics B.

*Work supported by Department of Energy contract DE–AC03–76SF00515.

1. Introduction

Recent interest in the three-dimensional Abelian Higgs model with a Chern-Simons term mainly lies in its possible connection with condensed matter physics, in particular, the fractional quantum Hall effect (FQHE) and the high temperature superconductivity. It has been clear that the charged vortices in the model serve as a field theoretical realization for the anyons of Wilczek [1-4]. The charged vortices obey fractional statistics, have fractional charges, and hence should describe the quasiparticles and the quasiholes of the FQHE [5]. These properties have had a firm foundation due to the work of Frohlich and Marchetti [4]. On the other hand, the nonrelativistic version of the model has been analyzed and proposed to be the effective theory of the Landau-Ginzburg type for the FQHE [6], after the original suggestion made by Girvin [7].

Based on the work of Cardy [8], Girvin has suggested that the hierarchical states observed in the FQHE could arise as a consequence of the duality property of the theory. The duality and its possible relations to the FQHE have recently been studied by Shapere and Wilczek [9], who have pointed out a similarity for the hierarchical structure of the FQHE with that of the two-dimensional Z_N model.

It is well-known that the four-dimensional QED has the electric and magnetic duality property, when the Dirac monopoles are allowed. Cardy has shown that the duality exists in four-dimensional Abelian gauge theories and two-dimensional spin theories with a theta term on the lattice [8]. It is also well-known that in three dimensions the electromagnetic self-duality of the kind mentioned above does not exists, because the monopoles in three dimensions are actually instantons (see, e.g., Ref. [10]). A recent attempt to study the duality in three dimensions has been made [11]. However, it failed to produce the relevant hierarchical structure.

The problem of duality in three-dimensional field theories is restudied in this paper. It is shown in the present work that the idea of duality can be actually realized in the three-dimensional Abelian Higgs model with a Chern-Simons term included, although it is only valid in a certain limit. Its possible relations to the FQHE is also discussed. It should be pointed out that the duality property and the phase structure in three-dimensional Abelian gauge theory has been studied by Peskin [8].

Our study is also motivated by the following analysis made by Laughlin [12]. According to Laughlin, in the "fractional statistics" representation the quasiparticles of the FQHE have exactly the same equations of motion as those of the electrons in the original theory. This observation suggests a possible self-duality for any field theory which is supposed to describe the FQHE. A useful analogy, in this respect, can be made with the self-duality of the (1+1)-dimensional Ising model [13], in which the kink variables are governed by the same dynamics as the original-Ising-spin variables. It has been clear, that the charged vortices in the three-dimensional Abelian Higgs Chern-Simons (AHCS) theory describe anyons, or quasiparticles in condensed matter physics. It is then natural to expect that a dual theory to the AHCS theory should exist with the necessary structure to produce the same dynamics for the anyons as that of the original theory. It turns out that this self-dual theory *does* exist in a certain limit.

This paper is organized as follows. In the next section, an approximate local field theory is constructed to describe the interactions of the charged vortices of the model. In the third section, we show that in certain limit, a duality exists for the local field theory of the vortices and the original theory. It is argued that in three dimensions the dual object of the electric charge is the vortex, rather than the magnetic monopole. The dual relations between the couplings of the dual and the original theories are thus obtained. In the fourth section, we discuss the possible relevance of the duality property to the Laughlin-Haldane-Halperin hierarchy of the FQHE. In the last section we discuss our results and draw our conclusions.

2. Local Field Theory of Charged Vortices

In this section a local field theory, which describes the interactions of the charged vortices-in the Abelian Higgs Chern-Simons theory, is constructed. Since the vortices are extended objects [2-4,16], our construction is approximate, and should be valid in the local field limit.

Our construction closely follows that of Refs. [14,15]. The starting theory has its Lagrangian given by

$$\mathcal{L} = -\frac{1}{4e^2}\, F_{\mu\nu}\, F^{\mu\nu} + \frac{\theta}{4\pi^2}\, \varepsilon^{\mu\nu\rho} A_\mu\, \partial_\nu A_\rho + |D_\mu \phi|^2 - V(\,|\phi|) \quad , \tag{1}$$

where $D_\mu\phi = (\partial_\mu + iA_\mu)\,\phi$, and $V(|\phi|) = (\lambda^2/4)\left(|\phi|^2 - v^2\right)^2$ We use the metric $\eta_{\mu\nu}$ = (1, -1, -1). The generating functional is written

