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# EXTENDED QUANTUM KINEMATICS AND A POSSIBLE ORIGIN OF INTERNAL-SYMMETRY GROUPS*

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## ABSTRACT

In order to locate the origin of internal–symmetry groups, we study extended quantum kinematics for internal spaces over various mathematical fields, *i.e.* commutative division algebras. We consider two possibilities:

(1) that internal symmetries—such as isospin symmetries—are kinematical isometries of underlying spaces which can depend on time; or

(2) that such internal symmetries are involutions (such as $C$, $P$, and $T$) or slight generalizations of them.

By inspecting several realizations of Extended Commutation Relations (ECR), we find that (1) is actually realized only on the non–commutative division algebras while internal symmetries in (2) are interpreted as Galois groups for appropriate field–extensions of a base field. A unique feature of our formalism is the appearance of a set of new quantum numbers, which characterizes similarity classes of simple algebras and are numerically equal to $\exp[2\pi i(m/n)]$ with relatively prime rational integers $m$ and $n$ ($0 \leq m < n$). The dynamical calculations based on ECR are briefly discussed.

## I. INTRODUCTION

If one asks basic questions in particle physics such as "How many neutrino–species are there in the universe?" or "What is the theoretical explanation for the value of $m_\mu/m_e$?" and so on, it would soon be realized that the traditional dynamical theories like quantum field theories are quite useless in answering these questions unless they are equipped with some *ad hoc* assumptions. This is obviously because the nature of problems is unprecedented from the conventional standpoint and thus most of the existing dynamical theories are practically neutral to them.

One heuristic approach to these problems would be to put aside all the useful yet unconfirmed hypotheses for a while (however useful they may be for phenomenological analyses) and then try to understand the consequences of a minimal number of assumptions, which are taken as weakly as possible. This strategy should be followed to its extremities as long as it is consistent with firmly established principles such as various conservation laws. The next step would be to determine an appropriate framework to answer the basic questions posed earlier in this paper, rather than quickly reproducing the existing data by employing widely available methods.

Among many alternatives, we have chosen the symmetry rule as our guiding principle. Keeping this in mind, let us consider basic internal symmetries such as isospin symmetry and $C, P$, and $T$. New approaches to these problems are presented. As will be shown, it would be nothing but a reconstruction of ordinary quantum mechanics, with the only difference being that the underlying space and its groups of automorphisms are now slightly generalized to incorporate internal–symmetry spaces in it.

In a previous work,[1] we considered discrete isospin groups and identified the hierarchy structures between various observed symmetries with the process of group extensions. Then the kinematics was introduced into internal space, which is entirely based on group concepts. The particular type of groups mentioned here has originated in the realization that the observed isospin symmetries can be appropriately described by finite subgroups[2] of $SU(2)$.

In this work, we proceed further along this line and will seek to find explicit representations for internal kinematics. Before attempting this, we have to be more specific about an isospin group, namely whether it is:

(Case A) a time–dependent isometry group in internal space just like the rotational or translational group in ordinary 3–space; or

(Case B) an involution like $C$, $P$, or $T$ or its suitable generalization.

As an example of the latter case, we note that the operation $C$ is not generally realizable as a physical process in the sense that electrons are not changed into positrons by emitting (absorbing) any known particles. In contrast, $e^-$ can go to $\nu_e$ by emitting $W^-$. So this may be classified under Case (A). Indeed, one can think that this is the very reason to put $\nu_e$ and $e_L$ into the same leptonic isomultiplet, as is familiar in electroweak theory. If Case (A) is valid, then isospin space is much the same as the ordinary 3–space except for an obvious lack of translational degree of freedom. The time–dependent transformation or rotation of vectors in it should actually take place.

We refer to Case (A) as the active type. According to conventional view, however, one cannot of course detect such rotations as far as isospin–symmetric interactions are concerned. This is because there is no way to determine a particular direction. Moreover, even in the presence of a large symmetry breaking, one usually assumes that the basic interaction is completely symmetric, to be broken only by "residual mechanism" for which one can choose a favorite one from various alternatives.

So, there is no need to speculate on possible internal rotations. This view is also included in Case (B) and will be referred to as the "passive" type. However, there would be no practical difference between "active" and "passive" types as long as one uses the concept only in a phenomenological way. We consider both types in the following, although our emphasis is on the active type because it seems more natural in explaining the origin of symmetry groups.

In Case (A), the kinematics can be introduced by identifying the automorphism groups of underlying space after a suitable topology is set up. Then one can establish ECR and a related algebra, which is subsequently decomposed into simple components, *i.e.* simple algebras.

