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INTRODUCTION TO CONFORMAL FIELD THEORY AND STRING THEORY*

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ABSTRACT

I give an elementary introduction to conformal field theory and its applications to string theory.

I. INTRODUCTION:

These lectures are meant to provide a brief introduction to conformal field theory (CFT) and string theory for those with no prior exposure to the subjects. There are many excellent reviews already available (or almost available), and most of these go in to much more detail than I will be able to here. Those reviews concentrating on the CFT side of the subject include refs. 1,2,3,4; those emphasizing string theory include refs. 5,6,7,8,9,10,11,12,13

I will start with a little pre-history of string theory to help motivate the subject. In the 1960's it was noticed that certain properties of the hadronic spectrum — squared masses for resonances that rose linearly with the angular momentum — resembled the excitations of a massless, relativistic string.[14] Such a string is char-

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acterized by just one energy (or length) scale,* namely the square root of the string tension T, which is the energy per unit length of a static, stretched string. For strings to describe the strong interactions $\sqrt{T}$ should be of order 1 GeV. Although strings provided a qualitative understanding of much hadronic physics (and are still useful today for describing hadronic spectra[15] and fragmentation[16]), some features were hard to reconcile. In particular, string theory predicted an exactly massless, spin-two particle that was nowhere to be found in the hadronic spectrum.[17]

The current incarnation of string theory as a quantum theory of gravity (and perhaps of all the fundamental interactions) came with the realization that the massless, spin two particle should be identified as the graviton rather than as some strongly interacting particle.[17] Since the characteristic energy scale of gravity is the Planck scale, $M_{\rm Pl} \approx 10^{19}$ GeV (Newton's constant is $G_N = M_{\rm Pl}^{-2}$), the new identification necessitated rescaling the string tension by some 38 orders of magnitude! Other problems with string theory, such as the existence of a tachyon (an unstable mode) and the absence of fermions, were cured by the introduction of supersymmetry.[18,19,20,21] The explosion of interest in superstrings in 1984 followed the observation that some superstring theories contain massless fermions with parity-violating gauge interactions, like the known quarks and leptons.[22,23,24] Indeed, it was possible to identify a subset of the massless fermions and vector bosons having the same gauge interactions as the quarks, leptons and gauge bosons of the standard model.[24] This opened up the possibility that superstring theory might unify all the known interactions and thereby relate all observed masses and coupling constants.

Obviously, this program has not yet been carried to completion. One of the initial obstacles was that the original formulation of superstrings required spacetime to be ten-dimensional, not four-dimensional. Since then many ways have been found to construct four-dimensional string theories, including ones whose low-energy physics might be realistic.[24,25,26,27] The constructions are generally termed *compactifications*, because the first such constructions[28,24] hid the extra six dimensions by having them live on a compact manifold with a size of order the Planck length. Later constructions have not required compact manifolds, and have led to more varied possibilities for low-energy physics. However, each construction has

* I set $\hbar = c = 1$ everywhere.

associated with it[29,30] a two-dimensional *conformal field theory*, which determines the full particle spectrum and all the coupling constants. Conformal field theory is therefore a necessary framework for understanding the types of physics that one can expect from string theory.

String theory is not the only motivation for studying two-dimensional conformal field theory, however. The long-distance behavior of a statistical mechanics system at a second-order phase transition (a critical point) is described by a conformal field theory.[31,4] Also, many connections have recently been established between conformal field theories and other exactly solvable (but not conformal) two-dimensional field theories,[32] as well as three-dimensional 'topological' field theories.[33,34] Here I will focus on the applications of conformal field theory to string theory.

The organization of these lectures is as follows. In chapter II the propagation of a free string is discussed via the Polyakov approach. In order to quantize the string, various symmetries of the action must be gauge-fixed. This provides a motivation for studying in chapter III some of the basic features of two-dimensional conformal field theory, along with some simple examples. Most of this material can also be found in ref. 3. In chapter IV the quantization of the bosonic string is carried out and its mass spectrum is discussed. Chapter V describes briefly the superstring, the heterotic string, and string interactions. Finally, chapter VI concludes with a few comments on the prospects for superstrings.

II. FREE STRINGS AND CONFORMAL INVARIANCE:

A. Analogy to Point Particles

A massless, relativistic string is the one-dimensional generalization of a point particle. String theory can therefore be developed in comparison to conventional theories of point particles.

Recall that the quantum mechanics of interacting, relativistic particles can be described in either a first- or second-quantized formalism. In a first-quantized formalism one first calculates the probabilities for free-particle propagation from point x to point y. Then interactions are put in by hand: A certain amplitude (a coupling constant) is assigned for one particle to emit another. These vertices allow all the amplitudes in the theory to be built up, as Feynman diagrams that connect the vertices together with free particle propagators. In a second-quantized approach, by contrast, one defines field operators $A_i(x)$ that create and destroy particles, and constructs a Lagrangian density in terms of those fields, $\mathcal{L}[A_i(x)]$. When $\mathcal{L}$ is quantized, its kinetic terms generate the propagators of the first-quantized approach, and its interaction terms generate the vertices, so that all the first-quantized Feynman diagrams are recovered. In addition, however, *nonperturbative* effects (such as instantons) that cannot be represented by Feynman diagrams are revealed by the second-quantized approach. Theories of point particles generally allow for many possible kinds of interactions, and arbitrary values of coupling constants. This is especially true if there are several different types of particles (fields) in the theory, but even a theory of a single scalar field A can have arbitrary A^n couplings.

The first-quantized formalism for string theory proceeds similarly: One first calculates amplitudes for free-string propagation, then assigns an amplitude (the *string coupling constant*, g_s) for one string to emit another. In contrast to the particle case, where there may be several kinds of particles and many possible coupling constants, in a given string theory there is only one kind of string and one coupling constant g_s, and only one 'Feynman diagram' at each order in g_s.* In fact, string interactions are *uniquely determined*, once the free-string propagation is specified! Again, only perturbative amplitudes are easily accessible to this first-quantized approach. The second-quantized approach to string theory is called *string field theory*;[35] it introduces string fields $\mathbf{A}(x(\sigma))$ that create and destroy strings (σ is a parameter along the string), and it (in some cases) generates 'Feynman diagrams' that agree with the first-quantized results. In principle, string field theory also contains nonperturbative information; in practice, no-one has yet been able to extract this information.

In these lectures only the first-quantized approach to string theory will be discussed. I will begin by quantizing the motion of a free string by analogy to the free-particle case, following ref. 5.

* For theories with open strings as well as closed strings, which will not be discussed in these lectures, there are a few 'Feynman diagrams' at a given order.

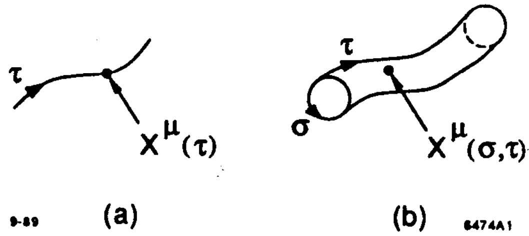

FIGURE 1

(a) A free particle propagating through space-time sweeps out a world-line.
(b) A free string sweeps out a world-sheet.

B. Free Particle Action

A free point particle is characterized by its position in spacetime, X^μ, where $\mu = 0, 1, 2, \ldots, D-1$ runs over the D space-time coordinates. As the particle propagates in time it sweeps out a *world-line* described by $X^\mu(\tau)$, where τ is some arbitrary parameter. See figure 1(a).

The free-particle equation of motion (in flat space-time) is

$$\frac{d^2 X^\mu}{d\tau^2} = 0,$$

which comes from varying the action

$$\begin{aligned} S(X) &= m(\text{length of world-line}) = m \int ds \\ &= -m \int d\tau \sqrt{\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}\,, \end{aligned} \tag{1}$$

where s is the proper time along the path.

The action (1) is difficult to quantize because of the square root. To avoid the square root we introduce an auxiliary field $g(\tau)$, which can be thought of as an intrinsic metric on the world-line, and replace the action $S(X)$ by

$$S(X,g) = \tfrac{1}{2} \int d\tau \left[\frac{1}{g(\tau)} \left(\frac{dX}{d\tau} \right)^2 - m^2 g(\tau) \right]. \tag{2}$$

Eliminating $g(\tau)$ from eq. (2) by its equations of motion leads back to the action $S(X)$ of eq. (1), so the two actions are classically equivalent. The second form is much easier to deal with quantum mechanically, however, because it is quadratic in X^μ. For example, the path integral $\int \mathcal{D}^D X^\mu \; \exp(-S(X,g))$ is Gaussian in X and can therefore be carried out exactly.

C. Free String Action and Symmetries

Rather than continuing with the particle example, let us now turn to the string, which sweeps out a two-dimensional (2d) surface, the *world-sheet*, as it moves through space-time. In these lectures I will discuss *closed strings* exclusively. A closed string is a loop with no free ends, and so the world-sheet is a cylinder, as shown in figure 1(b). (If the string had been an *open string*, with two free ends, the world-sheet would have been a strip.) A natural generalization of the particle action (1) is the Nambu-Goto action[36]

$$\begin{aligned} S(X) &= -T(\text{area of world-sheet}) = -T\int dA \\ &= -T\int d\tau d\sigma \sqrt{-(\partial_\tau X)^2(\partial_\sigma X)^2 + (\partial_\tau X\cdot\partial_\sigma X)^2}\ , \end{aligned} \tag{3}$$

where T is the *string tension* with dimensions of (length)$^{-2}$. For applications to hadronic physics, one would set $T \sim 1$ GeV^2 in eq. (3), whereas for applications to a unified field theory incorporating quantum gravity one sets $T \sim M_{\mathrm{Pl}}^2 \sim 10^{38}$ GeV^2 instead. As in the free particle case, the square root in eq. (3) can be avoided by introducing an auxiliary field $g_{\alpha\beta}(\tau,\sigma)$, which is also the *world-sheet metric*. Equation (3) can then be replaced by the classically equivalent action,[37]

$$S(X,g) = -\frac{T}{2}\int d^2\sigma\ \sqrt{-g}\ g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\,, \tag{4}$$

where $g \equiv \det(g_{\alpha\beta})$, $\alpha,\beta = 0,1$ and $(\sigma^0,\sigma^1) \equiv (\tau,\sigma)$.

Let us take the action (4) as the fundamental starting point for bosonic string theory, and quantize it by considering a path integral

$$\int \mathcal{D}^D X^\mu\ \mathcal{D}g_{\alpha\beta}\ e^{-S(X,g)}\,. \tag{5}$$

over the auxiliary field $g_{\alpha\beta}$ (the 2d metric) and over X^μ. The $g_{\alpha\beta}$ integral is an integral over the intrinsic shapes of 2d surfaces, whereas the X^μ integral is over the different ways of embedding a 2d surface into D-dimensional space-time. The boundary conditions on the path integral depend on the process being described.

To describe the propagation of a free, closed string, the 2d surface should be topologically equivalent to a cylinder, as in figure 1(b), and the boundary conditions on the ends of the cylinder need to be specified. For many purposes it suffices to consider the vacuum-to-vacuum amplitude, which is obtained by letting the cylinder become infinitely long ($\tau \in (-\infty, \infty)$). The tension in the string forces the vacuum state to have no spatial extent, so the ends of the cylinder are effectively collapsed to points. Thus the 2d surface appearing in the free-string vacuum amplitude is topologically a sphere.

