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An Improved Method for Setting Upper Limits with Small Numbers of Events*

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ABSTRACT

We note that most experimental searches for rare phenomena actually measure the ratio of the number of event candidates to the number of some normalizing events. These measurements are most naturally interpreted within the framework of binomial or trinomial statistics. We present a general expression, based upon a classical treatment, that accounts for statistical normalization errors and incorporates expected background rates. The solutions of this expression converge to the standard Poisson values when then number of normalizing events is larger than a few hundred.

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1. Introduction

In the search for rare processes, many experiments observe small numbers of event candidates. These observations are often accompanied by an expectation of some number of events from background sources. The interpretation of this information is normally based upon a Poisson statistical treatment.[1,2] An upper limit on the number of observed events is calculated and compared with the number that is expected from some theoretical calculation. Normally, this comparison is used to constrain the parameter space of the theoretical model.

We would like to point out that most experiments actually measure the ***ratio*** of the number of event candidates to the number of a different type of (normalizing) event. Similarly, all theoretical models predict ***event rates*** which must be normalized to the rate of the known process. This subtle distinction becomes important when the number of normalizing events is not large. Indeed, one can find examples in the literature of attempts to account for the ***statistical error on the theoretical expectation*** (often by subtracting ***one*** *sigma* from the theoretical value). In the following sections, we first review the standard Poisson treatment of the problem. We then show how to generalize the solution to include ***normalization*** errors by the use of binomial and trinomial statistics.

2. Poisson Statistics

In dealing with discrete statistical distributions, one is forced to adopt a ***point of*** *view*.[3] We choose to adopt the so-called classical approach.

Let us consider the case that N events are observed in an experiment. For the moment, we assume that there are no events expected from any source of background. Within the context of the classical approach, the upper limit on the number of signal events of confidence level β (μ_s^β) is defined as the solution of the following equation,

$$\alpha \equiv 1 - \beta = \sum_{n=0}^{N} e^{-\mu_s} \frac{\mu_s^n}{n!}, \tag{1}$$

where: α is the tail probability that corresponds to the confidence level β; μ_s is mean of the Poisson distribution; and $\boldsymbol{n}$ is an index. Equation (1) is derived by equating the Poisson probability of observing N or fewer events with the tail probability. The interpretation is that μ_s^β is the value of μ_s such that N or fewer events would be observed with frequency α in many repetitions of the experiment. An alternative interpretation is that μ_s^β is the value of μ_s such that more than N events would be observed with frequency β in many repetitions of the experiment.

Equation (1) has recently been generalized by Zech[1] to incorporate the case that μ_b background events are expected. In this case, the observed number of events is the sum, $N = n_s + n_b$, where n_s is some number of signal events and n_b is some number of background events. We now have *a priori* information that n_b must be less than or equal to N. The right-hand side of equation (1) is therefore redefined to be the joint Poisson probability that the sum $n_s + n_b$ is less than or equal to N where the number of background events is constrained to the interval $[0, N]$.* These requirements are shown graphically in Figure 1. The number of signal events is plotted along the horizontal axis and the number of background events is plotted along the vertical axis. The probability content of the region $n_b \leq N$ is renormalized to unity. The region enclosed by the constraint $n_s + n_b \leq N$ is indicated by the dashed line. The upper limit μ_s^β is found by requiring that the probability content of the enclosed region be α. Mathematically, the upper limit is the solution of the equation,

$$\alpha = \frac{\sum_{n=0}^{N} e^{-(\mu_s+\mu_b)}(\mu_s + \mu_b)^n/n!}{\sum_{n=0}^{N} e^{-\mu_b}\mu_b^n/n!}. \tag{2}$$

Equation (2) was first derived from a Bayesian approach.[2] The agreement

⋆ Alternatively, we could require that β be equal to the probability that $n_s + n_b > N$ given that n_b is constrained to the interval [0, N].

between the two approaches is accidental and does not apply to the generalization that follows.[4]

3. The Inclusion of Normalization Errors

As was stated in the Introduction, essentially all experiments measure the ratio of the number of interesting events N to the number of normalizing events D. We assume that the categorization of events as N-type and D-type is exclusive. An event can fall into one category or the other but not into both (these categories might refer to different decay modes of the same parent particle or they can refer to completely different physical processes such as Z^0 production and small-angle Bhabha scattering).