$$Z[J] = \int DA\, D\phi\, \exp\left(i \int d^3x \left[\mathcal{L}(x) + J_{\mu\nu}(x)\, F^{\mu\nu}(x)\right]\right) \delta\big(F_g(A,\phi) - c\big) \quad , \tag{2}$$

where $F_g(\,A,\phi)$ is a gauge function. We will only consider the case in the Higgs phase in the following, the symmetric phase corresponds to v = 0. The vortex solution to the field equations is characterized by the asymptotic behavior of the complex scalar ϕ: $\phi \rightarrow ve^{in\Theta}$, where n is the winding number. To proceed further we separate out the vortex configurations from the fields, which can be accomplished by writing

$$\begin{aligned} \phi \rightarrow \phi' &= e^{-in\Theta}\phi\,, & A, \rightarrow A'_\mu &= A, + \text{AA},\,, \\ \Delta A_i &= -n\varepsilon_{ij}\,\frac{x^j}{r^2}\,(1 - e^{-\beta r^2})\,, & \Delta A_0 &= \frac{n\theta e^2}{2}\,\beta r^2 e^{-\beta r^2}\,, \end{aligned} \tag{3}$$

where the smearing factor $e^{-\beta r^2}$ is used to simulate the asymptotic behavior of the vortex field at the spatial infinity and at the origin. The local limit is obtained in the end of the calculations by setting $\beta \to \infty$. Hence $\Delta F_{12} = 2\pi n\ \delta^2_\beta(x)$, with $\delta^2_\beta(x) = (\beta/\pi)\ e^{-\beta r^2} \to \delta^2(x)$ as $\beta \to \infty$. We will suppress the smearing factors in the following, whenever no confusion is caused, to simplify our notations. For arbitrary trajectories we have a covariant form

$$\Delta F_{\mu\nu} = 2\pi n \varepsilon_{\mu\nu\rho} \int d\tau \dot{\overline{x}}^\rho(\tau_m)\, \delta^3\big(x - \overline{x}_m(\tau_m)\big) \quad . \tag{4}$$

In the following we will consider the case n = 1 only, the case n = -1 corresponds to backtracing of the trajectory; and will assume that the higher winding number n can be formed by coalescence of the fundamental ones. The generalization of Eq. (4) to several vortices is

$$\Delta F_{\mu\nu} = \sum_{m=0}^{N} 2\pi \varepsilon_{\mu\nu\rho} \int d\tau \dot{\overline{x}}^\rho_m(\tau_m)\ \ \delta^3\big(x\ - \overline{x}_m(\tau_m)\big)\ , \tag{5}$$

where $\overline{x}_m(\tau_m)'s$ are the locations of the vortices. In the form of Eq. (3) the generating functional becomes

$$\begin{aligned} Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \int DA\ \ \overline{D}\phi\ \ DB\ \ DG \prod_{m=1}^{N} D\overline{x}_m \\ \times \exp\Big(i \int d^3x \big[J^{\mu\nu}F_{\mu\nu} + \mathcal{L}(x;\overline{x}_1,\cdot\cdot\cdot,\overline{x}_N)\big]\Big)\, \delta\big(F_g(A,\phi) - c\big) \quad , \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathcal{L}(x;\overline{x}_1,\cdots,\overline{x}_N) =& \mathcal{L}_B(x) \\ & + \frac{\theta}{8\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu(x)\,\varepsilon_{\nu\lambda\rho}2\pi \sum_{m=1}^{N} \int d\tau_m \dot{\overline{x}}^\rho_m(\tau_m)\ \ \delta^3\ \ (\mathrm{X} - \overline{x}_m(\tau_m)) \\ & - G^{\mu\nu}\Big(2\partial_\mu B_\nu - 2\pi \sum_{m=1}^{N} \varepsilon_{\mu\nu\lambda} \int d\tau_m \dot{\overline{x}}^\lambda_m(\tau_m)\ \ \delta^3\big(x - \overline{x}_m(\tau_m)\big)\Big)\ , \end{aligned} \tag{7}$$

and

$$\mathcal{L}_B(x) = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \left|(\partial_\mu + iA_\mu + iB_\mu)\phi\right|^2 - \frac{\lambda^2}{4} \left(|\phi|^2 - v^2\right)^2 \quad , \tag{8}$$