In the case of ordinary Canonical Commutation Relations (CCR) in quantum mechanics, this process can be performed straightaway, and gives us a clear physical insight. As this example provides us with a unique opportunity to acquaint ourselves with internal kinematics, the main features of this formalism are recalled somewhat in detail, from a new standpoint together with ECR for the $SU(2)$ case. This is shown in Section II. If the internal symmetry is that of Case (A), then it turns out that the non-trivial incorporation of this symmetry as the kinematical isometry-group of internal space generally requires a formalism which is based on various number fields. This combination of ECR with general linear groups $GL(R)$ opens a perspective with a dazzling extent and richness. We will only scratch the surface of this discipline for our limited purposes.

In contrast, the consequence of finite internal symmetries in Case (B) seems to be more modest than that of Case (A). The relevant groups are most naturally interpreted as Galois groups for appropriate field extensions in this case. The principle adopted here for Case (B) is a sequence of split extensions of finite groups. Consequently, we first focus on Case (A) and consider ECR over various fields. This is the subject of Section III.

In Section IV, we consider Case (B) and discuss tensor algebras and then the Clifford algebra. These concepts are clearly needed in formulating kinematics in a closed form. The Clifford algebra is defined only when a quadratic form is given. It will be essential to obtain anticommutation relations. The introduction of the concept of "particles" will not be undertaken yet on the grounds that it seems to require additional assumptions about Hamiltonians or quadratic forms.

We would therefore be content with the description of general framework in this paper. The choice of underlying fields is also a highly non-trivial problem in both Cases (A) and (B), although our basic formulas such as ECR in Eq. (20) below are as simple as the corresponding quantum mechanical ones. Any realistic use of our formalism has to be left to subsequent works.

Finally, in Section V, we summarize the results and make a few remarks.

## II. ALGEBRA OF CANONICAL COMMUTATION RELATIONS AND ITS EXTENSION TO *SU(2)*

The CCR in quantum mechanics is interpreted as a statement about Euclidean translation and takes the familiar form $[q,p] = i\,(\hbar = 1)$ in its simplest form. Writing it in the Weyl form[3]:

$$\exp\left[i\sigma p\right]\ \exp\left[i\tau q\right]\ \exp\left[-i\sigma p\right] = \exp\left[i\tau\left(q+\sigma\right)\right] \quad , \tag{1}$$

where $\sigma$, $\tau$ are real parameters ($\sigma$, $\tau$, $\in$ $\boldsymbol{R}$); one can interpret $\exp[-i\sigma p]$ as an operator of the translation $q \longrightarrow q+\sigma$. As this elementary fact is one of the basic motivations for subsequent developments, let us recall the consequences of Eq. (1) a little further including less familiar results.

If we introduce an operator:

$$S\ (\sigma,\ \tau) = \exp\left[i\,(\sigma p + \tau q)\right] \quad , \tag{2}$$

which is defined by a power series expansion, then it is also written as:

$$\begin{aligned} S\ (\sigma,\tau) &= \exp\left[-\frac{i}{2}\ \sigma\tau\right]\ \exp\left[i\sigma p\right]\ \exp\left[i\tau q\right] \\ &= \exp\left[\frac{i}{2}\ \sigma\,\tau\right]\ \exp\left[i\tau q\right]\ \exp\left[i\sigma p\right] \quad . \end{aligned} \tag{3}$$

The multiplication rule for $S\ (\sigma,\tau)$ follows from Eq. (1):

$$S\,(\sigma,\tau)\ S\,(\sigma',\tau') \ = \ \exp\left[\frac{i}{2}(\sigma\tau' - \sigma'\tau)\right]\ S\,(\sigma+\sigma',\ \tau+\tau') \quad . \tag{4}$$

This may be regarded as a special case of tensor product of algebras, which is considered later. It is one of the most basic steps in obtaining new algebras. When a classical physical quantity is expressed as a function of $p$ and $q$ as:

$$f(p,q) = \int\!\!\int_{-\infty}^{\infty} d\sigma\, d\tau\ \exp\left[i\tau p\right]\ \exp\left[i\tau q\right]\ F(\sigma,\tau) \quad , \tag{5}$$

in terms of a coefficient function $F$, then the corresponding quantum mechanical quantity (operator) is obtained as:

$$f(p,q)_Q = \int\!\!\int d\sigma\, d\tau\ S\,(\sigma,\tau)\ F\,(\sigma,\tau) \tag{6}$$

by using the same function $F(\sigma,\tau)$. This is, of course, the well-known prescription of Weyl[3] (see also J. Schwinger, Ref. 4). One may regard the expression (6) as a linear combination of $S(\sigma, \tau)$ over all possible values of $\sigma$ and $\tau$. It should be stressed, however, that the following discussion is independent of this prescription which connects $f(p, q)$ with $f(p, q)_Q$.