Although we intend to discuss first the quantization of free strings, the path-integral treatment of interactions is so similar that it bears mentioning here. Indeed, the path integral (5) also describes interacting strings with just a slight change of boundary conditions. If one requires string interactions to be local, then strings can only interact when they touch. Two strings that touch can join together to become one, or in the time-reversed process a single string can split into two. The vacuum amplitude will now have contributions in which the string splits and rejoins many times. Each time it does so it creates a 'handle' on the 2d surface. All of the surfaces are *closed* (they have no boundaries) like the sphere. The number of handles on a surface is also called its *genus* and is denoted by h, $h = 0, 1, 2, \ldots$ Two surfaces are in the same topological class — *i.e.,* they can be smoothly deformed into each other — if and only if they have the same genus. We take the 2d metric to have Euclidean signature $(+, +)$, because all surfaces with Minkowski signature and genus $h \geq 1$ are singular (the singularities occur wherever a time-like geodesic splits into two). The surfaces of lowest genus (shown in figure 2) are the sphere ($h = 0$), the torus ($h = 1$), and the aptly-named genus-two surface ($h = 2$).

Each handle on the surface requires one splitting and rejoining, and lends a factor of $(g_s)^2$ to the amplitude. Thus the vacuum-to-vacuum amplitude for interacting, closed bosonic strings is given by the *Polyakov path integral* [38]

$$Z = \sum_{\text{genus } h=0}^{\infty} (g_s)^{2h-2} \int \mathcal{D}^D X^\mu \, \mathcal{D} g_{\alpha\beta} \, e^{-S(X,g)} \,. \tag{6}$$

The leading term in eq. (6) (the sphere) is the classical (tree-level) contribution to the amplitude; the remaining terms are quantum (loop) corrections. Missing from the expression are any nonperturbative contributions to amplitudes, which could be proportional to say e^{-1/g_s^2}. Amplitudes with external states require the insertion

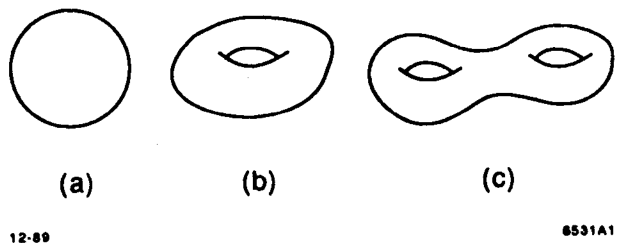

FIGURE 2

Closed 2d surfaces of lowest genus: (a) the sphere, with genus $h = 0$, (b) the torus, with $h = 1$, (c) the genus-two surface.

of 'vertex operators' into (6) in order to create the proper boundary conditions on the world-sheet; they will be discussed in section V.C.

The use of $S(X, g)$ rather than $S(X)$ in (6) again greatly simplifies the path integration over X^μ. The integral over $g_{\alpha\beta}$ can also be performed once we understand the gauge invariances of $S(X, g)$. The symmetries of $S(X, g)$ are:

(1) Two-dimensional reparametrizations (general coordinate transformations),

$$\begin{aligned} \sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha(\sigma), \\ g_{\alpha\beta}(\sigma) &\rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}\, g_{\gamma\delta}(\sigma), \\ X^\mu(\sigma) &\rightarrow X^\mu(\tilde{\sigma}). \end{aligned} \tag{7}$$

(2) Local changes of scale (Weyl transformations),

$$\begin{aligned} g_{\alpha\beta}(\sigma) &\rightarrow e^{\lambda(\sigma)} g_{\alpha\beta}(\sigma), \\ X^\mu(\sigma) &\rightarrow X^\mu(\sigma). \end{aligned} \tag{8}$$

(3) Global space-time Poincaré invariance,

$$\begin{aligned} g_{\alpha\beta}(\sigma) &\rightarrow g_{\alpha\beta}(\sigma), \\ X^\mu(\sigma) &\rightarrow \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu. \end{aligned} \tag{9}$$

Equation (9) is a global symmetry and as such it is rather easy to deal with. Equations (7) and (8) are local symmetries, however, and must be gauge-fixed in order to quantize $S(X, g)$.[38]

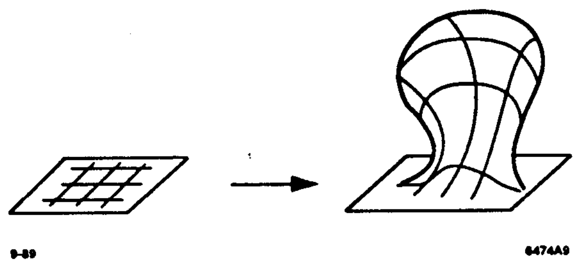

FIGURE 3

Example of a conformal transformation: a local change of scale that preserves angles.

There is a special class of coordinate transformations (7) under which the metric is merely rescaled by a scalar function of σ,

$$\begin{aligned} \sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha(\sigma), \\ g_{\alpha\beta}(\sigma) &\rightarrow \Omega(\sigma) g_{\alpha\beta}(\sigma). \end{aligned} \tag{10}$$

Such transformations are called *conformal transformations* because they are local changes of scale that preserve the angle between any two vectors u, v at a given point: $u \cdot v / |u||v| = g_{\alpha\beta} u^\alpha v^\beta \, (g_{\gamma\delta} u^\gamma u^\delta \, g_{\sigma\rho} v^\sigma v^\rho)^{-1/2}$ is invariant under the rescaling of $g_{\alpha\beta}$. An example of a conformal transformation is shown in figure 3.

Notice that the transformation of the metric in eq. (10) can be undone by a Weyl transformation (8), which does not affect $X^\mu(\sigma)$, so there is a combined symmetry of $S(X, g)$ that fixes $g_{\alpha\beta}$ and is a coordinate transformation acting on X^μ. This is significant because in many non-string-theory applications one is interested in two-dimensional field theories where the metric is fixed: usually it is fixed to the flat metric, either Minkowskian ($\eta_{\alpha\beta}$) or Euclidean ($\delta_{\alpha\beta}$). If the fields in such a theory are denoted by ϕ_i (replacing X^μ), and if the action $S(\phi_i)$ is invariant under coordinate transformations of the form (10) — where the fields ϕ_i are transformed but the metric appearing in S is left fixed — then the field theory is said to be conformally invariant. Like the Polyakov action, any Weyl-invariant and generally covariant action $S(\phi_i, g)$ reduces to a conformally invariant field theory with action $S(\phi_i)$ when the metric is considered fixed. Conversely, any conformal field theory gives rise to a Weyl-invariant theory when coupled to gravity. Conformal field theories play an important role in string theory because one can replace portions of the Polyakov action $S(X, g)$ by the action $S(\phi_i, g)$ for some other conformal field theory coupled to the 2d metric. The (phenomenological) reason for making this replacement is that, as will be seen in chapter IV, the Polyakov action only has a simple quantization if there are 26 space-time coordinates X^μ, which is 22 more than the four we observe. If the terms in the Polyakov action containing the 22 unwanted coordinates are replaced by an appropriate action $S(\phi_i, g)$, in a procedure known as compactification, then four-dimensional (super)string theories can be constructed. With this application in mind, I will now make a rather long digression to discuss the general properties of 2d conformal field theories. In chapter IV I will return to the problem of quantizing both the Polyakov action and its compactified counterparts (and the supersymmetric versions of these).

III. TWO-DIMENSIONAL CONFORMAL FIELD THEORY:

A. The Energy-Momentum Tensor and Primary Fields

The energy-momentum (or *stress*) tensor $T_{\alpha\beta}$ is of particular importance in a conformal field theory (CFT).[39] If we consider the CFT to be coupled to the 2d metric $g_{\alpha\beta}$, so its action has the form $S(\phi_i, g)$, then we can use the definition of $T_{\alpha\beta}$ suggested by general relativity,

$$T_{\alpha\beta} \equiv \frac{2\pi}{\sqrt{-g}} \cdot \frac{\delta S(\phi_i, g)}{\delta g^{\alpha\beta}}, \tag{11}$$

which is symmetric and (covariantly) conserved. Now as argued in the last section, $S(\phi_i, g)$ is invariant under infinitesimal Weyl transformations, which transform only the metric, $\delta g^{\alpha\beta} = -\delta\lambda \cdot g^{\alpha\beta}$. Thus the stress tensor is *traceless*:

$$0 = -\frac{2\pi}{\sqrt{-g}} \cdot \frac{\delta S}{\delta \lambda} = -\frac{2\pi}{\sqrt{-g}} \cdot \frac{\delta S}{\delta g^{\alpha\beta}} \cdot \frac{\delta g^{\alpha\beta}}{\delta \lambda} = T_{\alpha\beta}\, g^{\alpha\beta} = T^\alpha_\alpha. \tag{12}$$

If we don't couple the CFT to the metric, then the definition (11) is unavailable. However, it is still possible to construct a symmetric, conserved, traceless energy-momentum tensor.[40]

In two dimensions, the condition (12) that the stress tensor is traceless is quite powerful. To see this, take the 2d surface to have Euclidean signature $(+, +)$, and introduce complex coordinates, z and $\overline{z}$. The Polyakov path integral (6) receives contributions from surfaces of arbitrary genus; here I will concentrate mostly on the genus $h = 0$ term (the sphere), and later a little on the $h = 1$ term (the torus). If the surface has the topology of a sphere, it turns out that z and $\overline{z}$ can always be chosen so that the volume element has the form $d^2\sigma = e^{\lambda(z,\overline{z})} dz\, d\overline{z}$. A Weyl transformation can then be used to set $e^\lambda \equiv 1$, so that $d^2\sigma = dz\, d\overline{z}$. Thus the theory is effectively being considered on the complex z-plane. (The curvature of the sphere has been pushed off to $z = \infty$ in this description.) Alternatively, the theory might have been initially formulated on the plane with a flat metric, as would usually be the case in the statistical mechanics context. In any case, the components of the metric are

$$g_{z\overline{z}} = \tfrac{1}{2}, \qquad g_{zz} = g_{\overline{z}\overline{z}} = 0. \tag{13}$$

Now all tensor fields $\phi_{\alpha_1 \ldots \alpha_m}^{\beta_1 \ldots \beta_n}$ carry $z, \overline{z}$ indices, which can be lowered using the metric (13), to obtain $\phi_{z \ldots z \overline{z} \ldots \overline{z}}$ as the most general form.

In particular, the components of the stress tensor $T_{\alpha\beta}$ are

$$T_{zz} = \tfrac{1}{4}(T_{\tau\tau} - T_{\sigma\sigma} - 2iT_{\tau\sigma}), \quad T_{\overline{z}\overline{z}} = \tfrac{1}{4}(T_{\tau\tau} - T_{\sigma\sigma} + 2iT_{\tau\sigma}), \quad T_{z\overline{z}} = \tfrac{1}{4}(T_{\tau\tau} + T_{\sigma\sigma}). \tag{14}$$

Weyl-invariance then implies via eq. (12) that $T_{z\overline{z}} = 0$. Combining this with conservation of the stress tensor, $\partial^\alpha T_{\alpha\beta} = 0$, or in complex coordinates

$$\partial_{\overline{z}} T_{zz} + \partial_z T_{\overline{z}z} = \partial_z T_{\overline{z}\overline{z}} + \partial_{\overline{z}} T_{z\overline{z}} = 0, \tag{15}$$

one sees that the two remaining components of the stress tensor are a ***holomorphic*** component $T_{zz}(z) \equiv T(z)$ and an ***antiholomorphic*** component $T_{\overline{z}\overline{z}}(\overline{z}) \equiv \overline{T}(\overline{z})$. This feature allows one to say a great deal about the behavior of the stress tensor in any 2d CFT.

