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THE CABIBBO-KOBAYASHI-MASKAWA MIXING MATRIX*

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In the standard model with $SU(2) \times U(1)$ as the gauge group of electroweak interactions, both the quarks and leptons are assigned to be left-handed doublets and right-handed singlets. The quark mass eigenstates are not the same as the weak eigenstates, and the matrix relating these bases was defined for six quarks and given an explicit parametrization by Kobayashi and Maskawa[1] in 1973. It generalizes the four-quark case, where the matrix is parametrized by a single angle, the Cabibbo angle.[2]

By convention, the three charge 2/3 quarks (u, c, and t) are unmixed, and all the mixing is expressed in terms of a 3×3 unitary matrix V operating on the charge $-1/3$ quarks (d, s, b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \tag{1}$$

The values of individual matrix elements can in principle all be determined from weak decays of the relevant quarks, or, in some cases, from deep inelastic neutrino scattering. Using the constraints discussed below (in the full-sized edition only), together with unitarity, and assuming only three generations, the 90% confidence limits on the magnitude of the elements of the complete matrix are:

$$\begin{pmatrix} 0.9747 \text{ to } 0.9759 & 0.218 \text{ to } 0.224 & 0.001 \text{ to } 0.007 \\ 0.218 \text{ to } 0.224 & 0.9734 \text{ to } 0.9752 & 0.030 \text{ to } 0.058 \\ 0.003 \text{ to } 0.019 & 0.029 \text{ to } 0.058 & 0.9983 \text{ to } 0.9996 \end{pmatrix} . \tag{2}$$

The ranges shown are for the individual matrix elements. The constraints of unitarity connect different elements, so choosing a specific value for one element restricts the range of the others.

There are several parametrizations of the Cabibbo-Kobayashi-Maskawa matrix. In view of the need for a "standard" parametrization in the literature, we advocate:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \qquad (3)$$

proposed by Chau and Keung.[3] The choice of rotation angles follows earlier work of Maiani,[4] and the placement of the phase follows that of Wolfenstein.[5] The notation used is that of Harari and Leurer[6] who, along with Fritzsch and Plankl,[7] proposed this parametrization as a particular case of a form generalizable to an arbitrary number of "generations." The general form was also put forward by Botella and Chau.[8] Here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, with i and j being "generation" labels, $\{i, j = 1, 2, 3\}$. In the limit $\theta_{23} = \theta_{13} = 0$ the third generation decouples, and the situation reduces to the usual Cabibbo mixing of the first two generations with θ_{12} identified with the Cabibbo angle.[2] The real angles θ_{12}, θ_{23}, θ_{13} can all be made to lie in the first quadrant by an appropriate redefinition of quark field phases. Then all s_{ij} and c_{ij} are positive, $|V_{us}| = s_{12}c_{13}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}c_{13}$. As c_{13} is known to deviate from unity only in the fifth decimal place, $|V_{us}| = s_{12}$, $|V_{ub}| = s_{13}$, and $|V_{cb}| = s_{23}$ to an excellent approximation. The phase δ_{13} lies in the range $0 \le \delta_{13} < 2\pi$, with non-zero values generally breaking CP invariance for the weak interactions. The generalization to the n generation case contains $n(n-1)/2$ angles and $(n-1)(n-2)/2$ phases.[6,7,8] The range of matrix elements in Eq. (2) corresponds to 90% C.L. limits on the angles of $s_{12} = 0.218$ to $0.224, s_{23} = 0.030$ to 0.058, and $s_{13} = 0.001$ to 0.007.

[Continuation of this discussion found in full-sized edition of the Review of Particle Properties only.]