Now, let us consider the set of all possible linear combinations of $S(\sigma, \tau)$ with coefficients from $\boldsymbol{C}$, the complex number field. Under rule (4), this set is closed under multiplication as well as addition, and constitutes an algebra, to be called Weyl algebra (WA). Obviously, Eq. (6) tells us that any quantum mechanical operator is an element of WA. The structure of WA in the present case is summarized as follows:

I. WA corresponding to Eq. (1) is a direct sum of a countable infinite number of simple algebras; and

II. Each simple algebra is one–dimensional and has its own generating idempotent; that is, a projection operator. The eigenfunction of this operator in Schröedinger representation ($p = -i\, d/dq$) is identical to a harmonic oscillator wave function in one dimension.

To be more precise, let us consider an element of WA defined by

$$E_0 = \frac{1}{2\pi} \int\int d\sigma\, d\tau\ \exp\left[-\frac{1}{4}\left(\sigma^2 + \tau^2\right)\right] S(\sigma, \tau) \quad . \tag{7}$$

This element was originally considered by von Neuman[5] and has a remarkable property:

$$\begin{aligned} &E_0^2 = E_0 \quad , \\ &E_0^\dagger \text{ (hermitian conjugate)} = \mathrm{E}_0 \neq 0 \quad . \end{aligned} \tag{8}$$

This is easily confirmed by using Eq. (4). We may construct all possible projection operators $E_n$ ($n = 0,\ 1,\ 2,\ \ldots$) for WA, so that they are mutually orthogonal and constitute a complete set, *i.e.*

$$E_n^\dagger = E_n \quad ,$$
$$E_n\, E_m = \delta_{n,m}\, E_n \quad , \tag{9}$$

and

$$\sum_{n=0}^{\infty} E_n = I \text{ (identity)} \quad . \tag{10}$$

Our result for $n > 0$ together with Eq. (7) is summarized by the formula[6,7]:

$$E_n = \frac{1}{2\pi} \int\int d\tau\, d\tau\, \xi_n(\alpha,\tau)\, S(\alpha,\tau) \quad , \tag{11}$$

where

$$\xi_n = L_n\ (2\rho)\ e^{-\rho} \quad ,$$
$$\rho = (\sigma^2 + \tau^2)/4 \tag{12}$$

and

$$L_n(x) = \left(\frac{e^x}{n!}\right) \left(\frac{d}{dx}\right)^n (e^{-x} x^n)$$

are $n-th$ Lagurre polynomials ($n = 0,\ 1,\ 2,\ \ldots$). The property (II) given previously is easily confirmed by solving the equation $E_n \psi_n = \psi_n$ ($n = 0,\ 1,\ 2,\ \ldots$) with $p = -i\, d/d\, q$. As is understood from this derivation, we assumed Eq. (1) only, and considered its algebra over $\boldsymbol{C}$. Then all other results are implied. In this sense, the knowledge of simple algebra is almost equivalent to knowing the "primordial form" of physical states, namely harmonic oscillator wave functions in quantum mechanics. Although there will be no guarantee for such a situation to prevail in more general cases, there is certainly no obvious reason against the possibility that a simple algebra is a primordial form of a "particle" in the framework considered later.

If the Hamiltonian is given as an element of WA, then its eigenvalues are completely specified as usual. The privileged role of the harmonic oscillator Hamiltonian manifests itself in the fact that each of its eigenfunctions belongs to a single projection operator $E_n$. Any perturbation can generally induce a transition between simple algebras corresponding to distinct $E_n$. The lesson obtained from these considerations is as follows. When a space, either internal or external, with an isometry group is given and the corresponding Weyl–type algebra is constructed, it already contains all the necessary kinematical information within it. Establishing such kinematics for various isometry groups is one of the central themes of this paper.