In two dimensions there are an infinite number of conformal transformations of the form (10). (There are only a finite number in more than two dimensions.) In fact, any analytic transformation of the complex coordinates,

$$z \rightarrow z' = f(z), \qquad \overline{z} \rightarrow \overline{z}' = \overline{f}(\overline{z}), \tag{16}$$

is a conformal transformation, leaving $g_{zz} = g_{\overline{z}\overline{z}} = 0$ and rescaling $g_{z\overline{z}}$ by $|dz/dz'|^2$. An arbitrary tensor field $\phi_{z \ldots z \overline{z} \ldots \overline{z}}$ with n of the z indices and $\overline{n}$ of the $\overline{z}$ indices transforms under (16) as

$$\phi_{z\ldots z\bar{z}\ldots\bar{z}}(z,\bar{z}) \;\rightarrow\; \phi'_{z\ldots z\bar{z}\ldots\bar{z}}(z') = \left(\frac{dz}{dz'}\right)^{h}\left(\frac{d\bar{z}}{d\bar{z}'}\right)^{\bar{h}}\phi_{z\ldots z\,\bar{z}\ldots\bar{z}}(z,\bar{z})\,. \qquad (17)$$

Here the exponents $h = n + \Delta$, $\bar{h} = \bar{n} + \Delta$ allow for the possibility of a scaling dimension Δ for ϕ in addition to its fixed Lorentz properties. (In two dimensions the properties of a field under the Lorentz group, $SO(1,1)$ or $SO(2) = U(1)$, are characterized entirely by the field's 'spin' $n - \bar{n} = h - \bar{h}$.) In fact, equation (17) only states how a classical tensor field transforms. At the quantum level, ϕ is promoted to an operator, and if it is a composite field there may be issues of operator ordering, *etc.* which may spoil eq. (17), as will be seen in a while. Nevertheless some fields — called *primary fields* — will retain this transformation law and will play an important role in the CFT.

B. Radial Quantization and Conserved Charges

Now consider the components of the stress tensor $T(z)$ and $\overline{T}(\bar{z})$ as operators in a quantized CFT. If the space direction is considered periodic, *i.e.* it is a circle, then space-time is a (Euclidean) cylinder, which can be parameterized by $\zeta = \tau + i\sigma$, $\bar{\zeta} = \tau - i\sigma$. Instead of quantizing the conformal field theory on the cylinder, it proves convenient to conformally map the cylinder to the complex plane, defining the coordinates of the plane to be

$$z = e^{\zeta} = e^{\tau + i\sigma}\,, \qquad \bar{z} = e^{\bar{\zeta}} = e^{\tau - i\sigma}\,. \qquad (18)$$

The distant past ($\tau = -\infty$) is mapped to the origin $z = 0$, and the distant future ($\tau = +\infty$) to $|z| = \infty$. The τ or 'time' direction now points radially outward from the origin. We continue to treat this direction as the quantization direction, however. In particular, correlation functions in this *radial quantization* scheme[39] are defined via radially-ordered products of fields, namely

$$\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle \;\equiv \langle 0\,|\,R(\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n))\,|\,0\rangle\,, \qquad (19)$$

where

$$R(\phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2)) \;=\; \begin{cases} \phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2) & \text{if } |z_1| > |z_2|, \\ \phi_2(z_2,\bar{z}_2)\,\phi_1(z_1,\bar{z}_1) & \text{if } |z_2| > |z_1|. \end{cases} \qquad (20)$$

The importance of $T(z)$ and $\overline{T}(\bar{z})$ is as the generators of infinitesimal conformal transformations. Every continuous symmetry of a field theory has associated with it a conserved current, $J^{\alpha}(\sigma)$ with $\partial_{\alpha}J^{\alpha} = 0$. The conserved charge

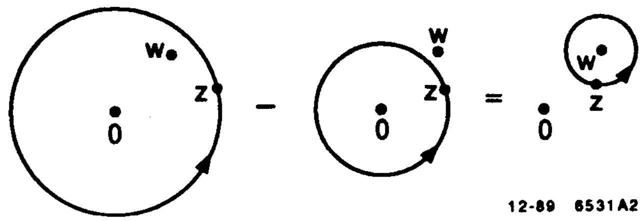

FIGURE 4

The difference between two contour integrals in z, one where z is radially outward from w and one where z is inward from w, is equal to the integral around w.

$Q = \int d^{d-1}\sigma\, J^0(t,\sigma)$ satisfies $dQ/dt = 0$ and generates the symmetry transformations of the fields: $[Q, \phi_i] = \delta\phi_i$. For an infinitesimal conformal transformation $\sigma^\alpha \to \sigma^\alpha - \epsilon^\alpha(\sigma)$ the conserved current is[39,40] $J^\alpha_{(\epsilon)} = T^\alpha{}_\beta \epsilon^\beta$. (One can easily check that $J^\alpha_{(\epsilon)}$ is conserved, using the symmetry, conservation and tracelessness of $T_{\alpha\beta}$, plus the relation $\partial_\alpha\epsilon_\beta + \partial_\beta\epsilon_\alpha = \frac{2}{d}(\partial\cdot\epsilon)\delta_{\alpha\beta}$, which ϵ^α must satisfy to be an infinitesimal conformal transformation (10).) In complex coordinates the conserved charges associated with $J^\alpha_{(\epsilon)}$ are

$$Q_{(\epsilon)} = \oint \frac{dz}{2\pi i}\,\epsilon(z)\,T(z) + \oint \frac{d\overline{z}}{2\pi i}\,\overline{\epsilon}(\overline{z})\,\overline{T}(\overline{z})\,, \tag{21}$$

where $\epsilon^z = \epsilon(z)$, $\epsilon^{\overline{z}} = \overline{\epsilon}(\overline{z})$, and the location of the contours depends on the location of $Q_{(\epsilon)}$ within a correlation function.

Now the conserved charges (21) must generate field transformations of the type (17) under $z \to z - \epsilon(z)$, $\overline{z} \to \overline{z} - \overline{\epsilon}(\overline{z})$. That is,

$$\begin{aligned}\delta_\epsilon\phi(w,\overline{w}) = {} & h(\partial_w\epsilon)\phi(w,\overline{w}) + \epsilon(w)\partial_w\phi(w,\overline{w}) + \overline{h}(\partial_{\overline{w}}\overline{\epsilon})\phi(w,\overline{w}) + \overline{\epsilon}(\overline{w})\partial_{\overline{w}}\phi(w,\overline{w}) \\ = {} & [Q_{(\epsilon)}, \phi(w,\overline{w})] = \oint_w \frac{dz}{2\pi i}\,\epsilon(z)\,T(z)\,\phi(w,\overline{w}) + \oint_{\overline{w}} \frac{d\overline{z}}{2\pi i}\,\overline{\epsilon}(\overline{z})\,\overline{T}(\overline{z})\,\phi(w,\overline{w})\,.\end{aligned} \tag{22}$$

The last expression follows from the (implicit) radial-ordering prescription (20): The commutator $[Q_{(\epsilon)}, \phi(w,\overline{w})]$ has two terms. In the term $Q_{(\epsilon)}\phi(w,\overline{w})$, the contour used in the integral expression (21) for $Q_{(\epsilon)}$ must lie radially outward from w, and in the term $\phi(w,\overline{w})$ it must lie inward from w. As depicted in figure 4, the two contours can be deformed into a small circle around w, denoted by $\oint_w$.

C. Operator Product Expansions and Commutation Relations

Equation (22) determines the short distance behavior of $T(z)$ and $\overline{T}(\overline{z})$ near $\phi(w,\overline{w})$. The short distance behavior of operators is usually described by an *operator product expansion*[41] (OPE),

$$\phi_i(x)\,\phi_j(y) \;\sim\; \sum_k C_{ijk}(x-y)\;\phi_k(y) \quad \text{as } x \to y. \tag{23}$$

The functions $C_{ijk}(x-y)$ are called operator product coefficients; their spatial dependence is determined by the Lorentz and scaling properties of the fields ϕ_i, ϕ_j, ϕ_k. In two dimensions they are all proportional to $(z-w)^{\alpha}(\overline{z}-\overline{w})^{\beta}$ for some α, β depending on i, j, k. In particular, the OPE of $T(z)$ with *any* field ϕ is constrained by the requirements of single-valuedness (as z moves around w) and $\overline{z}$-independence to have the form

$$T(z)\,\phi(w,\overline{w}) \;=\; \sum_{n\in\mathbf{Z}} (z-w)^n\;\mathcal{O}_n(w,\overline{w})\,. \tag{24}$$

Substitution of the expansion (24) into (22) leads to

$$\begin{aligned} T(z)\,\phi(w,\overline{w}) \;&\sim\; \frac{h}{(z-w)^2}\;\phi(w,\overline{w}) + \frac{\partial_w\phi(w,\overline{w})}{z-w} + \text{finite}, \\ \overline{T}(\overline{z})\,\phi(w,\overline{w}) \;&\sim\; \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\;\phi(w,\overline{w}) + \frac{\partial_{\overline{w}}\phi(w,\overline{w})}{\overline{z}-\overline{w}} + \text{finite}. \end{aligned} \tag{25}$$

A field ϕ that has an OPE with the stress tensor of the form (25) — or equivalently, that transforms like eqs. (17), (22) under conformal transformations — is called a *primary field* or *conformal field*, and $(h,\overline{h})$ are called its *conformal* or *scaling dimensions.*[39]

As remarked above, fields which transform like eq. (17) at the classical level may fail to do so at the quantum level. The prime example of this phenomenon is the stress tensor itself: $T(z)$ has $(h,\overline{h}) = (2,0)$, but its OPE with itself has an additional term:[39]

$$T(z)\,T(w) \;\sim\; \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{z-w} + \text{finite}. \tag{26}$$

Equation (26) is the most general form of the OPE consistent with the holomorphicity of $T(z)$ and its role as the generator of conformal transformations. Here c

is a real number, called the *central charge*, which depends on the conformal field theory and is nonzero in any nontrivial theory. Later in this chapter c will be calculated for some simple CFTs. There is an analogous OPE of $\overline{T}(\overline{z})$ with itself,

$$\overline{T}(\overline{z})\,\overline{T}(\overline{w}) \;\sim\; \frac{c/2}{(\overline{z}-\overline{w})^4} + \frac{2\overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\partial_{\overline{w}}\overline{T}}{\overline{z}-\overline{w}} + \text{finite}. \tag{27}$$

In section II.C it was mentioned that any CFT, when coupled to the 2d metric, gives rise to a Weyl-invariant theory. Actually this is not quite true, due to the presence of the central charge c. Notice that*

$$\partial_{\overline{z}} T_{z\bar{z}}(z)\, T_{ww}(w) \;=\; \partial_{\overline{z}}\left[\partial_z^2\partial_w\left(\frac{c/12}{z-w}\right)+\ldots\right] \;=\; c\cdot\frac{\pi}{12}\,\partial_z^2\partial_w\delta^{(2)}(z-w)+\ldots, \tag{28}$$

where the identity

$$\partial_{\overline{z}}\partial_z \ln|z-w|^2 \;=\; \partial_{\overline{z}}\left(\frac{1}{z-w}\right) \;=\; \pi\delta^{(2)}(z-w) \tag{29}$$

has been used. Taking $\partial_{\overline{w}}$ of eq. (28) and then using conservation of $T_{\alpha\beta}$ (15) to replace T_{zz}, T_{ww} by $T_{z\overline{z}}, T_{w\overline{w}}$, one gets

$$T_{z\overline{z}}(z)\, T_{w\overline{w}}(w) \;=\; c\cdot\frac{\pi}{12}\,\partial_z\partial_{\overline{w}}\,\delta^{(2)}(z-w). \tag{30}$$

On the other hand, $T_{z\overline{z}}$ is related via eq. (12) to the variation of the action — or at the quantum level, the effective action — under a Weyl transformation (8): $T_{z\overline{z}} \sim \delta S_{\text{eff}}/\delta\lambda$. If we write $g_{\alpha\beta}(\sigma) = e^{\lambda(\sigma)}\hat{g}_{\alpha\beta}(\sigma)$, where $\hat{g}_{\alpha\beta}$ is some reference metric and λ is the *conformal factor*, then equation (30) means that the quantum path integral Z for the CFT depends on the conformal factor, according to

$$Z(g_{\alpha\beta}) \;=\; Z(\hat{g}_{\alpha\beta})\cdot\exp\left[-\frac{c}{96\pi}\int d^2\sigma\,\partial_\alpha\lambda\partial^\alpha\lambda\right]. \tag{31}$$

This formula will be needed when we return to the quantization of the Polyakov action in chapter IV.