where $G_{\mu\nu} = -G_{\nu\mu}$, and the ϕ integration is performed for fields with no solitons; the $G_{\mu\nu}$ is a Lagrangian multiplier, and the B_μ is the gauge potential of the classical vortex. Note that the functional integration over the gauge fields has been explicitly split into two orthogonal parts for A, and B_μ respectively. The second line in Eq. (7) describes the coupling of the gauge field A, of the original theory with the effective current of the classical vortex described in terms of the vortex trajectory which carries a coefficient determined from the Gauss law constraint [9]. It is physically plausible that the mass M^2 does not vanish even in the local limit. Using Feynman's equivalence theorem (see, e.g., [14]) we introduce a scalar field χ to describe the vortices in the point-limit. Integrating over the $\overline{x}_m$'s, we arrive at

$$Z[J] = \int DA\, \overline{D}\phi\, DG\, DB\, D\chi \exp\left(i \int d^3x \left[J^{\mu\nu}F_{\mu\nu} + \mathcal{L}^{(\chi)}(x)\right]\right) \delta\left(F_g(A,\phi) - c\right) \quad ,$$

where

$$\mathcal{L}^{(\chi)} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - 2G^{\mu\nu}(\partial_\mu B_\nu) + |\partial_\mu\phi + i(A_\mu + B_\mu)\phi|^2 - \frac{\lambda^2}{4}(|\phi|^2 - v^2)^2 + \left|\partial_\mu\chi + 2\pi i\varepsilon_{\mu\nu\lambda}\left(G^{\nu\lambda} + \frac{\theta}{8\pi^2}\varepsilon^{\nu\lambda\rho}A_\rho\right)\chi\right|^2 + M^2|\chi|^2 \quad , \tag{9}$$

where χ is a complex scalar field. All the terms which vanish in the local limit have been dropped from the Eq. (9). Note that in obtaining Eq. (9) we had only considered the coupling of the vortices with the dual "gauge field." The self-interactions of the vortices have been ignored due to the lack of knowledge. The mass term of the x-field is put in by hand.

Now we define $V_\mu = A, + B_\mu$ and take the physical gauge Im $\phi = 0$, and write Re $\phi = (1/\sqrt{2})\,\phi_r + v$, the above Lagrangian can be written as

$$\begin{aligned}\mathcal{L}^{(\chi)} = &-\frac{1}{4e^2}\,F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2}\,\varepsilon^{\mu\nu\lambda}\,A,\,\partial_\nu A_\lambda + \frac{1}{2}\,(\partial_\mu\phi_r)^2 - \frac{\lambda^2}{4}\left(\frac{{\phi_r}^2}{2} + \frac{M_V}{e}\phi_r\right)^2 \\ &+ \frac{{M_V}^2}{2e^2}\left(1 + \frac{e}{M_V}\phi_r\right)^2 V_\mu V^\mu - G^{\mu\nu}\left(\partial_\mu V_\nu - \partial_\nu V_\mu - F_{\mu\nu}\right) \\ &+ \left|\partial_\mu\chi + 2\pi i\varepsilon_{\mu\nu\lambda}\left(G^{\nu\lambda} + \frac{\theta}{8\pi^2}\,\varepsilon^{\nu\lambda\tau}A_\tau\right)\chi\right|^2 + M^2\,|\chi|^2\ ,\end{aligned} \tag{10}$$

where $M_V^2 = e^2v^2$. The functional integration over V_μ is Gaussian, and can be performed to obtain

$$Z[J] = \int DA\ D\phi_r\ D\chi\ DW\ \exp\left(i\int d^3x\left[J_{\mu\nu}F^{\mu\nu} + \mathcal{L}_W(x)\right]\right)\ , \tag{11}$$

where

$$\begin{aligned}\mathcal{L}_W = &-\frac{1}{4e^2}\left(1 + \frac{e^4\theta^2}{16\pi^4M_V^2}\left(1 + \frac{e}{M_V}\phi_r\right)^{-2}\right)F_{\mu\nu}F^{\mu\nu} \\ &- \frac{e^2}{16\pi^2M_V^2}\left(1 + \frac{e}{M_V}\phi_r\right)^{-2} W_{\mu\nu}W^{\mu\nu} \\ &+ F^{\mu\nu}\left(J_{\mu\nu} + \frac{1}{4\pi}\,\varepsilon_{\mu\nu\lambda}W^\lambda + \frac{e^2\theta}{16\pi^3M_V^2}\left(1 + \frac{e}{M_V}\phi_r\right)^{-2} W_{\mu\nu}\right) \\ &+ \frac{1}{2}\,(\partial_\mu\phi_r)^2 - \frac{\lambda^2}{4}\left(\frac{{\phi_r}^2}{2} + \frac{M_V}{e}\phi_r\right)^2 + |\partial_\mu\chi + iW_\mu\chi|^2 + M^2\,|\chi|^2\ ,\end{aligned} \tag{12}$$

and

$$W_\mu = 2\pi\varepsilon_{\mu\nu\lambda}\left(G^{\nu\lambda} + \frac{\theta}{8\pi^2}\,\varepsilon^{\nu\lambda\tau}A_\tau\right)\ . \tag{13}$$