Consider next the case of $SU(2)$ rotations, in contrast to the one–dimensional translations. The analog of Eq. (1) in this case has been obtained previously by us in the angular Schröedinger representation.[7] Here we record only its form:

$$\exp\left[i\,\boldsymbol{n}\cdot\boldsymbol{J}\right]\widehat{D}^{(j)}\left(\phi\theta\psi\right)\exp\left[-i\,\boldsymbol{n}\cdot\boldsymbol{J}\right]\widehat{D}^{(j)}\left(\phi\theta\psi\right)^{-1} = \widehat{D}^{(j)}\left(\alpha\beta\gamma\right) \quad , \tag{13}$$

where

$$\exp\left[i\boldsymbol{n}\cdot\boldsymbol{J}\right]\equiv\exp\left[i\gamma J_z\right]\ \exp\left[i\beta J_y\right]\ \exp\left[i\alpha J_z\right]$$

is the operator for Euler rotation $(\alpha,\beta,\gamma)$ and $\widehat{D}^{(j)}$ is the standard rotation matrix[8] for $SU(2)$. In the simplest nontrivial case $j = 1/2$, the latter is explicitly given by:

$$\widehat{D}^{(1/2)}\ (\phi\theta\psi) = \begin{pmatrix} \cos\frac{\theta}{2}\ \exp\left[-i\left(\phi+\psi\right)/2\right] & -\sin\frac{\theta}{2}\ \exp\left[-i\left(\phi-\psi\right)/2\right] \\ \sin\frac{\theta}{2}\left[i\left(\phi-\psi\right)/2\right] & \cos\frac{\theta}{2}\ \exp\left[i\left(\phi+\psi\right)/2\right] \end{pmatrix} \quad . \tag{14}$$

As is easily confirmed for $j = 0,\ 1/2,\ 1,\ \ldots$, by power–series expansion, Eq. (13)

is satisfied by angular Schrödinger operators given by:

$$\begin{aligned} J_y &= -i\,[-\sin\phi\ \cos\theta\ (\partial/\partial\phi) \\ &\quad + \cos\phi\ (\partial/\partial\theta) + (\sin\phi/\cos\theta)\ (\partial/\partial\psi)] \quad , \\ J_z &= -i\ (\partial/\partial\phi) \quad . \end{aligned} \tag{15}$$

We notice several outstanding features of our ECR, Eq. (13):

(a) the phase factor $\exp[i\sigma\tau]$ in Eq. (1) is replaced here by a noncommutative matrix $\widehat{D}^{(j)}(\alpha,\beta,\gamma)$ ;

(b) $\widehat{D}^{(j)}$ for all irreducible representations of $SU(2)$ appear; and

(c) the operator $\exp[i\boldsymbol{n}\cdot\boldsymbol{J}]$ acts on each entry of the matrix $\widehat{D}^{(j)}\ (\phi\theta\psi)$ separately.

As for item (c), consider the (1,1) element of $\widehat{D}^{(j)}\ (\phi\theta\psi)$ as an example. The transform of it under $\exp(i\,\boldsymbol{n}\cdot\boldsymbol{J})$ is written explicitly as:

$$\begin{aligned} &\exp[i\,\boldsymbol{n}\cdot\boldsymbol{J}]\ \cos\frac{\theta}{2}\ \exp[-(\phi+\psi)/2]\ \exp[-i\,\boldsymbol{n}\cdot\boldsymbol{J}] \\ &= \cos\frac{\beta}{2}\ \exp[-i\,(\alpha+\gamma)/2]\ \cos\frac{\theta}{2}\ \exp[-i\,(\phi+\psi)/2] \\ &\quad + \sin\frac{\beta}{2}\ \exp[i\,(\alpha-\gamma)/2]\sin\frac{\theta}{2}\ \exp[i\,(\phi-\psi)/2] \\ &= (1,1)\text{ element of }\widehat{D}^{(1/2)}(\alpha\beta\gamma)\ \widehat{D}^{(1/2)}(\phi\theta\psi) \quad . \end{aligned} \tag{16}$$

This type of transformation formula for $SU(2)$ is possible only if each entry in matrices and $\exp(i\boldsymbol{n}\cdot\boldsymbol{J})$have functional forms as indicated here, as long as we adhere to conventional number systems such as $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$ (rational number field), etc. When one tries to represent Eq. (13) entirely by matrices without using any variables or derivatives, then each entry of matrices must itself be a matrix, or more properly an element of a certain ring. Therefore, the mechanism of the root of Eq. (13) should be understood in a more general concept than the mere matrix–multiplication, on which our previous ECR are based.[1]

Consider next the classical quaternion algebra $\boldsymbol{H}$ over $\boldsymbol{R}$ generated by three unit quaternions $j$, $k$, and $\ell\,(j^2 = k^2 = \ell^2 = -1,\ jk = -kj = \ell$ , etc.).