An important feature of holomorphic fields like $T_{zz}(z)$ is that they can be Laurent expanded. Given a holomorphic field $A(z)$ with dimension h (A need not

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* I follow the discussion of ref. 7 here.

be primary), the Laurent modes A_n are defined by

$$\begin{cases} A(z) &= \displaystyle\sum_{n\in\mathbf{Z}-h} A_n\, z^{-n-h}, \\ A_n &= \displaystyle\oint \frac{dz}{2\pi i}\, A(z)\, z^{n+h-1}. \end{cases} \tag{32}$$

The sum over the integers shifted by h in eq. (32) ensures that the field $A(z)$ is well-defined near the origin. The modes of the stress tensor are called L_n and $\overline{L}_n$:

$$\begin{cases} T(z) = \displaystyle\sum_{n\in\mathbf{Z}} L_n\, z^{-n-2}, & \overline{T}(\overline{z}) = \displaystyle\sum_{n\in\mathbf{Z}} \overline{L}_n\, \overline{z}^{-n-2}, \\ L_n = \displaystyle\oint \frac{dz}{2\pi i}\, T(z)\, z^{n+1}, & \overline{L}_n = \displaystyle\oint \frac{d\overline{z}}{2\pi i}\, \overline{T}(\overline{z})\, \overline{z}^{n+1}. \end{cases} \tag{33}$$

OPE's like (25) and (26) can be converted into (anti-)commutation relations for the Laurent modes, and vice-versa. For example, consider the commutator $[L_m, L_n]$. Insert the integral representation (33) for L_m and L_n, recalling that the location of the contours is specified by the radial-ordering prescription (20), and using the argument shown in figure 4:

$$\begin{aligned} [L_m, L_n] &= \left[\oint_{|z|>|w|} \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} - \oint_{|z|<|w|} \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \right] \left[T(z) z^{m+1}\, T(w) w^{n+1} \right] \\ &= \oint_0 \frac{dw}{2\pi i}\, w^{n+1} \oint_w \frac{dz}{2\pi i} z^{m+1}\, T(z)T(w)\,. \end{aligned} \tag{34}$$

Next insert OPE (26) into eq. (34), and expand $z^{m+1} = [w + (z-w)]^{m+1}$ around w in order to evaluate the z integral by residues,

$$[L_m, L_n] = \oint_0 \frac{dw}{2\pi i}\, w^{m+n+2} \left[\frac{c}{2} \binom{m+1}{3} w^{-3} + 2(m+1) w^{-1} T(w) + \partial_w T \right]. \tag{35}$$

Finally substitute the mode expansion (33) of $T(w)$ into (35) and do the w integral to get the *Virasoro algebra*[42] satisfied by the L_n's,

$$[L_m, L_n] = \frac{c}{12}\, (m^3 - m)\, \delta_{m+n,0} + (m-n) L_{m+n}. \tag{36}$$

Similarly one finds that the $\overline{L}_m$'s satisfy exactly the same algebra, and that L_m and $\overline{L}_n$ commute. Also, if $A(z)$ is now a holomorphic *primary* field of dimension

h, equations (25), (32) and (33) combine to give

$$[L_m, A_n] = ((h-1)m - n)A_{m+n}. \tag{37}$$

D. Primary and Descendant States

Using equation (21), one can see that the conformal transformations generated by L_n and $\overline{L}_n$ are $z \to z - \epsilon_n z^{n+1}$, where ϵ_n is a complex constant. Thus L_{-1}, $\overline{L}_{-1}$ generate uniform translations of the complex plane, $z \to z - \epsilon_{-1}$, and L_0, $\overline{L}_0$ generate dilatations and rotations, $z \to (1-\epsilon_0)z$. Since the radial direction is the 'time' direction, dilatations are time translations, and $L_0 + \overline{L}_0$ plays the role of the Hamiltonian. The energies of states are given by their L_0, $\overline{L}_0$ eigenvalues. 'In states' $|\phi\rangle$ are created by fields acting in the distant past, at $z = 0$: $|\phi\rangle = \phi(0)\,|0\rangle$. Similarly, $\langle\phi| \sim \langle 0|\,\phi(\infty)$. If $\phi(z,\overline{z})$ is a primary field, then $|\phi\rangle$ is said to be a *primary state*. Primary states are annihilated by the L_n modes with positive n:

$$\begin{aligned} L_n\,|\phi\rangle &= \oint_0 \frac{dz}{2\pi i}\, z^{n+1} T(z)\,\phi(0)\,|0\rangle = \oint_0 \frac{dz}{2\pi i}\, z^{n+1}\left[z^{-2}h\phi(0) + z^{-1}\partial\phi(0)\right]|0\rangle \\ &= \begin{cases} 0 & \text{if } n \geq 1, \\ h\,|\phi\rangle & \text{if } n = 0. \end{cases} \end{aligned} \tag{38}$$

Note from eq. (36) that $[L_0, L_n] = -nL_n$, so the operator L_n lowers the eigenvalue of L_0 by n units. (The same is also true of the more general operator A_n, as seen from eq. (37).) Thus a tower of *descendant states* above any primary state $|\phi\rangle$ is built up by acting with the L_0-raising operators L_{-n} $(n > 0)$ to get

$$(L_{-l})^{i_l} \ldots (L_{-2})^{i_2}(L_{-1})^{i_1}\,|\phi\rangle\,, \tag{39}$$

where the displayed state has $L_0 = h + \sum_n n i_n$. See figure 5.

In fact, *any* state in the CFT is either a primary state or a descendant of the form (39) (using also the modes $\overline{L}_{-n}$ of the antiholomorphic stress tensor $\overline{T}$), or is some linear combination thereof. In this way the Virasoro algebra is said to 'organize' the CFT. The properties of descendant states (fields) are closely related to, and are straightforward to determine from, the properties of the corresponding primary fields,[39] so most attention is usually focussed on the primaries. To summarize, a CFT is characterized (although not quite uniquely specified) by its

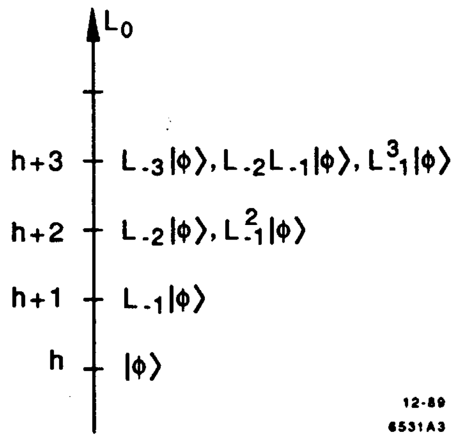

FIGURE 5

The tower of descendant states above a primary state $|\phi\rangle$.

central charge c and the spectrum of conformal dimensions $(h_i, \overline{h}_i)$ of its primary fields ϕ_i. If c is less than 1 it turns out that the CFT has only a finite number of primary fields,[43] and all correlation functions for the CFT can be determined exactly using only the Virasoro algebra![39,44] If $c \geq 1$ there is an infinite number of primary fields[45] and more information is needed to solve the theory.

If a CFT with $c \geq 1$ contains holomorphic fields $A^i(z)$ besides $T(z)$ they may be used to help solve the theory. First the tower of states in figure 5 can be enlarged by including states where the modes A^i_{-n} as well as L_{-n} act on $|\phi\rangle$. This organizes the CFT into 'primary' and 'descendant' states, not with respect to the Virasoro algebra, but with respect to the larger *chiral algebra* generated by the fields $A^i(z)$ and $T(z)$. If there are only a finite number of primary fields under this larger algebra, then correlation functions in the CFT can again be determined (at least in principle). The CFTs that can be solved in this way are called *rational conformal field theories.*[34]

E. Restrictions on Conformal Field Theories

There are various properties that a CFT must have in order to be of use in string theory. (Some of the properties are required by other CFT applications too.) I will describe three of the most important ones here.

(1) *Unitarity.* All states in the 2d CFT Hilbert space should have non-negative norm. In string theory, negative norm states in the 2d Hilbert space lead to negative norms for particle states in space-time. Unless the negative norm states decouple from scattering amplitudes of physical particles, it is impossible to have a unitary time evolution, as is required by quantum mechanics. Some negative-norm states are automatically decoupled in string theory, namely the time-like polarization states of vector and tensor particles, *etc.*, as will be seen in section IV.C. On the other hand, the CFTs that describe string 'compactifications' (see sections IV.B and IV.C) have to be unitary, because no mechanism exists for decoupling any negative norm states that they might contain.[30]

To check unitarity in a CFT, one has to calculate norms, which requires knowing how Hermitian conjugation acts on operators. The action is easiest to determine in Minkowski space, that is by mapping the z-plane back to the cylinder, and then Wick-rotating so that it becomes Minkowskian. One finds that the modes A_n of any real, holomorphic field $A(z)$ satisfy $A_n^\dagger = A_{-n}$. In particular, the Virasoro modes satisfy $L_n^\dagger = L_{-n}$. Using this information it is possible to show,[43] for example, that there is only a discrete set of unitary CFTs with central charge $c < 1$, which have

$$c \;=\; 1 - \frac{6}{m(m+1)}\,, \qquad m = 2,3,4,\ldots, \tag{40}$$

and a finite number of primary fields for each m, with conformal dimensions

$$h_{p,q}(m) \;=\; \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}\,, \qquad 1 \le p \le m-1, \quad 1 \le q \le p. \tag{41}$$

The restrictions imposed by unitarity alone on CFTs with central charge $c \ge 1$ are very weak; however, unitarity in combination with other constraints can be useful.

(2) *Single-valued correlation functions.* The quantity $\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle$ should be well-defined on the complex plane, which means that it should not change when any point z_i is carried continuously around any other point z_j.

(3) *Modular invariance.* The Polyakov path integral (6), when generalized to include external states, requires correlation functions to be well-defined on any

closed 2d surface, and different parametrizations of the same 2d surface should give the same answer. An important feature of surfaces with genus $h \geq 1$ is a set of complex parameters, or *moduli*,[46] which are global obstructions to the choice of coordinates that we made on the sphere ($h = 0$), namely $z, \bar{z}$ with $d^2\sigma = e^{\lambda(z,\bar{z})} dz d\bar{z}$. Surfaces with $h \geq 2$ have $3h - 3$ complex moduli, denoted by τ_i, $i = 1, 2, \ldots, 3h - 3$. The torus ($h = 1$) has one complex modulus, denoted by τ (which should not be confused with the world-sheet time parameter τ used earlier). For each surface with $h \geq 1$ there are 'large' reparametrizations, called *modular transformations*, which are not continuously connected to the identity, and which change the values of the moduli, but which do not change the shape of the surface. Any correlation function on the surface should therefore be invariant under modular transformations.