3.1. Measurements Without Background

It was shown by James and Roos[5] that the statistical distribution of the ratio N/D can be derived from the binomially distributed quantity $N/(N + D)$. If r_s is the $(D \to \infty)$ asymptotic value of N/D, then the quantity $N/(N + D)$ has a binomial distribution that is characterized by the probability $p_s = r_s/(1 + r_s)$. To define an upper limit on p_s (and therefore on r_s), we lose no generality by fixing the binomial denominator to the sum $N + D$. The upper limit is then defined by requiring that the sum of the binomial probabilities for all ratios $n/(N + D) \leq N/(N + D)$ be the tail probability α. The upper limit, r_s^β, is therefore given by the solution of the following equation,

$$\alpha = \sum_{n=0}^{N} \frac{(N+D)!}{n!(N+D-n)!} \left(\frac{r_s}{1+r_s}\right)^n \left(1 - \frac{r_s}{1+r_s}\right)^{N+D-n}. \qquad (3)$$

Although equation (3) and equations 2-3 of Reference 5 appear to differ, they are completely equivalent.[6] We note that the product $r_s^\beta D$ converges to the Poisson value μ_s^β in the limit $D \gg N$. To show this explicitly, we make the replacement $r_s \rightarrow \mu_s/D$ in equation (3). Using the identity,

$$\lim_{D\to\infty}\left(1 - \frac{\mu_s}{D}\right)^D = e^{-\mu_s}, \tag{4}$$

it is straightforward to show that equation (3) converges to equation (1) in the limit of large D.

3.2. Measurements With Background

In the case that background events are expected, the binomial treatment must be modified. The problem now contains three types of events: signal events, background events, and normalizing events. It can therefore be described by a ***trinomial*** statistical distribution. We assume that the expected rate of background events is normalized to the rate of D-type events. If r_b is this ratio and r_s is ratio of signal events to D-type events, the trinomial distribution is described by the probability of measuring a signal event, $p_s = r_s/(1 + r_s + r_b)$, and the probability of measuring a background event, $p_b = r_b/(1 + r_s + r_b)$.

In analogy to the Poisson case, we define the tail probability α to be the sum of all trinomial probabilities for the ratios $n_s/(N + D)$ and $n_b/(N + D)$ such that the sum $(n_s + n_b)/(N + D)$ is less than or equal to $N/(N + D)$ where n_b is constrained to the interval $[0, N]$.* These requirements are shown graphically in Figure 2. The number of signal events is plotted along the horizontal axis and the number of background events is plotted along the vertical axis. The trinomial treatment requires that the sum of the numerators, $n_s + n_b$, be less than or equal to the total denominator $N + D$. This constraint is shown as the dotted line. The number of background events is constrained to the region below the solid line ($n_b \leq N$). The probability content of the allowed region is renormalized to unity. The region

* Alternatively, we could require that β be equal to the probability that $N/(N + D) < (n_s + n_b)/(N + D) \leq 1$ given that n_b is constrained to the interval [0, N].

enclosed by the constraint $n_s + n_b \leq N$ is indicated by the dashed line. The upper limit on r_s is found by requiring that the probability content of the enclosed region be α. It is straightforward to show that r_s^β is given by the solution of the following equation,

$$\alpha = \frac{\sum_{n_b=0}^{N} \sum_{n_s=0}^{N-n_b} \frac{(N+D)!}{n_s! n_b! (N+D-n_s-n_b)!} p_s^{n_s} p_b^{n_b} [1-(p_s+p_b)]^{N+D-n_s-n_b}}{\sum_{n_b=0}^{N} \frac{(N+D)!}{n_b!(N+D-n_b)!} p_b^{n_b} [1-p_b]^{N+D-n_b}}, \qquad (5)$$

where both p_s and p_b depend upon r_s.

Equation (5) is the trinomial analog of equation (2). If we make the replacements, $r_b \longrightarrow \mu_b/D$ and $r_s \longrightarrow \mu_s/D$, and then use equation (4), we can show that equation (5) converges to equation (2) in the limit $D \gg N$. Note that both equations reduce to their simpler forms (equations (1) and (3), respectively) when the number of observed events is zero. This reflects the fact that ***we*** have ***a priori*** knowledge that there are no background events in the sample.

3.3. Numerical Results

Equations (2) and (5) are straightforward to solve by numerical methods. In order to compare the trinomial solutions with the Poisson solutions, we examine the quantity $r_s^\beta D$. It must be stressed that it is more natural to use r_s^β to compare measurements with expectations and that $r_s^\beta D$ is used solely for the purpose of comparison. The $\beta = 0.90$ values of $r_s^\beta D$ are listed in Table I for several values of N, $\mu_b = r_b D$, and D . The Poisson solution ${}^{\beta}_{s}$ is listed in the last column. The $\beta = 0.95$ values are listed in Table II. Note that the $N = 0$ values of $r_s^\beta D$ and μ_s^β are independent of the background expectation.