The functional integration over the $F_{\mu\nu}$ can now be performed in the gauge

$$\varepsilon^{\mu\nu\lambda}\partial_\lambda\left(\varepsilon_{\mu\nu\tau}W^\tau + \frac{\theta e^2}{4\pi^2M_V^2}\left(1 + \frac{e}{M_V}\phi_r\right)^{-2} W_{\mu\nu}\right) = 0\ .$$

We finally have the Lagrangian which describes the gauge interactions of the vortices.

$$\mathcal{L}_{dual} = -\frac{\pi^2}{4e^2A}\, W_{\mu\nu}W^{\mu\nu} + \frac{\theta}{4A}\,\varepsilon^{\mu\nu\lambda}W_\mu\,\partial_\nu W_\lambda + \frac{\pi^2 M_V^2}{2e^2A}\left(1+\frac{e}{M_V}\phi_r\right)^2 W_\mu W^\mu$$
$$+\frac{1}{2}\,(\partial_\mu\phi_r)^2 - \frac{\lambda^2}{4}\left(\frac{{\phi_r}^2}{2}+\frac{M_V}{e}\phi_r\right)^2 + |\partial_\mu\chi + iW_\mu\chi|^2 + M^2\,|\chi|^2 + \dots\ , \tag{14}$$

where

$$A = \left(1+\frac{e}{M_V}\phi_r\right)^2\left(\frac{2\pi^2 M_V}{e^2}\right)^2 + \left(\frac{\theta}{2}\right)^2 ,$$

and the dots represent the source terms, and the gauge condition is $\partial_\mu W^\mu = 0$.

In the above we have taken a set of specific smearing factors and have assumed that the mass of the vortex does not vanish even in the local limit. This problem is studied in Ref. [15] for the case $\theta = 0$. In the present context, due to the lack of knowledge of explicit solutions, we could only rely on physical plausibilities. As discussed in [6], the intrinsic problem with the Landau-Ginzburg approach involves the physics at distance scale which is comparable with the inverse mass of the vector bosons. Hence it is natural to expect that, by taking the local limit, one would not lose too much relevant physical information. There also seems no compelling reason which is strong enough to force a vanishing soliton mass in that limit.

It is clear that, at distances far away from the vortex core, the ϕ_r field can be effectively frozen. Thus the Lagrangian of Eq. (14) can be approximated by

$$\mathcal{L}_{dual} = -\frac{1}{4e'^2}\, W_{\mu\nu}W^{\mu\nu} + \frac{\theta'}{4\pi^2}\,\varepsilon^{\mu\nu\lambda}W_\mu\,\partial_\nu W_\lambda + \frac{{M_V}'^2}{2}\, W_\mu W^\mu$$
$$+|\partial_\mu\chi + iW_\mu\chi|^2 + M^2\,|\chi|^2 + \dots\ , \tag{15}$$

where we have redefined ${M_V}'^2 = M_V^2/e'^2$. This Lagrangian describes the gauge interactions of the charged vortices in the original theory, which is described by the Lagrangian in Eq. (1). Note that Eq. (15) is already written in a gauge-fixed form (with the gauge condition $\partial_\mu W^\mu = 0$), and it is gauge invariant under $W_\mu \rightarrow W_\mu + \partial_\mu \Lambda$ with $\partial_\mu \partial^\mu \Lambda = 0$. We now turn to the discussions for the relations between the two theories.

3. Duality in Three-Dimensional Abelian Gauge Theories

In this section we discuss a dual relation between the two theories. In order to compare the two Lagrangians given by Eqs. (1) and (15), we rewrite Eq. (1) in the following parametrization of the Heisenberg fields

$$\phi \rightarrow e^{i\Theta} \left(v + \frac{1}{\sqrt{2}} \phi_r \right) \quad , \qquad A_\mu \rightarrow \partial_\mu \Theta + A_\mu \quad . \tag{16}$$

The Lagrangian is then described, in terms of the new fields, by

$$\begin{aligned} \mathcal{L} = &- \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{1}{2} M_V^2 \left(1 + \frac{1}{M_V} \phi_r \right)^2 A_\mu A^\mu \\ &+ \frac{1}{2} \partial_\mu \phi_r \partial^\mu \phi_r - \frac{\lambda^2}{4} {\phi_r}^2 \left(\frac{\phi_r}{2} + M_V \right)^2 \quad , \end{aligned} \tag{17}$$

with $M_V^2 = v^2$. The Lagrangian has the similar appearance as that of Eq. (15).