The simplest modular transformations are those of the torus. Also, modular invariance of the vacuum amplitude on the torus (usually called the *partition function*) is one of the strongest constraints on a CFT.[45,47,25] A two-dimensional torus can be constructed by identifying points on the complex z-plane that differ by vectors of a 2d lattice. It is conventional to set the second of the two basis vectors of the lattice to 1 by a scale transformation. Then the identification of points is

$$z \equiv z + \tau, \qquad z \equiv z + 1, \tag{42}$$

where τ is the modulus of the torus (see figure 6(a)).

In equation (42), *any* pair of basis vectors for the lattice — not just the pair $(\tau, 1)$ — will yield the same torus. The most general basis is $(a\tau + b, c\tau + d)$, where a, b, c, d are integers and $ad - bc = 1$. Again rescaling the second vector to 1, all moduli of the form $(a\tau + b)/(c\tau + d)$ for fixed τ are seen to give equivalent tori. Thus the modular transformations of the torus are

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1. \tag{43}$$

All of these can be generated by the two transformations

$$T: \ \tau \to \tau + 1\,, \qquad S: \ \tau \to -\frac{1}{\tau}\,, \tag{44}$$

which are shown in figures 6(b,c). T corresponds to cutting the torus in figure 2(b) along a circle, twisting one end by 2π, and then rejoining; S corresponds to interchanging the two intersecting noncontractible circles on the torus. So modular

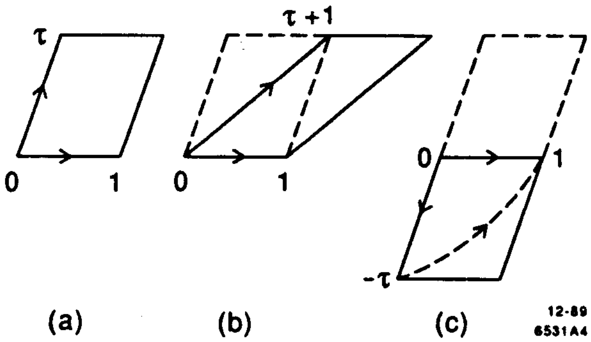

FIGURE 6

(a) A torus with modular parameter τ. (b) The modular transformation $\tau \to \tau + 1$. (c) The modular transformation $\tau \to -1/\tau$, which includes a rescaling of the plane by $-1/\tau$.

invariance of the partition function $Z(\tau, \overline{\tau})$ requires

$$Z(\tau, \overline{\tau}) \;=\; Z(\tau + 1, \overline{\tau} + 1) \;=\; Z(-1/\tau, -1/\overline{\tau}). \tag{45}$$

One can think of the torus with modulus τ as a cylinder of length $2\pi(\mathrm{Im}\tau)$ in the Euclidean time direction* (the $\mathrm{Im}z$ direction in the z-plane), whose ends have been glued together with periodic boundary conditions. This gluing corresponds to taking a trace over the CFT Hilbert space, so that

$$Z(\tau, \overline{\tau}) \;=\; (q\overline{q})^{-c/24}\,\mathrm{Tr}(q^{L_0}\overline{q}^{\overline{L}_0}), \qquad q = e^{2\pi i \tau}, \quad \overline{q} = e^{-2\pi i \overline{\tau}}. \tag{46}$$

The factors inside the trace evolve the states for a Euclidean time $2\pi(\mathrm{Im}\tau)$ using the Hamiltonian $H = L_0 + \overline{L}_0$, and translate them spatially by a distance $2\pi(\mathrm{Re}\tau)$ using the translation generator $P = L_0 - \overline{L}_0$, as is required by the identification of points in (42). The prefactor accounts for a vacuum energy of $-c/12$ when the

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⋆ The 2π rescaling is required here to agree with our previous convention that the spatial direction of the cylinder has period 2π, not 1 as in (42).

torus is constructed as in (42). The condition that the expression (46) for $Z(\tau,\overline{\tau})$ is invariant under T is easy to impose:

$$Z(\tau+1,\overline{\tau}+1) \;=\; (q\overline{q})^{-c/24}\,\mathrm{Tr}(q^{L_0}\overline{q}^{\overline{L}_0}\; e^{2\pi i(L_0-\overline{L}_0)}), \tag{47}$$

so the requirement is that $L_0-\overline{L}_0$ must be an integer for every state in the theory. The S invariance condition is less trivial; we will see in the next few sections how it is satisfied by partition functions for some simple CFTs.

F. Massless Free Boson Example

Now that I have described some of the general properties of 2d CFTs, I turn to a few simple examples, which are just free field theories. Even though they are rather trivial as 2d field theories, they nevertheless have nontrivial string theory applications.

First consider a massless free boson X, with the action

$$S(X) \;=\; \frac{1}{8\pi}\int d^2\sigma\; \partial_\alpha X \partial^\alpha X. \tag{48}$$

This action is precisely the Polyakov action (4), with the 2d metric fixed to be flat, and with X interpreted as a single space-time coordinate. The propagator for X is obtained as usual by inverting the kinetic term, so it satisfies $\partial_\alpha\partial^\alpha\langle X(\sigma)X(\sigma')\rangle = -4\pi\delta^{(2)}(\sigma-\sigma')$, or $\partial_{\overline{z}}\partial_z\langle X(z,\overline{z})X(w,\overline{w})\rangle = -\pi\delta^{(2)}(z-w)$, which is solved by

$$\langle X(z,\overline{z})X(w,\overline{w})\rangle \;=\; -\ln|z-w|^2. \tag{49}$$

From the propagator (49) one can also read off

$$\begin{aligned} \langle \partial_z X \partial_w X\rangle &= -\frac{1}{(z-w)^2}\,, \\ \langle \partial_{\overline{z}} X \partial_{\overline{w}} X\rangle &= -\frac{1}{(\overline{z}-\overline{w})^2}\,, \\ \langle \partial_z X \partial_{\overline{w}} X\rangle &= \pi\delta^{(2)}(z-w)\,. \end{aligned} \tag{50}$$

The vacuum expectation values (50) also give the singular terms in the corresponding OPEs. The reason is that the products on the left-hand side of OPEs are implicitly 'time'-ordered (radially-ordered), whereas the composite operators appearing on the right-hand side of such OPEs are generally normal-ordered. Wick's theorem states that the difference between the two kinds of products is obtained by Wick contractions, which for $\partial_z X \partial_w X$, *etc.*, is just the corresponding vacuum expectation value. For example,

$$\begin{aligned} \partial_z X \, \partial_w X &= \frac{-1}{(z-w)^2} + :\partial_z X \partial_w X: \\ &\sim \frac{-1}{(z-w)^2} + :\partial_w X \partial_w X: + O(z-w). \end{aligned} \tag{51}$$

As usual, (51) can be converted to commutation relations for the modes α_n of $\partial_z X$, which are defined by

$$\left\{ \begin{aligned} \partial_z X &= -i \sum_{n \in \mathbf{Z}} \alpha_n \, z^{-n-1}, \\ \alpha_n &= i \oint \frac{dz}{2\pi i} \, \partial_z X \, z^n. \end{aligned} \right. \tag{52}$$

They obey $[\alpha_m, \alpha_n] = m \delta_{m+n,0}$. Similarly the modes $\overline{\alpha}_n$ of $\partial_{\overline{z}} X$ obey $[\overline{\alpha}_m, \overline{\alpha}_n] = m \delta_{m+n,0}$, and $[\alpha_m, \overline{\alpha}_n] = 0$. These are just the commutation relations of an infinite set of harmonic oscillator creation and annihilation operators, $a_n^\dagger \equiv \frac{1}{\sqrt{n}} \alpha_{-n}$, $\quad a_n \equiv \frac{1}{\sqrt{n}} \alpha_n$.

To show that the massless free boson theory actually is conformally invariant, it suffices to show that the component T_{zz} of the stress tensor is holomorphic and has the OPE (26). T_{zz} can be obtained either by the Noether procedure or by the definition $T_{zz} \sim \delta S / \delta g^{zz}$. One finds (after removing the factor of $\frac{1}{2\pi}$ which appears explicitly in eq. (21) and defining the composite operator via normal-ordering) the free boson stress tensor

$$T_{zz} = T(z) = -\tfrac{1}{2} :\partial_z X \partial_z X: \, . \tag{53}$$

First let's check the OPE of $T(z)$ with $\partial_w X$, using (two) Wick contractions to obtain the singular terms:

$$\begin{aligned} T(z) \, \partial_w X = -\frac{1}{2} :\partial_z X \partial_z X: \, \partial_w X &\sim -\frac{1}{2} \cdot 2 \cdot \frac{-1}{(z-w)^2} \, \partial_z X + \dots \\ &\sim \frac{1}{(z-w)^2} \, \partial_w X + \frac{1}{z-w} \, \partial_w (\partial_w X) + \dots \end{aligned} \tag{54}$$

Equation (54) verifies (*cf.* eq. (25)) that $\partial_z X$ is a primary field of dimension $h = 1$. The OPE of $\overline{T}$ with $\partial_z X$ is finite (up to delta functions), so $\overline{h} = 0$. These are precisely the dimensions of the classical tensor field $\partial_z X$, so the free field has acquired no anomalous dimension.

The OPE of T with itself is computed similarly. There are two ways to do the double Wick contractions for the most singular term, and four ways to do a single contraction:

$$\begin{aligned} T(z)T(w) &= \frac{1}{4} : \partial_z X \partial_z X :: \partial_w X \partial_w X : \sim \frac{\frac{1}{4}\cdot 2}{(z-w)^4} + \frac{\frac{1}{4}\cdot 4 \cdot (-1)}{(z-w)^2} \partial_z X \partial_w X + \dots \\ &\sim \frac{1/2}{(z-w)^4} + \frac{2(-\frac{1}{2}\partial_w X \partial_w X)}{(z-w)^2} + \frac{\partial_w(-\frac{1}{2}\partial_w X \partial_w X)}{z-w} + \dots , \end{aligned} \tag{55}$$

which shows that the massless free boson is a CFT, with central charge $c = 1$ (*cf.* eq. (26)). Note that the Polyakov action with D space-time dimensions contains D free bosons, X^μ, $\mu = 0, 1, \dots, D-1$, and that such a system has central charge $c = D$.

Besides ∂X^μ and $\overline{\partial} X^\mu$, there is another important set of primary fields that one can build out of D free bosons, namely the normal-ordered exponentials

$$V_k(z,\overline{z}) = : \exp\left(ik \cdot X(z,\overline{z})\right) : . \tag{56}$$

One can easily use Wick contractions to derive the OPE

$$\partial_z X^\mu \, e^{ik\cdot X(w,\overline{w})} \sim \frac{-ik^\mu}{z-w} \, e^{ik\cdot X(w,\overline{w})} + \dots , \tag{57}$$

and from it

$$T(z) \, e^{ik\cdot X(w,\overline{w})} \sim -\frac{1}{2}\left(\frac{-ik^\mu}{z-w}\right)^2 e^{ik\cdot X(w,\overline{w})} + \dots \tag{58}$$

(and an analogous OPE with $\overline{T}$), so that the conformal dimensions of V_k are $h = \overline{h} = k^2/2$.