For the range of N that is considered ($N \leq 5$), we see that the trinomial limits are larger than the Poisson limits when D is small. At $D = 100$, the difference is reduced to a few percent or less. It is clear that attempts to account for the normalization error within the Poisson framework by changing either μ_s^β or the theoretical expectation by the fraction $1/\sqrt{D}$ are incorrect.

4. Conclusions

We have noted that most experimental searches for rare phenomena actually measure the ratio of the number of event candidates to the number of some normalizing events. These measurements are most naturally interpreted within the framework of binomial or trinomial statistics. We have presented a general expression (equation (5)), based upon a classical treatment, that accounts for statistical normalization errors and incorporates expected background rates. The solutions of this expression converge to the standard Poisson values when then number of normalizing events is larger than a few hundred.

Although equation (5) is formally correct in all cases, it becomes sensitive to computer roundoff error when D becomes larger than a few hundred. Happily, the two approaches converge in this regime. It is important to note that the Poisson solutions are correct as they stand in this regime ($D \gtrsim 100$). Further attempts to correct for statistical normalization error are not only unnecessary but are incorrect.

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Table I

The $\beta = 0.90$ trinomial confidence limit $r_s^\beta D$ for several values of N, $\mu_b = r_b D$, and $\boldsymbol{D}$. The Poisson limit μ_s^β is included for comparison.

N	μ_b	D = 10	D = 50	D = 100	Poisson
0		2.59	2.36	2.33	2.30
1	0.0	4.50	4.00	3.95	3.89
1	0.5	4.09	3.62	3.56	3.51
1	1.0	3.82	3.37	3.32	3.27
2	0.0	6.27	5.50	5.44	5.32
2	1.0	5.35	4.61	4.53	4.43
2	2.0	4.67	4.22	3.95	3.88
3	0.0	7.99	6.93	6.79	6.68
3	1.5	6.54	5.52	5.40	5.29
3	3.0	5.41	4.55	4.46	4.36
4	0.0	9.68	8.32	8.16	7.99
4	2.0	7.71	6.39	6.24	6.09
4	4.0	6.09	5.01	4.89	4.78
5	0.0	11.35	9.67	9.47	9.27
5	2.5	8.87	7.23	7.04	6.85
5	5.0	6.73	5.43	5.29	5.15

Table II

The $\beta = 0.95$ trinomial confidence limit $r_s^\beta D$ for several values of N, $\mu_b = r_b D$, and D. The Poisson limit μ_s^β is included for comparison.

N	μ_b	D = 10	D = 50	D = 100	Poisson
0	—	3.49	3.09	3.04	3.00
1	0.0	5.73	4.93	4.84	4.74
1	0.5	5.32	4.53	4.45	4.36
1	1.0	5.02	4.28	4.20	4.11
2	0.0	7.80	6.57	6.44	6.30
2	1.0	6.86	5.67	5.54	5.41
2	2.0	6.13	5.06	4.94	4.82
3	0.0	9.79	8.13	7.95	7.75
3	1.5	8.32	6.71	6.53	6.36
3	3.0	7.11	5.70	5.54	5.40
4	0.0	11.74	9.63	9.39	9.15
4	2.0	9.76	7.70	7.47	7.24
4	4.0	8.04	6.26	6.07	5.89
5	0.0	13.67	11.10	10.81	10.51
5	2.5	11.18	8.65	8.37	8.09
5	5.0	8.92	6.77	6.54	6.33

REFERENCES

1. **G.** Zech, *Nucl.* ***Instr. and*** *Meth.* **A277**, 608 (1989).

2. O. Helene, *Nucl.* ***Instr. and*** *Meth.* **212**, 319 (1983).

3. Note that if we were able to use a continuous estimator to find our confidence interval, we could avoid the classical versus Bayesian controversy. In such cases, the Monte Carlo method can be used to generate intervals that are independent of assumptions about ***a priori*** distributions.

4. The difference between a particular Bayesian treatment of binomial confidence levels and the classical treatment is discussed in O. Helene, *Nucl.* ***Instr. and*** *Meth.* ***228,*** 120 (1984) and in F. James, *Nucl.* ***Instr. and*** *Meth.* **A240**, 203 (1985).

5. F. James and M. Roos, *Nucl.* ***Phys.*** **B172**, 475 (1980).

6. Equation (3) can be transformed into a one-sided version of equations 2 and 3 of Reference 5 in two steps. First, the complement of the right-hand side of equation (3) can be transformed into the Incomplete Beta function $I_p(N+1, D)$ (where $p = r_s/(1+r_s)$) using equation 26.5.24 of M. Abramowitz and I.A. Stegun, ***Handbook of*** *Mathematical* ***Functions,*** Dover Publications, Inc. (New York, 1964). The application of equation 26.6.2 from the same reference then converts the result into that given in Reference 5.