We make the following remarks concerning the above discussion. Firstly, the scalar field part of the vortex configuration is given asymptotically in Eq. (3) plus some exponentially damping term. At distances far away from the vortex core the latter term should be negligibly small. Thus, as a first approximation, one can take $\phi_r \approx 0$ in Eq. (17) in the large distance, or equivalently, the local limit. Note that the U(1) current is given, in the parametrization of Eq. (17), by $J_\mu = (M_V + \phi_r)^2 A_\mu$. Thus in approaching the local limit, one approximates the current by $J_\mu \approx M_V^2 A_\mu$, which is the approximation we will use repeatedly in the following discussion. Secondly, the dual

Lagrangian, as given by Eq. (15), has been obtained under the point approximation for the charged vortices, while the latter are clearly extended objects. We should stress that we do not have any knowledge about the self-interaction of the vortices. A physically plausible conjecture is that the dual theory has a similar structure as the the original one, and a Lagrangian similar to Eq. (17) could be obtained for the dual theory.

The self-duality property of the theory concerns the relationship of the coupling parameters in the two theories. The relevant parameters are contained in the gauge fields parts in their Lagrangian, given by

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{1}{2} M_V^2 A_\mu A^\mu , \tag{18}$$

and

$$\mathcal{L}_{dual} = -\frac{1}{4e'^2} W_{\mu\nu}W^{\mu\nu} + \frac{\theta'}{4\pi^2} \varepsilon^{\mu\nu\lambda} W_\mu \partial_\nu W_\lambda + \frac{1}{2} M_V'^2 W_\mu W^\mu , \tag{19}$$

where

$$e'^2 = \frac{e^2}{\pi^2} \left[\left(\frac{2\pi^2 M_V}{e} \right)^2 + \left(\frac{\theta}{2} \right)^2 \right] , \tag{20}$$

$$\theta' = \pi^2 \theta \left[\left(\frac{2\pi^2 M_V}{e} \right)^2 + \left(\frac{\theta}{2} \right)^2 \right]^{-1} , \tag{21}$$

and

$$M_V'^2 = \pi^2 M_V^2 \left[\left(\frac{2\pi^2 M_V}{e} \right)^2 + \left(\frac{\theta}{2} \right)^2 \right]^{-1} . \tag{22}$$

It follows that the dual form of the dimensionless ratio of e/M_V is given by

$$\frac{e'}{M_V'} = \frac{e}{\pi^2 M_V} \left[\left(\frac{2\pi^2 M_V}{e} \right)^2 + \left(\frac{\theta}{2} \right)^2 \right] . \tag{23}$$

It should also be noted that the following products are equal in the two theories.

$$e'^2\theta' = e^2\theta \ , \qquad e'^2 M_V'^2 = e^2 M_V^2 \quad .$$

Thus the mass parameters in the two theories are also the same.

It is clear that the Eqs. **(18)** and (19) lead to the same equations of motion and canonical commutation relations. It is then expected that the same dynamics would produce the same physical spectrum. Hence this formulation of the three-dimensional Abelian Higgs Chern-Simons theory is self-dual. The dual relations between the couplings of the theory are given by Eqs. (20-23). This self-duality is only an approximate property of the theory, which should be exact only in the local limit.

In the absence of the Chern-Simons term, the dual relation of Eq. (23) reads

$$\frac{e'}{2\pi M_V'} = \frac{2\pi M_V}{e} \ .$$

That is, the weak coupling regime of the original theory is mapped into the strong coupling regime, and vice versa. This fact is characteristic of the self-duality for the systems mentioned in the Introduction. In the presence of the theta terms, the self-duality takes the form given by Cardy and Rabinowicz in Ref. [8]. The dual relationship of Eqs. (20-23) can be reexpressed by defining a complex parameter ζ

$$\zeta = \frac{2\pi M_V}{e} + i\frac{\theta}{2\pi} \ , \tag{24}$$

and by defining the generalized duality transformation

$$DT \ : \ \zeta \rightarrow \left(\zeta^{-1}\right)^* \ . \tag{25}$$

where D is the duality transformation defined by D: $\zeta \rightarrow \zeta^{-1}$, and T is the time reversal transformation, which is not a symmetry of (18), neither of (19). Note that $D^2 = (DT)^2 = 1$.