Acting on the vacuum state $|0\rangle$, V_k creates a state $|k\rangle \equiv e^{ik\cdot\hat{x}}\,|0\rangle$, with 'momentum' k. Here $\hat{x}$ is the zero-mode of X, which is canonically conjugate to the 'momentum' $\hat{p} \equiv \alpha_0 = \overline{\alpha}_0$: $[\hat{x}, \hat{p}] = i$. 'Momentum' here is not 2d momentum (we always work in position space in 2d), but rather a global internal symmetry of the 2d CFT, which in string theory is interpreted as *space-time* momentum. The states $|k\rangle$, like the vacuum $|0\rangle$, are annihilated by all of the positive frequency modes (annihilation operators) $\alpha_m, \overline{\alpha}_n$. The complete free boson Hilbert space is obtained by acting on the states $|k\rangle$ with all the negative frequency modes (creation operators); it consists of all states of the form

$$(\alpha_{-l})^{i_l} \dots (\alpha_{-1})^{i_1} (\overline{\alpha}_{-m})^{j_m} \dots (\alpha_{-1})^{j_1} |k\rangle , \quad k \in \mathbf{R}. \tag{59}$$

One can check that the free boson theory is unitary by showing explicitly that every state in the Hilbert space has positive norm, using the Hermitian conjugation rules $\alpha_n^\dagger = \alpha_{-n}$, $\overline{\alpha}_n^\dagger = \overline{\alpha}_{-n}$, the oscillator commutation relations, and the norm $\langle k|k\rangle = 1$ for the momentum states.

With the full Hilbert space (59) one can also compute the partition function $Z(\tau, \overline{\tau})$ and verify that it is modular invariant. It is easy to see that the trace (46) is a product of independent contributions from the different oscillators α_{-n} and the 'momentum' k. The contribution of the states $\{(\alpha_{-n})^i |0\rangle , \; i = 0, 1, 2, \dots\}$, with $L_0 = in$, is

$$\sum_{i=0}^{\infty} q^{in} \;=\; 1 + q^n + q^{2n} + \dots \;=\; \frac{1}{1 - q^n}. \tag{60}$$

The contribution of the continuum of states $|k\rangle$, with $L_0 = k^2/2$, is

$$\int_{-\infty}^{\infty} dk \, (q\overline{q})^{k^2/2} \;=\; \int_{-\infty}^{\infty} dk \, e^{-2\pi \, \mathrm{Im}\tau \, k^2} \;=\; (2\, \mathrm{Im}\tau)^{-1/2}. \tag{61}$$

Putting all the factors together, the free boson partition function is

$$Z(\tau, \overline{\tau}) \;=\; (2\, \mathrm{Im}\tau)^{-1/2} \left| q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right|^2 \;=\; (2\, \mathrm{Im}\tau)^{-1/2} |\eta(\tau)|^{-2}, \tag{62}$$

where $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function. The modular transformation properties of the eta function are well known,[5]

$$\eta(\tau + 1) \;=\; e^{i\pi/12} \eta(\tau) \, , \quad \eta(-1/\tau) \;=\; (-i\tau)^{1/2} \eta(\tau) \, .$$

Using also $\mathrm{Im}(-1/\tau) = (\tau\overline{\tau})^{-1} \mathrm{Im}\tau$, we see that $Z(\tau, \overline{\tau})$ is indeed modular invariant.

G. Free Boson on a Circle

The free boson CFT can be modified slightly by changing the boundary conditions on X. Before it was assumed that the field X could take on any real value. Now let us take the values of X to lie on a circle of radius R. That is, we identify

$$X \equiv X + 2\pi R. \tag{63}$$

This identification has two consequences. First, the 'momenta' k are quantized, because the operator $\exp(i\hat{p}\cdot 2\pi R)$ that translates states by $2\pi R$ must now be trivial, *i.e.,* equal to 1. Thus for any k, $\exp(ik\cdot 2\pi R) = 1$, or $k = m/R$ for some integer m. Second, there are new 'winding' states in the theory, because the boundary conditions

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R\cdot n, \qquad n \in \mathbf{Z} \tag{64}$$

are allowed by the identification (63).

To describe the new states we define left-moving and right-moving components of $X(z,\overline{z})$,

$$X(z,\overline{z}) = X_L(z) + X_R(\overline{z}), \tag{65}$$

with OPEs

$$X_L(z)X_L(w) \sim -\ln(z-w), \qquad X_R(\overline{z})X_R(\overline{w}) \sim -\ln(\overline{z}-\overline{w}), \tag{66}$$

and mode expansions

$$\begin{aligned} X_L(z) &= \hat{x}_L - i\hat{p}_L \ln z + i\sum_{n\neq 0} \frac{1}{n}\,\alpha_n z^{-n}, \\ X_R(\overline{z}) &= \hat{x}_R - i\hat{p}_R \ln \overline{z} + i\sum_{n\neq 0} \frac{1}{n}\,\overline{\alpha}_n \overline{z}^{-n}. \end{aligned} \tag{67}$$

For momentum $k = m/R$ and winding number n in (64) the eigenvalues of the zero-modes $(\hat{p}_L, \hat{p}_R)$ are

$$(k_L, k_R) = \left(\frac{m}{R} + \frac{nR}{2}, \frac{m}{R} - \frac{nR}{2}\right), \qquad m, n \in \mathbf{Z}. \tag{68}$$

None of the nonzero modes $\alpha_n, \overline{\alpha}_n$ are affected by the change in boundary conditions on X, so the Hilbert space (59) is unaltered except for the replacement of $|k\rangle$ by $|k_L, k_R\rangle$, with (k_L, k_R) given by (68).

The field $X_L(z)$ has a multi-valued OPE (66) with itself. However, $X_L(z)$ does not appear by itself in the CFT of a free boson on a circle, and the fields that do appear are cleverly arranged to have single-valued OPEs and hence single-valued correlation functions. Let us check that the primary fields $e^{i(k_L X_L + k_R X_R)}$ making the states $|k_L, k_R\rangle$ have single-valued OPEs. Using the Wick contractions (66) repeatedly, one can show that

$$\begin{aligned} & e^{i(k_L X_L + k_R X_R)}(z,\bar{z})\; e^{i(k'_L X_L + k'_R X_R)}(w,\bar{w}) \\ & \quad \sim\; (z-w)^{k_L\cdot k'_L}(\bar{z}-\bar{w})^{k_R\cdot k'_R}\; e^{i((k_L+k'_L)X_L + (k_R+k'_R)X_R)}(w,\bar{w}) + \ldots . \end{aligned} \tag{69}$$

If z is carried around w, so that $z-w \to (z-w)e^{2\pi i}$, $\bar{z}-\bar{w} \to (\bar{z}-\bar{w})e^{-2\pi i}$, then the OPE (69) develops a phase $\exp 2\pi i(k_L\cdot k'_L - k_R\cdot k'_R)$, which for *arbitrary* (k_L, k_R) and (k'_L, k'_R) is not equal to 1. But the values of k_L, k_R should be taken from the set (68), in which case the phase is

$$\begin{aligned} & \exp 2\pi i\left[\left(\tfrac{m}{R}+\tfrac{nR}{2}\right)\left(\tfrac{m'}{R}+\tfrac{n'R}{2}\right)-\left(\tfrac{m}{R}-\tfrac{nR}{2}\right)\left(\tfrac{m'}{R}-\tfrac{n'R}{2}\right)\right] \\ & \quad = \exp 2\pi i(mn' + m'n) = 1, \end{aligned} \tag{70}$$

so the OPE is single-valued.

Let us also compute the new partition function $Z_R(\tau,\bar{\tau})$ and show that it is modular invariant. The only difference from the free boson with $X \in \mathbf{R}$ is that the integral over continuous momenta k is replaced by a discrete sum over momenta (labelled by m) and winding numbers (labelled by n). So the modular-invariant partition function $Z_0(\tau,\bar{\tau})$ in equation (62) is replaced by

$$Z_R(\tau,\bar{\tau}) \;=\; (2\,\mathrm{Im}\tau)^{1/2}\cdot \sum_{m,n=-\infty}^{\infty} q^{(\frac{m}{R}+\frac{nR}{2})^2/2}\bar{q}^{(\frac{m}{R}-\frac{nR}{2})^2/2}\cdot Z_0(\tau,\bar{\tau}). \tag{71}$$

Z_R is invariant under $\tau \to \tau+1$ because $\frac{1}{2}[(\frac{m}{R}+\frac{nR}{2})^2 - (\frac{m}{R}-\frac{nR}{2})^2] = mn$ is an integer. To show that Z_R is invariant under $\tau \to -1/\tau$ one has to 'Poisson resum' the expression in (71), as follows:[5] For any function $f(x)$, the sum over lattice points can be rewritten as a sum of the Fourier transform $\tilde{f}(p)$ over the points of the dual lattice,

$$\sum_{x\in\mathbf{Z}^n} f(x) \;=\; \sum_{p\in\mathbf{Z}^n}\tilde{f}(p), \quad \text{where } \tilde{f}(p) \;=\; \int d^n x\, f(x)\, e^{-2\pi i p\cdot x}. \tag{72}$$

Here we let $f(x) = \exp(-\pi x^T A x)$, where

$$A(\tau) \;=\; \begin{pmatrix} \frac{2\,\mathrm{Im}\tau}{R^2} & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & \frac{R^2\,\mathrm{Im}\tau}{2}, \end{pmatrix} \tag{73}$$

which gives $\widetilde{f}(p) = (\det A)^{-1/2} \exp(\pi p^T A^{-1} p)$. But $\det A = |\tau|^2$ and

$$A^{-1}(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A(-1/\tau) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{74}$$

After one exchanges m and n in the sum in (71) and uses $(\mathrm{Im}\tau)^{1/2} = |\tau|(\mathrm{Im}(-1/\tau))^{1/2}$, one sees that Z_R is indeed invariant under S.

H. Massless Free Fermion Example

Our last example of a CFT is a massless free Majorana-Weyl fermion. A 2d spinor Ψ has two components, $\Psi^T = (\psi \;\; \overline{\psi})$. Choose a basis for the gamma matrices where γ_3 is diagonal, so the components ψ and $\overline{\psi}$ describe Weyl fermions of opposite chirality. The Majorana condition is that the components are real (in Minkowski space), $\psi^* = \psi$, $\overline{\psi}^* = \overline{\psi}$. The action is

$$S = \frac{1}{8\pi} \int d^2\sigma \, \overline{\Psi} \gamma^\alpha \partial_\alpha \Psi = \frac{1}{4\pi} \int d^2 z \, (\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi}). \tag{75}$$

The equations of motion $\overline{\partial}\psi = \partial\overline{\psi} = 0$ show that $\psi = \psi(z)$ and $\overline{\psi} = \overline{\psi}(\overline{z})$ are holomorphic and antiholomorphic fields, respectively. Again the kinetic term is inverted to obtain the propagator: $\partial_{\overline{z}} \langle \psi(z)\psi(w) \rangle = \partial_z \langle \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) \rangle = \pi \delta^2(z-w)$ leads to

$$\langle \psi(z)\psi(w) \rangle = \frac{-1}{z-w}, \qquad \langle \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) \rangle = \frac{-1}{\overline{z}-\overline{w}}.$$

The stress tensor is now

$$T_{zz} = \tfrac{1}{2} : \psi \partial_z \psi : , \qquad T_{\overline{z}\overline{z}} = \tfrac{1}{2} : \overline{\psi} \partial_{\overline{z}} \overline{\psi} : .$$

Similar calculations to the boson case show that ψ and $\overline{\psi}$ are primary fields with $(h, \overline{h}) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively (the classical values for a spinor field). Also, T_{zz} obeys the correct OPE (26) with itself, with central charge $c = \frac{1}{2}$.