FIGURE CAPTIONS

1) The region of signal and background events that is included in the Poisson probability for N observed events. The number of signal events **(n,)** is plotted along the horizontal axis and the number of background events (n_b) is plotted along the vertical axis. The number of background events is constrained to the region below the horizontal solid line ($n_b \leq N$, $0 \leq n_s < \infty$). The total probability is renormalized to unity in the allowed region. The region incorporated in the sum $n_s + n_b \leq N$ is indicated by the dashed line. The upper limit μ_s^β is found by requiring that the probability content enclosed by the dashed line to be $\alpha = 1 - \beta$.

2) The region of signal and background events that is included in the trinomial probability for N observed events and $\boldsymbol{D}$ normalizing events. The number of signal events (n_s) is plotted along the horizontal axis and the number of background events (n_b) is plotted along the vertical axis. The trinomial treatment requires that the sum of the numerators, $n_s + n_b$, be less than or equal to the total denominator $N + \boldsymbol{D}$. This constraint is shown as the dotted line. The number of background events is constrained to the region below the solid line ($n_b \leq N$). The total probability is renormalized to unity in the allowed region. The region incorporated in the sum $n_s + n_b \leq \boldsymbol{N}$ is indicated by the dashed line. The upper limit on r_s is found by requiring that the probability content enclosed by the dashed line to be $\alpha = 1 - \beta$.