However the shift in the θ is generally not a symmetry of the theory under consideration, since it changes the physical spectrum. In fact, we do *not* need a symmetry in relating the model to FQHE, although in the next section we will point out an approximate periodicity in the θ, thereby to partially recover the $SL(2, Z)$ symmetry of Refs. [8,9].

From the construction given above it is clear that it is possible to build an entire tower of theories, in which each theory describes the interactions of the vortex excitations of the closest previous theory in the tower, except for the lowest one. However, the tower only possesses two sets of couplings related by the Eqs. (20-23). The experimentally observed hierarchical states of the FQHE are interpreted by Haldane as a sequence of condensations of the quasiparticles [17]. The possible relevance of the self-duality discussed above to the Laughlin-Haldane-Halperin hierarchy will be discussed in the next section.

It is useful to have the explicit correspondence between the two fields A, and W_μ in the Eqs. (18,19). We write the field W_μ in terms of the A, field by requiring that they satisfy the same equations of motion and the same equal-time commutation relations (or classical Poisson bracket). In the following we will treat the problem classically, and hence the Poisson bracket will be always used. Let us assume the dual gauge field W_μ can be formed by linear combinations of the gauge field and the topological current in the original theory. It is obvious that the W_μ defined by

$$W_\mu = \frac{2\pi}{e^2}\, \varepsilon_{\mu\nu\lambda}\, \partial^\nu A^\lambda + aA_\mu \,, \tag{26}$$

satisfies the same equations of motion as that of the A, for any choice of a. By imposing the requirement on the Poisson bracket, we find that

$$a = -\frac{e}{2\pi} \ , \tag{27}$$

where we have used the equations of motion derived from Eq. (18), in rewriting the canonical momentum Π^i_{dual},

$$\Pi^i_{dual} = -\frac{1}{e'^2}\, W^{0i} + \frac{\theta'}{4\pi^2}\, \varepsilon^{ij} W_j = -\frac{1}{2\pi}\, \varepsilon^{ij} A_j \ , \tag{28}$$

for the choice of a given in Eq. (27). It is clear that the dual gauge field W_i is proportional to the canonical momentum of the original theory, and the dual canonical momentum is proportional to the original gauge field A_i. We mention that this is not a derivation, but only a convenient representation for the dual fields. Inversely, we have

$$A, \ = \frac{2\pi}{e'^2}\, \varepsilon_{\mu\nu\lambda}\, \partial^\nu\, W^\lambda - \frac{\theta'}{2\pi}\, W_\mu \ ,$$

and

$$\Pi^i \ = -\frac{1}{e^2}\, F^{0i} \ + \ \frac{\theta}{4\pi^2}\, \varepsilon^{ij} A_j = -\frac{1}{2\pi}\, \varepsilon^{ij} W_j \ \ .$$

As mentioned in the introduction the dual object of the U(1) current should be the topological current both in the original theory, while the latter, in turn, is described by the-U(l) current in the dual theory. Using equations of motion and the dual relation Eq. (26), it can be shown that

$$J_\mu^{dual} = \frac{1}{4\pi}\,\varepsilon_{\mu\lambda\tau}\ \mathbf{w}^{\lambda\tau} = \frac{1}{2}\,j_\mu = j_\mu^{eff} \quad .$$

where J_μ and j_μ are the topological and the **U(1)** currents respectively; the j^{eff} is the effective U(**1)** current, which is equal to one half of the corresponding **U(1)** current [9], in the strong coupling limit ($e^2 \to \infty$).

In this section we have discussed the dual relations between the two theories of Eqs. (18,19), which is valid in the local limit. In order to study the possible relevance of this gauge system to the FQHE, along the lines suggested by Girvin [7], we have to consider some extreme cases which are discussed in the next section.

4. Relations to the FQHE

It is clear from the above discussions that there is a major difference between the theory under consideration and the theories studied by Cardy and Rabinowicz: we do not, in general, have the periodicity of θ in three dimensional theories. This is simply due to the following fact. The θ-terms in four- and two-dimensions are surface terms, hence the shift in θ does not change the equations of motion, and physical quantities are periodic in θ. In three dimensions, however, the B-term is not a surface term and it appears in the equations of motion (more precisely, it appears in the Gauss law constraint). In particular, we do not know the O-dependence of the physical spectrum. This could be a difficulty to accomplish the (2+1)-dimensional analog of the oblique confinement for the FQHE hierarchy suggested by Girvin. We will only study two extreme cases in the following, and will point out a possible way to improve our understanding of this problem.