Note that the fields ψ and $\overline{\psi}$ themselves cannot appear in a modular-invariant partition function because $h - \overline{h}$ is not an integer. All fields in the CFT with an even number of ψ's plus $\overline{\psi}$'s have integer $h - \overline{h}$, however. The projection onto this subset of fields is known (in the string theory context) as the GSO projection.[19] It leads to a modular-invariant theory, which contains additional *spin fields*[6] that are *double*-valued with respect to ψ and $\overline{\psi}$. (This is acceptable because ψ and $\overline{\psi}$ are not present in the modular-invariant theory, and the spin fields are single-valued with respect to the combinations of ψ's and $\overline{\psi}$'s that survive the projection.)

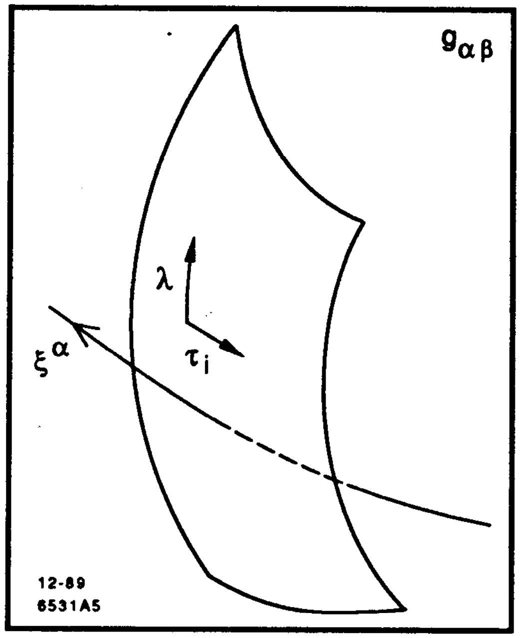

FIGURE 7

Decomposition of the 2d metric $g_{\alpha\beta}(\sigma)$ into the reparametrizations $\xi^\alpha(\sigma)$, which sweep out the gauge orbits, and the conformal factor $\lambda(\sigma)$ and moduli τ_i, which parametrize the gauge slice.

IV. QUANTIZATION OF THE POLYAKOV ACTION:

A. Gauge-fixing, Ghosts and the Critical Dimension

Now that we have seen some of the properties of conformal field theories, and a few relevant explicit examples, let us return to the problem of quantizing the bosonic string. Since the Polyakov action has a local 'gauge' symmetry, 2d reparametrization invariance, we may use the Fadeev-Popov procedure to evaluate the path integral over metrics $g_{\alpha\beta}$ in (6), by writing the integral as an integral along a 'gauge orbit' times an integral over a 'gauge slice'.[38] (See figure 7.)

The gauge orbit is parametrized by the infinitesimal gauge transformations (reparametrizations) $\xi^\alpha(\sigma)$, which take $\sigma^\alpha \to \sigma^\alpha + \xi^\alpha$. The gauge slice is parametrized by the degrees of freedom left after using reparametrization invariance to fix a gauge. A convenient gauge choice is conformal gauge, $g_{\alpha\beta}(\sigma) = e^{\lambda(\sigma)}\delta_{\alpha\beta}$, or in complex coordinates,

$$g_{z\overline{z}} = e^{\lambda(\sigma)}, \qquad g_{zz} = g_{\overline{z}\overline{z}} = 0. \tag{76}$$

As remarked in section III.E, the gauge choice (76) cannot be made globally on surfaces with genus $h > 0$, due to the $3h-3$ complex moduli τ_i (one modulus τ for the torus with $h=1$). Thus the gauge slice is parametrized[48] by the function $\lambda(\sigma)$ and the finite set of parameters τ_i. The dependence on the moduli does not change the essence of the analysis, and so I will suppress much of the moduli-dependence in the following.

The change of path-integration variables that implements the gauge-orbit/gauge-slice splitting is the replacement

$$\int \mathcal{D}g_{\alpha\beta}(\sigma) \;\rightarrow\; \int \mathcal{D}\xi^{\alpha}(\sigma)\cdot\mathcal{D}\lambda(\sigma)\left(\prod_{i=1}^{3h-3} d^2\tau_i\right). \tag{77}$$

The Jacobian J for the change of integration variables is given by the variation of the gauge-fixed quantities g_{zz}, $g_{\overline{z}\overline{z}}$ under the gauge transformation (reparametrization) by ξ:*

$$J \;=\; \det(\delta g_{zz}/\delta\xi)\cdot\det(\delta g_{\overline{z}\overline{z}}/\delta\xi). \tag{78}$$

The infinitesimal version of the transformation (7) of the metric under $z \rightarrow z+\xi^z$, $\overline{z}\rightarrow\overline{z}+\xi^{\overline{z}}$ is

$$\delta g_{zz} \;=\; 2\nabla_z\xi^{\overline{z}}, \qquad \delta g_{\overline{z}\overline{z}} \;=\; 2\nabla_{\overline{z}}\xi^{z}, \tag{79}$$

where $\nabla_z, \nabla_{\overline{z}}$ are covariant derivatives with respect to the metric (76). The determinants appearing in the Jacobian (78) can be represented[6] as path integrals over 2d anticommuting fields (ghosts) b_{zz}, c^z and $b_{\overline{z}\overline{z}}, c^{\overline{z}}$:

$$\begin{aligned}\det(\nabla_{\overline{z}}) \;&=\; \int \mathcal{D}c^z\mathcal{D}b_{zz}\exp\left(\frac{1}{\pi}\int d^2z\, b_{zz}\nabla_{\overline{z}}c^z\right),\\ \det(\nabla_z) \;&=\; \int \mathcal{D}c^{\overline{z}}\mathcal{D}b_{\overline{z}\overline{z}}\exp\left(\frac{1}{\pi}\int d^2z\, b_{\overline{z}\overline{z}}\nabla_z c^{\overline{z}}\right).\end{aligned} \tag{80}$$

The integral $\mathcal{D}\xi^{\alpha}$ over reparametrizations in eq. (77) is an infinite overall factor that can be ignored, leaving us with a path integral measure of the form

$$\int \mathcal{D}\lambda\,\left(\prod d^2\tau_i\right)\mathcal{D}c^z\,\mathcal{D}b_{zz}\,\mathcal{D}c^{\overline{z}}\,\mathcal{D}b_{\overline{z}\overline{z}}\,\mathcal{D}^D X^{\mu}. \tag{81}$$

The integral over the conformal factor λ cannot be ignored in general, because λ is coupled to the total central charge of the remaining fields through eq. (31).

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* I neglect here some purely λ-dependent factors in the Jacobian that are (in the end) unimportant, as well as some moduli-dependence of J.

Implicit in the last remark is the fact that the b, c ghost system is also a conformal field theory, possessing a holomorphic stress tensor $T^{\rm gh}_{zz}$, with a central charge $c^{\rm gh}$ that will now be computed. The ghost Lagrangian appearing in eq. (80) is first order in derivatives, $\mathcal{L}_{\rm gh} = -\frac{1}{\pi} b_{zz} \nabla_{\overline{z}} c^z$, like the free fermion Lagrangian described in section III.H. It leads to a similar propagator,

$$\langle c^z(z) b_{ww}(w) \rangle = \frac{1}{z-w} = \langle b_{zz}(z) c^w(w) \rangle . \tag{82}$$

The stress tensor is given by

$$T^{\rm gh} = -2b\partial_z c - (\partial_z b)c . \tag{83}$$

One can check that $T^{\rm gh}$ gives rise to the proper conformal dimensions for both $c \equiv c^z$ ($h_c = -1$) and $b \equiv b_{zz}$ ($h_b = 2$). There is an identical construction for $\overline{T}^{\rm gh}$ in terms of $\overline{c} \equiv \overline{c^{z}}$ and $\overline{b} \equiv \overline{b_{zz}}$.

In fact, one can define[38,6] a more general conformal 'b, c system' of fields $\hat{b}, \hat{c}$, with dimensions $h_{\hat{b}} = j$, $h_{\hat{c}} = 1 - j$ chosen so that $\mathcal{L}_{\hat{b}\hat{c}} = -\frac{1}{\pi} \hat{b} \nabla_{\overline{z}} \hat{c}$ continues to have dimension (1,1). Another example of such a system will appear when we discuss the ghosts for the superstring (except that in this case the ghosts will be commuting rather than anticommuting objects). For the more general system, the stress tensor

$$T^j = -j\, \hat{b} \partial_z \hat{c} + (1-j)(\partial_z \hat{b})\hat{c} \tag{84}$$

generates the correct conformal dimensions. (The propagators (82) are unchanged.) The calculation of the central charge c^j for this stress tensor again proceeds via Wick contractions, using (82):

$$T^j(z) T^j(w) \sim (z-w)^{-4} \left[j^2(-1) + (1-j)^2(-1) + 2j(j-1)(-2) \right] + \ldots , \tag{85}$$

so $c^j = -2(6j^2 - 6j + 1)$ ($\hat{b}, \hat{c}$ anticommuting). If $\hat{b}, \hat{c}$ commute, then $\langle \hat{b}\hat{c} \rangle$ has the opposite sign from (82), while $\langle \hat{c}\hat{b} \rangle$ still has the same sign, and one gets $c^j = +2(6j^2 - 6j + 1)$ ($\hat{b}, \hat{c}$ commuting).

For the anticommuting ghosts of the bosonic string, b_{zz}, c^z, set $j = 2$ to get $c^{\rm gh} = -26$. It is now apparent that a single space-time dimension D is picked out by the Polyakov action (4): For the *critical dimension* $D = D_{\rm crit} = 26$, the central charge $c^{\rm gh}$ of the ghost system cancels the central charge c^X of the X^μ system.

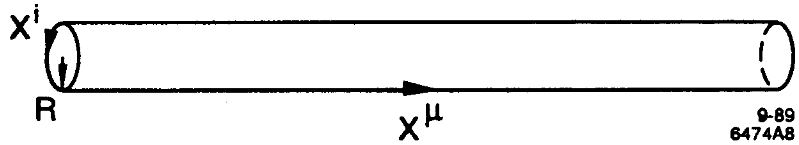

FIGURE 8

Compactification of a single extra coordinate X^i on a circle with radius R of order the Planck length. The coordinates X^μ of Minkowski space-time are represented by the long direction of the "drinking straw".

That is, the total stress tensor $T^{\rm tot} \equiv T^{\rm gh} + T^X$ has zero central charge,

$$c^{\rm tot} \;=\; c^{\rm gh} + c^X \;=\; -26 + D \;= 0 \qquad \text{for } D = D_{\rm crit} = 26. \tag{86}$$

Only for $D = 26$ does the λ-dependence of the combined ghost (b, c) and matter (X^μ) system cancel in eq. (31), and so only then does the path integral over the conformal factor λ decouple completely from the integrals over b, c and X^μ in (81).[38] This decoupling makes the remaining analysis much simpler, because the b, c, X systems are free field theories; whereas the so-called Liouville action for λ is not free, and indeed it has proved to be rather difficult to deal with. For this reason, almost all discussions of string theory — including this one — assume that $c^{\rm tot} = 0$.