Kobayashi and Maskawa[1] originally chose a parametrization involving the four angles, θ_1, θ_2, θ_3, δ:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} . \qquad (4)$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ for $i = 1, 2, 3$. In the limit $\theta_2 = \theta_3 = 0$, this reduces to the usual Cabibbo mixing with θ_1 identified (up to a sign) with the Cabibbo angle.[2] Slightly different forms of the Kobayashi-Maskawa parametrization are found in the literature. The C-K-M matrix used in the 1982 Review of Particle Properties is obtained by letting $s_1 \to -s_1$ and $\delta \to \delta + \pi$ in the matrix given above. An alternative is to change Eq. (4) by $s_1 \to -s_1$ but leave δ unchanged. With this change in s_1, the angle θ_1 becomes the usual Cabibbo angle, with the "correct" sign (i.e. $d' = d\cos\theta_1 + s\sin\theta_1$) in the limit $\theta_2 = \theta_3 = 0$. The angles θ_1, θ_2, θ_3 can, as before, all be taken to lie in the first quadrant by adjusting quark field phases. Since all these parametrizations are referred to as "the" Kobayashi-Maskawa form, some care about which one is being used is needed when the quadrant in which δ lies is under discussion.

Other parametrizations, mentioned above, are due to Maiani[4] and to Wolfenstein.[5] The latter emphasizes the relative sizes of the matrix elements by expressing them in powers of the Cabibbo angle. Still other parametrizations[9] have come into the literature in connection with attempts to define "maximal CP violation". No physics can depend on which of the above parametrizations (or any other) is used as long as a single one is used consistently and care is taken to be sure that no other choice of phases is in conflict.

Our present knowledge of the matrix elements comes from the following sources:

(1) Nuclear beta decay, when compared to muon decay, gives[10–13]

$$|V_{ud}| = 0.9744 \pm 0.0010 \ . \tag{5}$$

This includes refinements in the analysis of the radiative corrections, especially the order $Z\alpha^2$ effects, which have brought the ft-values from low and high Z Fermi transitions into good agreement.

(2) Analysis of K_{e3} decays yields[14]

$$|V_{us}| = 0.2196 \pm 0.0023 \ . \tag{6}$$

The isospin violation between K^+_{e3} and K^0_{e3} decays has been taken into account, bringing the values of $|V_{us}|$ extracted from these two decays into agreement at the 1% level of accuracy. The analysis of hyperon decay data has larger theoretical uncertainties because of first order $SU(3)$ symmetry breaking effects in the axial-vector couplings, but due account of symmetry breaking[15] applied to the WA2 data[16] gives a corrected value[17] of 0.222 ± 0.003 . We average these two results to obtain:

$$|V_{us}| = 0.2205 \pm 0.0018 \ . \tag{7}$$

(3) The magnitude of $|V_{cd}|$ may be deduced from neutrino and antineutrino production of charm off valence d quarks. The dimuon production cross sections of the CDHS group[18] yield $\overline{B}_c \ |V_{cd}|^2 = 0.41 \pm 0.07 \times 10^{-2}$, where $\overline{B}_c$ is the semileptonic branching fraction of the charmed hadrons produced.

The corresponding preliminary value from a recent Tevatron experiment[19] is $\overline{B}_c\ |V_{cd}|^2 = 0.534^{+0.052}_{-0.078} \times 10^{-2}$. Averaging these two results gives $\overline{B}_c\ |V_{cd}|^2 = 0.47 \pm 0.05 \times 10^{-2}$. Supplementing this with measurements of the semileptonic branching fractions of charmed mesons,[20] weighted by a production ratio of $D^0/D^+ = (60 \pm 10)/(40 \mp 10)$, to give $\overline{B}_c = 0.113 \pm 0.015$, yields

$$|V_{cd}| = 0.204 \pm 0.017 \tag{8}$$

(4) Values of $|V_{cs}|$ from neutrino production of charm are dependent on assumptions about the strange quark density in the parton-sea. The most conservative assumption, that the strange-quark sea does not exceed the value corresponding to an $SU(3)$ symmetric sea, leads to a lower bound,[18] $|V_{cs}| > 0.59$. It is more advantageous to proceed analogously to the method used for extracting $|V_{us}|$ from K_{e3} decay; namely, we compare the experimental value for the width of D_{e3} decay with the expression[21] that follows from the standard weak interaction amplitude:

$$\Gamma(D \to \bar{K} e^+ \nu_e) = |f_+^D(0)|^2 |V_{cs}|^2 (1.54 \times 10^{11} \mathrm{sec}^{-1}). \tag{9}$$