It is interesting to notice that the situation can be partially improved by considering the case when M_V is large and e is small. In this case one has a weak dependence on θ, for small θ. The duality transformation reads

$$\frac{e'}{2\pi M_V'} = \frac{2\pi M_V}{e} \quad ,$$

Thus one could formally proceed according to Cardy. The approximate periodicity is defined by $\theta \rightarrow \theta' = \theta + 2\pi$, and which is described by

$$\text{A} \ : \quad \zeta \rightarrow \zeta + i \ . \tag{29}$$

Since the symmetry operations DT and A do not commute, the general symmetry operation is of the following form

$$\Delta^{p_0} \ DT \ \Delta^{p_1} \ DT \ \Delta^{p_2} \ . \ . \ . \ DT \ \Delta^{p_N} \quad , \tag{30}$$

where p_i's are integers, and $p_i \neq$ Cl for i $\neq 0$, provided that θ is smaller than M_V and l/e. Under the transformation (30) the ζ' takes the following form

$$\zeta' = ip_0 + \cfrac{1}{ip_1 + \cfrac{1}{ip_2 + \cfrac{1}{\ddots \cfrac{1}{ip_N \quad + \zeta}}}} \quad . \tag{31}$$

This symmetry can be expressed in terms of a group $SL(2, Z)$ on the complex ζ plane, by redefining

$$\zeta = \frac{\theta}{2\pi} + i\frac{2\pi M_V}{e} \quad ,$$

$$D \ : \quad \zeta \rightarrow -1/\zeta \ ,$$

$$\text{A} \ : \quad \zeta \rightarrow \zeta + 1 \ .$$

and the group action

$$\zeta \rightarrow \frac{a\zeta + b}{c\zeta + d} \quad , \qquad ad - bc = 1 \quad ; \qquad a, b, c, d \ \ \epsilon Z \quad . \tag{32}$$

However, we do not have an infinite continued fraction, or the full group $SL(2, Z)$. The symmetry operation (30) [or (32)] terminates when the θ is comparable with M_V or 1/e. At this point the approximation breaks down. Hence we do not have the complete phase diagram given in Refs. [8,9] in our case.

In another extreme case when $M_V/e \rightarrow 0$, one has the duality in θ

$$DT: \ \frac{\theta'}{2\pi} = \frac{2\pi}{\theta} \quad . \tag{33}$$

The periodicity in the 2π-shift of θ is not valid in this case. Note, however, the mass gap in this limit is infinite.

It should be mentioned at this point that, even if the periodicity in θ is an exact symmetry, the full group $SL(2, Z)$ or the infinite continued fraction as given in Eq. (31) is still not quite what one actually wants. As pointed out in Ref. [9], what one really needs to obtain the Laughlin-Haldane-Halperin hierarchy is the subgroup $\Gamma_{1,2}$ of the $SL(2,$ Z), defined by

$$\zeta \rightarrow \frac{a\zeta + b}{c\zeta + d} \quad , \qquad ad - bc = 1 \quad ; \qquad a, b, c, d \ \ \epsilon Z \quad .$$

with even *ab* and *cd*, rather than Eq. (32).

On the other hand, one does *not* really need the shift in θ to be a symmetry operation, since the Laughlin-Haldane-Halperin hierarchical states are really different states.

Then for the case of Eq. (33), and proceed according

$$\left(\frac{\theta}{2\pi}\right)^{-1} = \nu^{-1} - 1 \quad ,$$

where ν is the filling fraction, one could obtain all the observed fractions in the lowest Landau level by starting with, say, $\theta = \pi$, along with few fractions with even denominators. The above relationship between the θ and the filling fraction ν is discussed in Refs. [9,18]. The following results might be interesting.

$$\begin{array}{llllll}
\frac{\theta}{2\pi} = \frac{1}{2} & \longrightarrow & & & & \nu = \frac{1}{3} \ , \\
 & \xrightarrow{DT} & \frac{\theta}{2\pi} = 2 & \longrightarrow & & \nu = \frac{2}{3} \ , \\
 & \xrightarrow{\Delta} & \frac{\theta}{2\pi} = \frac{3}{2} & \longrightarrow & & \nu = \frac{3}{5} \ , \\
 & \xrightarrow{\Delta} & \frac{\theta}{2\pi} = \frac{5}{2} & \xrightarrow{DT} & \frac{\theta'}{2\pi} = \frac{2}{3} & \longrightarrow \ \nu = \frac{2}{5} \ , \\
 & \xrightarrow{\Delta^2} & \frac{\theta}{2\pi} = \frac{}{5} & \xrightarrow{DT} & \frac{\theta'}{2\pi} = \frac{2}{} & \longrightarrow \ \nu = \frac{2}{7} \ .
\end{array} \tag{34}$$

However, the even denominator fractions are not consistent with the experiments.