By using Euler parameters $(\alpha, \beta, \gamma)$as before and writing:

$$R(\alpha\beta\gamma) = \exp[-\gamma\ell/2]\ \exp[-\beta k/2]\ \exp[-\alpha\ell/2] \quad , \tag{17}$$

one can put Eq. (13) into the form:

$$R(\alpha\beta\gamma)\begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix} R(\alpha\beta\gamma)^{-1} = \widehat{D}^{(1)}(\alpha\beta\gamma)\begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix} \quad . \tag{18}$$

where $(i^2 = -1)$;

$$\begin{aligned} x_+ &= -(\ell + ik)/\sqrt{2} \\ x_0 &= j \\ x_- &= (\ell - ik)\,\sqrt{2} \quad . \end{aligned} \tag{19}$$

Notice that Eq. (18) is purely an algebraic relation and expresses the fact that the division algebra $\boldsymbol{H}$ has $SU(2)\ /\{\pm 1\} \approx SO(3)$ as a group of automorphisms. The operator $R(\alpha\beta\gamma)$, which is an element of $\boldsymbol{H}$, induces a rotation in 3–dimensional quaternion space spanned by $j, k, \ell$. Obviously, this type of representation is preferred over the angular Schröedinger representation given in Eq. (15) for our purpose because no measurable, continuous angular parameters seem to exist in internal space. However, it should be noted that, when a group is given, an irreducible representation of it by the relation like Eq. (18) is possible only if its dimension is at most four due to the fact that the dimension of $\boldsymbol{H}$ is only four.

A higher dimensional representation of ECR requires the use of division algebras over fields more general than $\boldsymbol{C}$ or $\boldsymbol{R}$. The formation of tensor algebra followed by decomposition to simple algebras will be performed in a parallel way to the case of Eq. (1), although we do not explicitly show this here. When the base field is $\boldsymbol{R}$, the simple algebra is either of the type $M_n(\boldsymbol{H})$ or $M_n(\boldsymbol{R})$ where $M_n$ denotes an $n \times n$ matrix with entries from a division algebra in the bracket. This completes our review and observation from a new standpoint of ECR for $SU(2)$ over $\boldsymbol{R}$.

The problem before us is to see whether a similar ECR and its representation are possible in the case of particular types of finite internal–symmetry groups. In the next section, we attempt to formulate this problem more thoroughly, including the results summarized here.

## III. EXTENDED COMMUTATION RELATIONS (ECR) OVER VARIOUS NUMBER FIELDS

As we have seen in Section II, a full knowledge of ECR over a given space can provide us with complete kinematical information. Moreover, we may expect that it can even suggest to us the primordial form of physical states, even without the knowledge of Hamiltonians. We also learned that the matrix representation of ECR requires, in general, the use of noncommutative division algebras instead of fields (which equal commutative division algebras).

To understand the general mathematical framework which is used in the following, let us first recall some elementary notions.[9–13] Consider a division algebra $D$ of finite dimension $d^2$ over any field. Many types of division algebras are mathematically known, depending on the choice of field $F$. Let $V$ be an $n$–dimensional vector–space over $D$. The space of homomorphisms of $V$ into itself constitutes a ring, denoted by $End(V)$. This ring is isomorphic to the matrix algebra $M_n(D)$, which was already introduced for the choice $D = \boldsymbol{H}$ or $\mathbf{R}$ (see Section II). $End(V)$ is isomorphic to a simple algebra of dimension $n^2d^2$ over $F$. An element of this algebra is an $n \times n$ matrix whose entries belong to a division algebra $D$, which in turn is $d^2$–dimensional over $F$. We formulate two mathematical problems as follows:

(I) When a field $F$ is given, what types of division algebras do exist; and

(II) What is the full automorphism group of the algebra $M_n(D)$?

Question (II) may be replaced by a slightly different one, *i.e.* to know the structure of the group $M_n(D)^\times$ consisting of invertible elements of $M_n(D)$. This is because $M_n(D)^\times$ is homomorphic to the full automorphism group of $M_n(D)$.[9,11,12] Moreover, it is necessary to include not only automorphisms, but also anti–automorphisms when one considers unitary groups.[14,15] The answer to (I) is completely known for special types of fields including all local (such as $p$–adic) and global (such as algebraic number) fields. Some of these fields are ubiquitous in recent physical literature,[16] although they are used in different contexts from ours. The answer to (II) seems to be generally more complicated than (I). As both problems are purely mathematical in character, one can use the known results if necessary. Therefore, we suppose that both (I) and (II) have been solved.