Here $f_+^D(q^2)$, with $q = p_D - p_K$, is the form factor relevant to D_{e3} decay; its variation has been taken into account with the parametrization $f_+^D(t)/f_+^D(0) = M^2/(M^2 - t)$ and $M = 2.1\ GeV/c^2$, a form and mass consistent with Mark III and E691 measurements.[22,23] Combining data on branching ratios for $D_{\ell 3}$ decays[22,23] with accurate values[24] for τ_{D^+} and τ_{D^0}, gives the value $0.78 \pm 0.11 \times 10^{11}\ \mathrm{sec}^{-1}$ for $\Gamma(D \to \bar{K} e^+ \nu_e)$. Therefore

$$|f_+^D(0)|^2 |V_{cs}|^2 = 0.51 \pm 0.07. \tag{10}$$

A very conservative assumption is that $|f_+^D(0)| < 1$, from which it follows that $|V_{cs}| > 0.66$. Calculations of the form factor either performed[25,26] directly at $q^2 = 0$, or done[27] at the maximum value of $q^2 = (m_D - m_K)^2$ and interpreted at $q^2 = 0$ using the measured q^2 dependence, yield $f_+^D(0) = 0.7 \pm 0.1$. It follows that

$$|V_{cs}| = 1.02 \pm 0.18 \ . \tag{11}$$

The constraint of unitarity when there are only three generations gives a much tighter bound (see below).

(5) The ratio $|V_{ub}/V_{cb}|$ can be obtained from the semileptonic decay of B mesons by fitting to the lepton energy spectrum as a sum of contributions involving $b \to u$ and $b \to c$. The relative overall phase space factor between the two processes is calculated from the usual four-fermion interaction with one massive fermion (c quark or u quark) in the final state. The value of this factor depends on the quark masses, but is roughly one-half (in suppressing $b \to c$ compared to $b \to u$). Both the CLEO[28] and ARGUS[29] collaborations have reported evidence for $b \to u$ transitions in semileptonic B decays. The interpretation of the result in terms of $|V_{ub}/V_{cb}|$ depends fairly strongly on the theoretical model used to generate the lepton energy spectrum, especially for $b \to u$ transitions.[26,27,30] Combining the experimental and theoretical uncertainties, we quote

$$|V_{ub}/V_{cb}| = 0.09 \pm 0.04 \tag{12}$$

(6) The magnitude of V_{cb} itself can be determined if the measured semileptonic bottom hadron partial width is assumed to be that of a b quark decaying

through the usual $V - A$ interaction:

$$\Gamma(b \to c\ell\bar{\nu}_\ell) = \frac{BR(b \to c\ell\bar{\nu}_\ell)}{\tau_b} = \frac{G_F^2 m_b^5}{192\pi^3} F(m_c/m_b)|V_{cb}|^2 , \qquad (13)$$

where τ_b is the b lifetime and $F(m_c/m_b)$ is the phase space factor noted above as approximately one-half. Most of the error on $|V_{cb}|$ derived from Eq. (13) is not is not from the experimental uncertainties, but in the theoretical uncertainties in choosing a value of m_b and in the use of the quark model to represent inclusively semileptonic decays which, at least for the B meson, are dominated by a few exclusive channels. Instead we quote the value derived from $B_{\ell 3}$ decay, $\bar{B} \to D\ell\bar{\nu}_\ell$, by comparing the observed rate with the theoretical expression that involves a form factor, $f_+^B(q^2)$. This is analogous to what gives the most accurate values for $|V_{us}|$ (from K_{e3} decay) and $|V_{cs}|$ (from $D_{\ell 3}$ decay). It avoids all questions of what masses to use, and the heavy quarks in both the initial and final states give more confidence in the accuracy of the theoretical calculations of the form factor. With account of a number of models of the form factor, the data[31] yield

$$|V_{cb}| = 0.044 \pm 0.009 . \qquad (14)$$

The central value and the error are now comparable to what is obtained from the inclusive semileptonic decays, but ultimately, with more data and more confidence in the calculation of the form factor, exclusive semileptonic decays should provide the most accurate value of $|V_{cb}|$.

The results for three generations of quarks, from Eqs. (5), (7), (8), (11), (12), and (14) plus unitarity, are summarized in the matrix in Eq. (2). The ranges given there are different from those given in Eqs.(5)-(14) (because of the inclusion of unitarity), but are consistent with the one standard deviation errors on the input matrix elements.