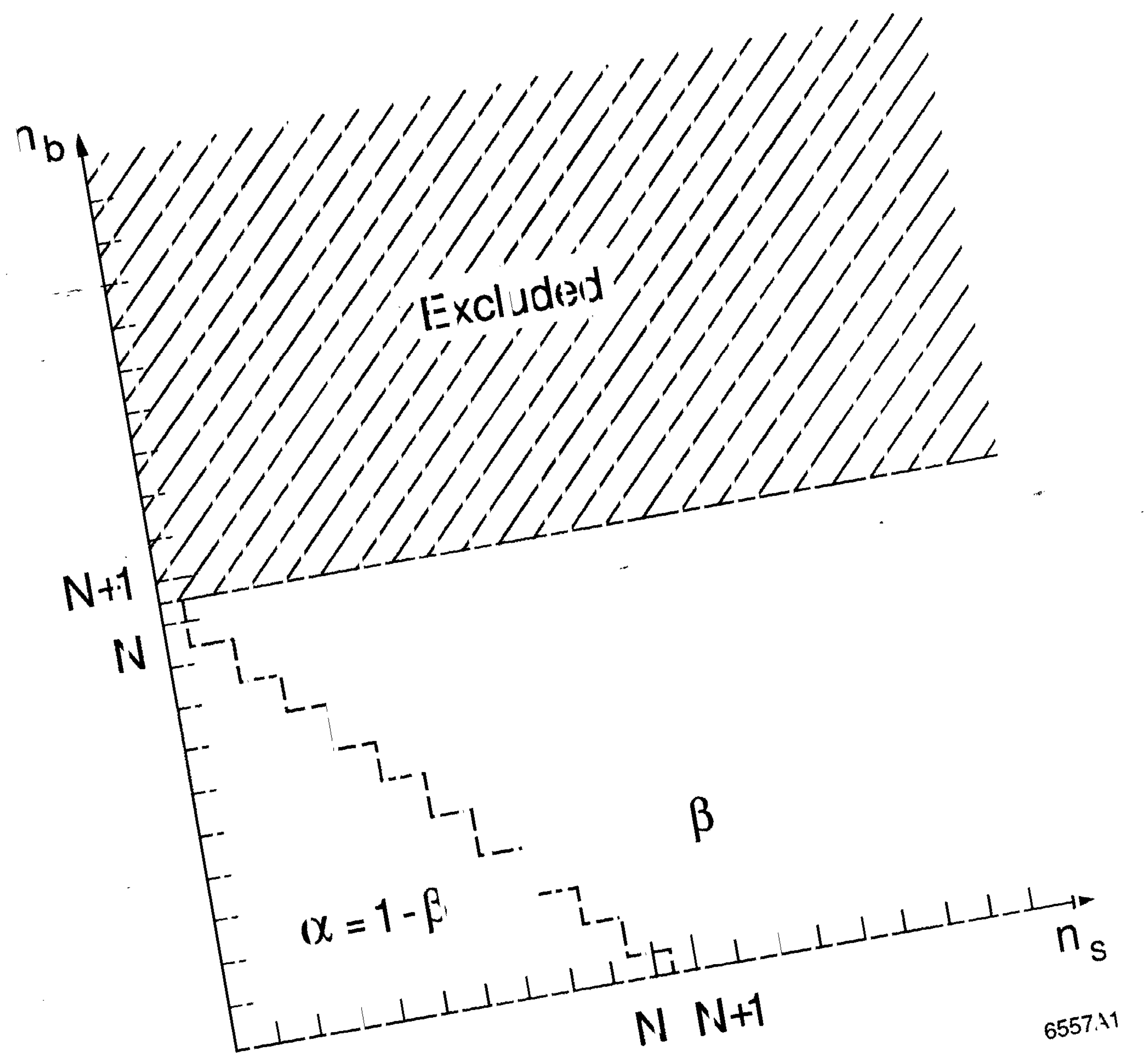

Fig. 1

1-90

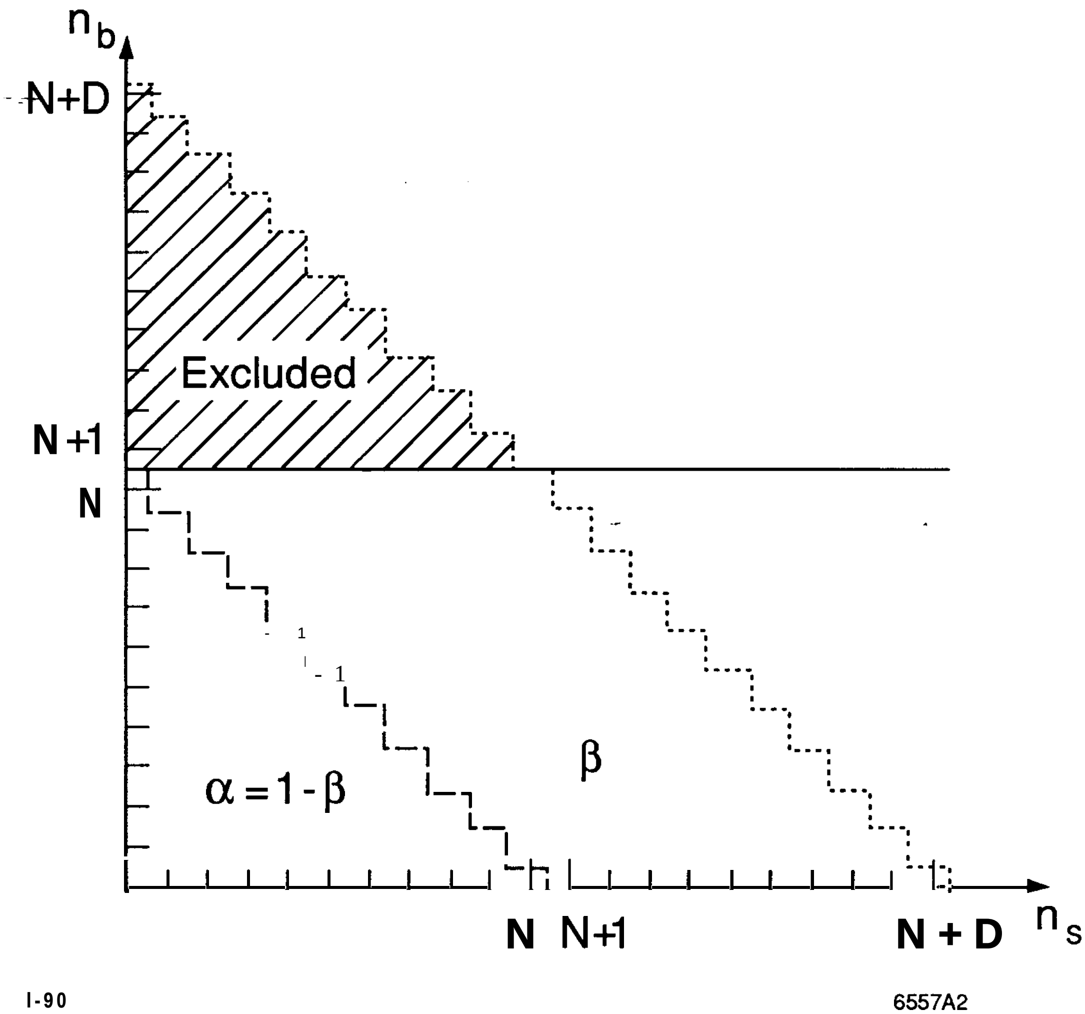

Fig. 2