With these preparations in mind, let us consider the analog of Eqs. (1) and (13). It will be of the form ($x \epsilon D$):

$$x^{-1}\, M_n\ (D) x = M_n\ (F_x)^\times M_n\ (D) \quad , \tag{20}$$

where $F_x$ indicates a set of elements of $F$, which uniquely specifies an element $x$ over a suitable basis of $D$. Further multiplication by $y^{-1}$ and $y$ indicates that

$M_n\,(F_x)^{\times} M_n\,(F_y)^{\times} = M_n\,(F_{xy})^{\times}$. So $M_n\,(F_x)^{\times}$ constitutes a representation of a group which is homomorphic to $D^{\times} \equiv D - \{0\}$. We notice that Eq. (20) may be replaced by a little more general relation. Indeed, $x$ in Eq. (20) actually stands for a diagonal $n \times n$ matrix:

$$x = \begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{pmatrix} \tag{21}$$

which is only a special subalgebra of $M_n(D)$.

Therefore one may consider the group $M_n(D)^{\times}$ and its subgroups. Here one should be aware of two points. Firstly, the division algebra $D$ used in Eq. (20) has to be non–commutative in general. Otherwise the ECR is reduced to a trivial relation, as was noticed before. Secondly, subgroups generated by $x$ of the form (21) need not be normal subgroups of $M_n(D)^{\times}$. Therefore, ECR expressed as Eq. (20) is still possible even if $M_n(D)^{\times}$ is a simple group. Now we recall that $M_n(F)^{\times}$ corresponds to $\widehat{D}^{(j)}\,(\alpha, \beta, \gamma)$ in $SU(2)$ case. Therefore, what we expect for ECR is the situation such that $M_n(F)^{\times}$ is simply an irreducible representation over $F$ of a given internal–symmetry group $G$. We already confirmed that this is actually realized for $SU(2)$ and $O(3)$; namely, such ECR exist explicitly, as shown in Eqs. (13) and (18).

This requirement seems to be quite natural because the internal symmetry group is understood to be the result of internal transformations which can generally depend on time. The central problem in this section is: suppose one starts with a given field $F$. Construct a division algebra $D$. Then, is there any choice of the field $F$ such that the automorphism group of $D$ is isomorphic to the given internal–symmetry group $G$? This is identical to the $n = 1$ case of problem (II), which was already solved by our assumption. In practice, we identify $G$ with finite group $< 3,3,2 >$ or $< 4,3,2 >$, which are finite subgroups of $SU(2)$ of order 24 and 48, respectively.[1,2,17]

Before considering examples, let us briefly summarize the answer to (II) in several cases. In the standard notation,[15] the invertible elements of $M_n(D)$ constitute a multiplicative group $GL_n(V)$ which is called a general linear group and is obviously identical to $M_n(D)^\times$. As the central of $M_n(D)^\times$ is $F^\times$ (multiplicative group of $F$), one can set up a homomorphism called the reduced norm:

$$N_{rd} \ : \ GL_n(V) \ \rightarrow \ F^\times \quad . \tag{22}$$

Now, the norm one group $SL_n(V)$ is defined to be the kernel of this map. One more important subgroup of $GL_n(V)$, denoted by $E_n(V)$, is generated by a class of particular elements of $GL_n(V)$, called transvections. It is a normal subgroup of $GL_n(V)$. We do not need to enter into a detailed discussion of it. Suffice it to say that these groups satisfy:

$$E_n \ \subseteq \ SL_n(V) \ \subseteq \ GL_n(V) \quad . \tag{23}$$

Several salient features of these groups are[15]:

(a) If $D$ is a finite field $\boldsymbol{F}_d$, then all these groups are finite, and $E_n(V) \ = \ SL_n(V)$.

(b) Assume $n \geq 2$ and the center $F$ of $D$ is a local or global field. Then $E_n(V) \ = \ SL_n(V)$. Moreover, $GL_n(V)/E_n(V)$ is isomorphic to $F^\times$ (local field case) or a known subgroup of $F^\times$ (global field case). In particular, for the classical quaternion algebra over $\boldsymbol{Q}(\boldsymbol{R})$, $GL_n(V)/E_n(V)$ is isomorphic to $\boldsymbol{Q}_+^\times(\boldsymbol{R}_+^\times)$, the multiplicative group of positive numbers of $\boldsymbol{Q}(\boldsymbol{R})$. Therefore, all remaining structure lies in the group $E_n(V)$ for these fields.

(c) The quotient $E_n(V)$ / {Center of $E_n(V)$} is a simple group in all but these two cases: $n = 2$ with $D$ the finite field $\boldsymbol{F}_2$ or $\boldsymbol{F}_3$.

Among these properties, (c) is the most relevant to our purpose. As the center of $E_n(V)$ is $F$ in our case, $E_n(V)$ is practically identical to the group of automorphisms $Aut\ M_n(D)$.