The data do not preclude there being more than three generations. Moreover, the entries deduced from unitarity might be altered when the C-K-M matrix is expanded to accommodate more generations. Conversely, the known entries restrict the possible values of additional elements if the matrix is expanded to account for additional generations. For example, unitarity and the known elements of the first row require that any additional element in the first row have a magnitude $|V_{ub'}| < 0.07$. When there are more than three generations the allowed ranges (at 90% C.L.) of the matrix elements connecting the first three generations are

$$\begin{pmatrix} 0.9728 \text{ to } 0.9757 & 0.218 \text{ to } 0.224 & 0.001 \text{ to } 0.007 & \cdots \\ 0.182 \text{ to } 0.227 & 0.865 \text{ to } 0.975 & 0.030 \text{ to } 0.058 & \cdots \\ 0 \text{ to } 0.13 & 0 \text{ to } 0.45 & 0 \text{ to } 0.9995 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

where we have used unitarity (for the expanded matrix) and Eqs. (5), (7), (8), (11), (12), and (14).

Further information on the angles requires theoretical assumptions. For example, $B_d - \bar{B}_d$ mixing, if it originates from short distance contributions to ΔM_B dominated by box diagrams involving virtual t quarks, gives information on $V_{tb}\, V_{td}^*$ once hadronic matrix elements and the t quark mass are known. A similar comment holds for $V_{tb}\, V_{ts}^*$ and $B_s - \bar{B}_s$ mixing.

Direct and indirect information on the C-K-M matrix is neatly summarized in terms of the "unitarity triangle." The name arises since unitarity of the 3×3 C-K-M matrix applied to the first and third columns yields

$$V_{ud}\, V_{ub}^* \;+\; V_{cd}\, V_{cb}^* \;+\; V_{td}\, V_{tb}^* = 0. \tag{15}$$

In the parametrization adopted above, V_{cb} is real and V_{cd} is real to a very good approximation. Setting cosines of small angles to unity, Eq. (15) becomes

$$V_{ub}^* \;+\; V_{td} = |V_{cd}\, V_{cb}|\;. \tag{16}$$

The unitarity triangle is just a geometrical presentation of this equation in the complex plane.[32]

CP-violating processes will involve the phase in the C-K-M matrix, assuming that the observed CP violation is solely related to a nonzero value of this phase. This allows additional constraints to be brought to bear. More specifically, a necessary and sufficient condition for CP violation with three generations can be formulated in a parametrization independent manner in terms of the non-vanishing of the determinant of the commutator of the mass matrices for the charge 2e/3 and charge -e/3 quarks.[33] CP violating amplitudes or differences of rates all are proportional to the C-K-M factor in this quantity. This is the product of factors $s_{12}s_{13}s_{23}c_{12}c_{13}^2c_{23}s_{\delta_{13}}$ in the parametrization adopted above, and is $s_1^2s_2s_3c_1c_2c_3s_\delta$ in that of reference 1. With the approximation of setting cosines to unity, this is just twice the area of the unitarity triangle. While hadronic matrix elements whose values are imprecisely known generally now enter, the constraints from CP violation in the neutral Kaon system are tight enough to very much restrict the range of angles and the phase of the C-K-M matrix. For CP-violating asymmetries of neutral B mesons decaying to CP eigenstates, there is a direct relationship between the magnitude of the asymmetry in a given decay and $\sin 2\phi$, where ϕ is an appropriate angle of the unitarity triangle.[32] The combination of all the direct and indirect information can be used to find the overall constraints on the C-K-M matrix and thence the implications for future measurements of CP violation in the B system.[34]

REFERENCES

1. M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
2. N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963).
3. L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. 53, 1802 (1984).
4. L. Maiani, Phys. Lett. 62B, 183 (1976) and in *Proceedings of the 1977 International Symposium on Lepton and Photon Interactions at High Energies* (DESY, Hamburg, 1977), p. 867.
5. L. Wolfenstein, Phys. Rev. Letters 51, 1945 (1983).
6. H. Harari and M. Leurer, Phys. Lett. 181B, 123 (1986).
7. H. Fritzsch and J. Plankl, Phys. Rev. D35, 1732 (1987).
8. F. J. Botella and L.-L. Chau, Phys. Lett. 168B, 97 (1986).
9. See, for example, M. Gronau and J. Schechter, Phys. Rev. Letters 54, 385 (1985), where various parametrizations are discussed, including one equivalent to that in Eq. (3)
10. W. J. Marciano and A. Sirlin, Phys. Rev. Lett. 56, 22 (1986).
11. A. Sirlin and R. Zucchini, Phys. Rev. Lett. 57, 1994 (1986).