The program initiated by Girvin in Ref. [7] is thus partially justified. The unpleasant aspect of this study is that the mass gap is infinite in the local limit.

It should be mentioned that a related difficulty has also been observed by Zhang, Hansson and Kivelson in Ref. [6], i.e., the difference between the profile and the creation energies of the quasiparticles and the quasiholes is not understood at this stage. This issue is apparently related to the physical spectrum of the theory, in particular, its θ-dependence.

At this point, we speculate on the possibility of enlarging the theory by embedding it into a larger one. It is known that the 2π-shift of the θ-angle is an exact symmetry in four dimensions. The θ-term in three dimensions is related to that in four dimensions.

If one could understand the process of this dimensional reduction physically, one might be able to draw some conclusions concerning the e-dependence of the physical quantities in three dimensions. One of the possibility is the following: there might be a one-to-one correspondence for the spectra in the different θ-sector, although one does not expect that they are exactly the same in three-dimension. If that would be the case, it is then possible to build a larger theory which includes all the shifted and unshifted θ-sectors. This idea would not be considered too unrealistic, if one notes that, before the dimensional reduction, this large set of theories originates from one single theory in four dimensions. One could then expect a phase diagram of the overall shape as that given by Cardy, but with some deformations. In general, the phase diagram in our case is expected to be more complicated.

It is believed that the difficulties and the possible way-outs, discussed above, provide a useful clue to the understanding of the Landau-Ginzburg approach to the FQHE.

5. Discussions

In this work we have constructed a local field theory which describes the interactions of charged vortices in the ordered phase of the three- dimensional Abelian Higgs Chern-Simons theory. The dual theory has the same appearance as that of the original theory, with their couplings related by the dual relations given in Eqs. (20-23). This self-duality is the type discussed by Cardy in Ref. [8] for gauge theories with θ-terms.

However, the periodicity of θ is not a symmetry in three dimensions, and thus the hierarchy obtained in Ref. [8] for four- and two-dimensions does not exist in three dimensions. The difficulties to implement Girvin's program in (2+1)-dimension have been focussed on the need for understanding of the physical spectrum, in particular, the d-dependence of the physical quantities. We have proposed a possibility to resolve this problem.

On the other hand, it has become apparent that the three-dimensional Abelian Higgs Chern-Simons theory has attracted tremendous attention in recent years. The relevance of its study to the FQHE and the high temperature superconductivity has recently been pursued by March-Russell and Wilczek [19], and by Wen, Wilczek and Zee [20]. It is believed that we have found the correct dual symmetry of the theory in this work. Better understanding of its physical spectrum and the θ dependence of physical quantities would certainly shed new lights on this exciting field. Further study in this direction is currently underway and will be reported elsewhere.

Acknowledgments

This work was started when the author worked in the ITP, Academia Sinica in Beijing. The author has benefitted very much from the talks presented there by A. Zee, R. Laughlin and many others. The author gratefully acknowledges the opportunity and hospitality provided by the theory group at SLAC, Prof. R. Blankenbecler, Prof. M. Peskin, Prof. H. Quinn, Prof. M. Weinstein, and all the others. Otherwise this work would have never been finished. The author thanks Prof. L. Susskind for numerous discussions which have clarified many ideas; he also thanks the referee for critical comments. The author also gratefully acknowledges the support by a grant from DOE, under the contract DE-AC03-76SF00515.

NOTES ADDED

-This work was submitted first in May 1989. Since then we have noticed that there appeared several preprints on the same problem of duality in three-dimensional gauge theory with a θ term. Notably, a preprint by S. Rey and A. Zee discussed the case of a compact U(1) group on the lattice, and a duality relation is obtained, which is the same as ours. However, it is not clear in their work what the dual theory describes. We have shown in this work that the dual theory precisely describes the vortices in the original theory, which is on line with Girvin's proposal.

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