The case of finite fields should be excluded in order to obtain a non–trivial ECR because they are commutative. Then $E_n(V)/\{\text{Center } E_n(V)\}$ are all simple. Yet, these groups still admit many subgroups which are, however, not normal. This is not surprising. In fact, finite subgroups of $SU(2)$ used previously[1,2] are not normal either. One may thus expect to have many finite subgroups of $E_n(V)$ (and $M_n(D)^\times$). The detailed study of these subgroups is not undertaken here. Once ECR's are explicitly found, then the complete formulations of resultant kinematics as well as dynamics should be possible by using those fields described above. Namely, the dynamics is then based on a Weyl–type algebra consisting of the set $\{M_n(D)\}$ in analogy with ordinary quantum mechanics. As it seems to require additional assumptions on Hamiltonians or quadratic forms, it must be left to subsequent work. We mention merely that a global field such as an algebraic function field contains a transcendental element $t$ over finite field. This suggests to us a possibility for introducing a space–time variable through division algebras. In the final part of the next section, we will discuss another step which is necessary in formulating kinematics for states satisfying anticommutation relations. We also speculate that a highly unconventional approach to basic problems undertaken by Noyes[18] can possibly be submerged into our theoretical environment, as far as a field with non–zero characteristics can be chosen at our starting point.

## IV. ISOSPIN SYMMETRIES AND TENSOR ALGEBRAS

As we have seen in Section II, it is necessary to form a tensor algebra in order to get a whole Weyl–type algebra. In the notation used in the previous section, we now describe a general method to get a tensor product of two simple algebras over the same base field, since this is required in any realistic formulation. Then we shall consider Case (B). According to Wedderburn's theorem of associated algebra, there is a unique division algebra $D$ to each central simple algebra $M_n(D)$. Indeed, this justifies the notation used here.[9] Two such algebras are similar to each other if they are $M_n(D)$ and $M_m(D)$; namely, if they correspond to the same $D$. Therefore, central simple algebras are classified into several distinct similarity classes. These classes constitute a group under tensor product, and this group is called a Brauer group of field $F$. It is denoted by $B(F)$. For a local field $F$, it is isomorphic to $\boldsymbol{Q}/\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the ring of rational integers.[9–13] This is an additive group modulo one consisting of all rational numbers. Or equivalently, each element of $B(F)$ is uniquely specified by a root of 1 in $\boldsymbol{C}$, *i.e.* by giving $\exp[2\pi i(m/n)]$ with relatively prime integers $m, n$ satisfying $0 \leq m < n$. It is called a Hasse invariant.[9] For all matrix–algebras considered in our previous work,[1] this invariant is trivial; namely, it is equal to 1 due to the fact that $B(\boldsymbol{C}) = 0$ (identity). Also, a set of these invariants can uniquely characterize division algebras over global fields. They can be interpreted as new quantum numbers for our theory as long as these familiar fields can be employed as base fields.

We notice that when one starts with a given field, tensor products of $M_n(D)$ over $F$ corresponding to all elements of the relevant Brauer group appear. This is a completely solved problem for all fields which we discussed in Section III. These observations apply to both Cases (A) and (B) of Section II. Keeping this in mind, let us consider Case (B) for internal symmetry groups. If the isospin groups are special types of finite groups, as were considered previously,[1,2] they are solvable groups—*i.e.* constructed out of cyclic groups. Moreover, these cyclic groups are of order 2 or 3. This suggests to us that they are not very different from $C$, $P$, $T$ transformations.

In quantum mechanics, we know examples of involutions; namely, operations of taking complex conjugation, or hermitian conjugation, and so on. This operation represents a certain type of (anti–) automorphism of underlying number systems or algebras. Indeed, $C, P, T$ are closely connected to these (anti–) automorphisms (involutions, in case of order 2) of the number system. Therefore, a natural question arises: Can one represent finite isospin–symmetry groups as (anti–) automorphisms of underlying number fields? As it turns out, this is possible if we start with some base field $F$ which is distinct from an algebraically closed field such as $\boldsymbol{C}$. Namely, starting with $F$, one can consider its finite–dimensional Galois extension $E$ such that Galois group $\simeq G$, where $G$ is a solvable group, used as an internal symmetry group. Indeed, it has been known for a long time in the theory of elliptic curves that the finite isospin group such as $< 3,3,2 >$ arises quite naturally in this way.[19] But we refrain from entering into this subject any further because the physical picture behind it has not been sufficiently clarified.