12. W. Jaus and G. Rasche, Phys. Rev. D35, 3420 (1987).

13. A. Sirlin, Phys. Rev. D35, 3423 (1987).

14. H. Leutwyler and M. Roos, Z. Phys. C25, 91 (1984).

15. J. F. Donoghue, B. R. Holstein, and S. W. Klimt, Phys. Rev. D35, 934 (1987)

16. M. Bourquin *et al.*, Z. Phys. C21, 27 (1983).

17. J. M. Gaillard and G. Sauvage, private communication .

18. H. Abramowicz *et al.*, Z. Phys. C15, 19 (1982).

19. M. Shaevitz, talk at the Twelfth International Workshop on Weak Interactions and Neutrinos, Ginosar, Israel, April 9 - 14, 1989, and Nevis Report No. 1415, 1989 (unpublished).

20. D. Hitlin, *Proceedings of the 1987 International Symposium on Lepton and Photon Interactions at High Energies*, Hamburg, July 27 - 31, 1987, edited by W. Bartel and R. Rückl (North Holland, Amsterdam, 1988), p. 179.

21. The result for $M = 2.2$ GeV is found in F. Bletzacker, H. T. Nieh, and A. Soni, Phys. Rev. D16, 732 (1977).

22. D. M. Coffman, California Institute of Technology Ph.D. thesis, 1986 (unpublished).

23. J. C. Anjos *et al.*, Phys. Rev. Lett. 62, 1587 (1989).

24. J. R. Raab *et al.*, Phys. Rev. D37, 2391 (1988).

25. T. M. Aliev *et al.*, Yad. Fiz. 40, 823 (1984) [Sov. J. Nucl. Phys. 40, 527 (1984)].

26. M. Bauer, B. Stech, and M. Wirbel, Z. Phys. C29, 637 (1985).

27. B. Grinstein, N. Isgur, and M. B. Wise, Phys. Rev. Lett. 56, 298 (1986); B. Grinstein, N. Isgur, D. Scora, and M. B. Wise, Phys. Rev. D39, 799 (1989).

28. R. Fulton *et al.*, Cornell report CLNS-89/951, 1989 (unpublished).

29. H. Albrecht *et al.*, DESY report DESY 89/152, 1989 (unpublished).

30. G. Altarelli *et al.*, Nucl. Phys. B208, 365 (1982).

31. H. Albrecht *et al.*, Phys. Lett. 229B, 175 (1989).

32. L.-L. Chau and W.-Y. Keung, Ref. 3; J. D. Bjorken, private communication and Phys. Rev. D39, 1396 (1989); C. Jarlskog and R. Stora, Phys. Lett. 208B, 268 (1988); J. L. Rosner, A. I. Sanda, and M. P. Schmidt, in *Proceedings of the Workshop on High Sensitivity Beauty Physics at Fermilab*, Fermilab, November 11-14, 1987 , edited by A. J. Slaughter, N. Lockyer, and M. Schmidt (Fermilab, Batavia, 1988), p. 165; C. Hamzaoui, J. L. Rosner and A. I. Sanda, *ibid.*, p. 215.

33. C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985) and Z. Phys. C29, 491 (1985).

34. C. O. Dib *et al.*, SLAC report SLAC-PUB-5109, 1989 (unpublished); A. I. Sanda, invited talk at the KEK Topical Conference on Electron-Positron Collision Physics, Tsukuba, Japan, May 17-19, 1989 and KEK report 89-70, 1989 (unpublished); C. S. Kim, J. L. Rosner, and C.-P. Yuan, University of Chicago report EFI 89-47, 1989 (unpublished).