As discussed in Section I, we do not need to consider any time–dependent kinematical structure of this group. The central problem here is: Why is there just such symmetry group as $G$ and not others? A possible answer would be given by saying that $G$ is the result of a sequence of split extensions of an elementary cyclic group of order 2. This process has already been described elsewhere.[1] It is noteworthy here to point out that the extended field $E$ over $F$ is considered to be an $n$–dimensional vector space $V$ over $F$, where $n = \text{card Gal}(E/F)$, *i.e.* order of Galois group. It is then tempting to form a tensor algebra, which is defined to be:

$$T(V) = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots \quad . \tag{28}$$

If a suitable quadratic form $Q(x)$, $(x \epsilon V)$ is given, one can construct a Clifford algebra $C(V,Q) = T(V)/K_Q$ of $2^n$ dimension, where $K_Q$ is an ideal of $T(V)$ generated by $Q$. In our case, $n$ is always even ($n = 24$ for $G =< 3, 3, 2 >$). Then the Clifford algebra is decomposed into a tensor product of quaternion algebras in a unique way.

One may also think that this is the most natural way to introduce anticommutation relations in the framework described in Section III. It is possible to identify $Q(x)$ with Hamiltonians of the system or some other conserved quantity. However, it seems difficult to tell more about the consequence of the assumption (B) at the present stage. This is simply because the present framework is not yet restrictive enough to allow physically meaningful predictions about masses, flavors, etc. In a subsequent work, we will try to work out the details of this formalism for both Cases (A) and (B).

## V. CONCLUDING REMARKS

When one encounters new physical phenomena, the first thing to do is to try to understand those phenomena by using existing, well–established theories. If this turns out to be unsuccessful, then the next step is to extend or modify existing methods in order to cover new phenomena, taking care to retain the logical and physical structures of established theories as much as possible.

We applied this policy to the basic problems related to internal symmetries. As is clear from our considerations, an important problem in this approach is to find out a base field $F$ from physical conditions, over which one can construct division algebras for Case (A) or Galois extensions for Case (B) of Section II. Although we attempted in a previous work[1] to introduce "flavors" and considered also Gell–Mann–Okubo type mass–formulas for mesons, the framework used there is now to be superseded by the one presented here. The consideration of space–time dependence as well as a concrete formulation of dynamics will be reported in subsequent works.

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## REFERENCES


1. K. Yamada, Phys. Rev. **D35**, 2559 (1987).
2. K. Yamada, Phys. Rev. **D18**, 935 (1978).

   See also K. M. Case, R. Karplus, and C. N. Yang, Phys. Rev. **101**, 874 (1956);

   W. M. Fairbairn, T. Fulton, and W. H. Klink, J. Math. Phys. **5**, 1038 (1964).
3. H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931).
4. J. Schwinger, Proc. Nat. Acad. Sci. USA **46**, 570 (1960); **46**, 883 (1960); **46**, 1401 (1960); **47**, 1075 (1961). See also J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970).
5. J. von Neumann, Math. Ann. **104**, 570 (1931).
6. K. Yamada, unpublished report TIT/HEP–71 (1982).
7. K. Yamada, Phys. Rev. **D25**, 3256 (1982).
8. D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
9. A. Weil, *Basic Number Theory*, 3rd ed. (Springer–Verlag, New York, 1974).
10. J. P. Serre, *Local Fields* (Springer–Verlag, New York, 1979).
11. N. Jacobson, *Basic Algebra II* (Freeman, San Francisco, 1980).
12. R. S. Pierce, *Associative Algebras* (Springer–Verlag, New York, 1982).
13. K. Iwasawa, *Local Class Field Theory* (Oxford University Press, New York, 1986).
14. W. Scharlau, *Quadratic and Hermitian forms* (Springer–Verlag, Berlin, Heidelberg, 1985).

15. A. J. Hahn and O. T. O'Meara, *The Classical Groups and K–Theory* (Springer–Verlag, Berlin, Heidelberg, 1989).

16. See, *e.g.* B. Grossman, Phys. Lett. **B197**, 101 (1987);
P. G. O. Freund and M. Olson, Phys. Lett **B199**, 186 (1987);
P. G. O. Freund and E. Witten, Phys. Lett. **B199**, 191 (1987);
P. H. Frampton and Y. Okada, Phys. Rev. Lett. **60**, 484 (1988);
L. Brekke, P. G. O. Freund, M. Olson, and E. Witten, Nucl. Phys. **B302**, 365 (1988); F. Lev, J. Math. Phys. **30**, 1985 (1989), and papers quoted therein.

17. H. S. M. Coxeter, *Regular Complex Polytopes* (Cambridge University Press, London, 1974).

18. H. P. Noyes, SLAC–PUB–3566, January 1985 (unpublished).

19. S. Lang, *Elliptic Functions*, Second Edition (Springer–Verlag, New York, 1987).