B. Compactification

The assumption $c^{\rm tot} = 0$ does not necessarily mean that space-time is 26-dimensional, however, because one can replace some of the 26 X^μ fields in the Polyakov action, say $D_{\rm int}$ of them, with some other 'internal' CFT with the same central charge $c_{\rm int} = D_{\rm int}$. Then the conformal factor will continue to decouple, but the number of flat (Minkowski) space-time dimensions will now be $D = 26 - D_{\rm int}$. In some cases the internal CFT can be interpreted as representing $D_{\rm int}$ compactified spatial directions with Planck-length sizes, but this need not be the case.

The simplest example where the 'internal' CFT does represent a compactified dimension is the case of a single free boson on a circle of radius R, described in section III.G. In this case the compactified dimension is just the circle, and space-time looks roughly like a 'drinking straw' (see figure 8)..

If R is of order the Planck-length, the extra dimension will not be directly observable by the naked eye, or for that matter by any particle physics experiment in the near future. However, its presence can affect the spectrum of particles. To compactify more dimensions, one can use more free bosons (say $D_{\rm int}$ of them) living on a $D_{\rm int}$-dimensional torus (the product of $D_{\rm int}$ circles),[28] or on some other appropriate compact manifold.[24,25]

A simple 'internal' CFT which cannot be interpreted literally as a compactification of extra dimensions is the free fermion example of section III.H, or rather the system of $2D_{\rm int}$ free fermions — so that the central charge of the system is an integer, $c = D_{\rm int}$.[26]

C. Physical States

In order to understand the properties of compactified (or uncompactified) strings, we first need to describe the physical states. Consider the bosonic string in the critical dimension $D = 26$. Because of the gauge symmetry (reparametrization plus Weyl invariance), not all the states in the (b, c, X) Hilbert space are physical. This is just as well, because some of them have negative norm, such as $\alpha^0_{-1}\,|0\rangle$.* Fortunately, such unphysical states will decouple from the physical, gauge-invariant states in scattering amplitudes.

There are various ways to identify the gauge-invariant physical subspace of states. Perhaps the most elegant one is BRST invariance:[49,6] The gauge-fixed action has a fermionic 'BRST' symmetry, for which one can construct an anti-commuting BRST charge Q satisfying $Q^2 = 0$ and $[Q, L_0] = 0$. Here Q is given by

$$Q = \oint \frac{dz}{2\pi i}\ :c(z)\left(T^X(z) + \tfrac{1}{2}T^{\rm gh}(z)\right): \,. \tag{87}$$

Similarly there is a charge $\overline{Q}$ satisfying $\overline{Q}^2 = 0$, $[\overline{Q}, \overline{L}_0] = 0$, and $\{Q, \overline{Q}\} = 0$. (The condition $c^{\rm tot} = 0$ is crucial for showing that $Q^2 = \overline{Q}^2 = 0$.) Physical states $|\psi\rangle$ are required to satisfy

$$Q\,|\psi\rangle \;=\; \overline{Q}\,|\psi\rangle \;=\; 0. \tag{88}$$

Note that any state of the form $Q\overline{Q}\,|\chi\rangle$ satisfies eq. (88), so $|\psi\rangle + Q\overline{Q}\,|\chi\rangle$ is physical

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⋆ The Minkowski metric $\eta^{\mu\nu}$ implicit in the action (4) leads to the commutation relations $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}$, so that the norm $\|\,\alpha^\mu_{-1}\,|0\rangle\,\|^2 = \langle 0|\,\alpha^\mu_1\alpha^\mu_{-1}\,|0\rangle = \eta^{\mu\mu}$ is negative for the time-like mode.

if $|\psi\rangle$ is. In fact $|\psi\rangle$ and $|\psi\rangle + Q\overline{Q}\,|\chi\rangle$ are physically equivalent states for any $|\chi\rangle$, so it is possible to choose any member of this (cohomology) class of states (any χ) to represent a given physical state. In particular, because Q carries ghost number, it is possible to choose a (virtually) ghost-free representative for each physical state. For this representative, the conditions (88) only involve the matter (X) part of the Hilbert space and become, using eq. (87),

$$\begin{aligned} L_0^X\,|\psi\rangle &= 1\,|\psi\rangle\,, \qquad L_n^X\,|\psi\rangle = 0 \text{ for } n>0, \\ \overline{L}_0^X\,|\psi\rangle &= 1\,|\psi\rangle\,, \qquad \overline{L}_n^X\,|\psi\rangle = 0 \text{ for } n>0. \end{aligned} \tag{89}$$

But this is just the condition (38) that $|\psi\rangle$ is a primary state with $(h,\overline{h}) = (1,1)$. The dimension (1,1) primary fields that make physical states are called *vertex operators* and are generally denoted by $V(z,\overline{z})$. It is not too surprising that (1,1) primary fields should make the physical states: To describe a scattering process in string theory, in which states can be emitted at any point of the world-sheet, one has to integrate the vertex operators V over their locations on the world-sheet; but the integral $\int d^2z\, V(z,\overline{z})$ will be invariant under reparametrizations $z \to z'$ only if V transforms as a (1,1) primary field.

BRST invariance can also be used to show that the unphysical states (those which Q does not annihilate) decouple from the physical ones in any scattering process. Basically this is because Q commutes with the 'Hamiltonian' L_0, so that a physical in-state, $|\psi_{\rm in}\rangle$ with $Q\,|\psi_{\rm in}\rangle = 0$, always evolves to a physical out-state, $|\psi_{\rm out}\rangle$ with $Q\,|\psi_{\rm out}\rangle = 0$.

Now we can describe explicitly the spectrum of the bosonic string. For instance, the vertex operator

$$V_T(z,\overline{z}) \;=\; e^{ik\cdot X(z,\overline{z})} \tag{90}$$

makes a physical state if $h = \overline{h} = k^2/2 = 1$. The parameter k^μ in the exponential is the spacetime momentum of the state. To see this, notice that the momentum operator P^μ that generates space-time translations $X^\mu \to X^\mu + a^\mu$ is

$$P^\mu \;=\; \oint \frac{dz}{2\pi i}\, i\partial_z X^\mu + \oint \frac{d\overline{z}}{2\pi i}\, i\partial_{\overline{z}} X^\mu, \tag{91}$$

and that the eigenvalue of P^μ acting on V_T is k^μ (using eq. (57)). Thus the state $|k\rangle$ can be identified with a Lorentz-scalar particle of mass m_T, where

$$m_T^2 \;=\; -(\pi T)k^2 \;=\; -2\pi T. \tag{92}$$

(The minus sign comes from our convention for the Minkowski space-time metric

$\eta^{\mu\nu} = (- + \ldots +)$. Also, we have reinserted the dimensionful string tension T which was previously set to $1/\pi$.)

Equation (92) is rather disturbing because it shows there is a particle with negative $(\text{mass})^2$, a *tachyon*, in the bosonic string spectrum. A tachyon is physically unacceptable; it indicates some kind of instability in the theory. For this reason the bosonic string is generally considered to be a toy model rather than a viable string theory. Fortunately, tachyons are not present in the supersymmetric string, as will be seen in the next section.

The bosonic string states with the next smallest masses (which also appear in the superstring spectrum) are made by the vertex operators

$$V_g(z,\overline{z}) \;=\; \zeta_{\mu\nu}\partial_z X^\mu \partial_{\overline{z}} X^\nu e^{ik\cdot X(z,\overline{z})}\;. \qquad (93)$$

The primary state condition (89) restricts the polarization tensor $\zeta_{\mu\nu}$. Computing the OPE of V_g with T,

$$T(z)\,V_g(w,\overline{w}) \;\sim\; \frac{ik^\sigma\zeta_{\sigma\nu}}{(z-w)^3}\partial_{\overline{w}}X^\nu e^{ik\cdot X(w,\overline{w})} + \frac{1+k^2/2}{(z-w)^2}\,V_g(w,\overline{w}) + \ldots \qquad (94)$$

(and also with $\overline{T}$) one obtains the transversality conditions $k^\mu\zeta_{\mu\nu} \;=\; k^\nu\zeta_{\mu\nu} \;=\; 0$ and the mass-shell condition $k^2 = 0$. Thus the states made by V_g are *massless.* They fall into three different Lorentz multiplets: the graviton $g_{\mu\nu}$ (with $\zeta_{\mu\nu}$ symmetric and traceless), the antisymmetric tensor field $B_{\mu\nu}$ ($\zeta_{\mu\nu}$ antisymmetric), and the dilaton ϕ (the trace piece of $\zeta_{\mu\nu}$). Note that the states with time-like polarization, which we saw to have negative norms, are removed from the physical spectrum by the transversality condition. The spontaneous appearance of the graviton in the string spectrum provides one of the principal motivations for studying string theory; it suggests that strings may be a consistent theory of quantum gravity (once we find a way to remove the tachyon from the spectrum!).

The remaining states in the bosonic string spectrum all have positive masses, with $m^2 \;=\; (2\pi T)n, \quad n \;=\; 1,2,3,\ldots$ They are made by vertex operators like V_g but with more factors of ∂X^μ and $\overline{\partial} X^\nu$; hence they form increasingly high rank tensor representations of the Lorentz group (higher spin particles). Since their masses are all of order the Planck mass, these particles are not of direct interest experimentally. However, they do play a key role in radiative corrections (loops), giving string theory its nice ultraviolet behavior. (Superstrings are free of ultraviolet divergences at one loop[21,23], and this finiteness is expected to persist to all orders in perturbation theory.)

We just described the spectrum of the uncompactified ($D = 26$) bosonic string. To construct a compactified spectrum we have to combine the fields $e^{ik\cdot X}$, ∂X^μ, $\overline{\partial} X^\nu$ with fields ϕ from the internal CFT to make dimension (1,1) vertex operators V. Every primary field $\phi(z,\overline{z})$ with conformal dimension (h,h) (a 2d scalar field) gives rise to a space-time-Lorentz scalar particle with mass $m^2 = -k^2 = (2\pi T)(h-1)$, through the vertex operator $V_\phi = \phi\, e^{ik\cdot X}$. Higher mass and spin particles are generated by adding ∂X^μ's and $\overline{\partial} X^\nu$'s to V_ϕ. The most interesting particles are again usually the massless ones; massless scalars arise only from internal fields with $(h,\overline{h}) = (1,1)$, and massless vectors from fields with $(h,\overline{h}) = (1,0)$ or (0,1). For example, if the internal CFT is a free boson X on a circle of radius R, $\phi = \partial X \overline{\partial} X$ leads to a massless scalar and ∂X, $\overline{\partial} X$ lead to massless vectors. (The partition function (71) for the circle CFT shows that for special values of R there can be additional massless vectors and scalars, obtained from vertex operators of the form $e^{i(k_L X_L + k_R X_R)}$.)

V. SUPERSTRINGS AND INTERACTIONS:

A. Superstrings and Superconformal Invariance

As we have seen, the bosonic string is an unacceptable theory because of its tachyon, but it is also phenomenologically unacceptable as a unified theory because it has no spacetime fermions in its spectrum. To remedy the situation, we now describe the superstring, which incorporates space-time supersymmetry and therefore leads to a fermionic state for each bosonic state. (Of course, for phenomenological reasons space-time supersymmetry must be spontaneously broken at an energy scale of at least the weak scale (≈ 100 GeV).) There are two different descriptions of the superstring. In the Green-Schwarz[20] (GS) formulation space-time supersymmetry is manifest, but Lorentz covariance is hard to maintain in the quantization procedure. I will describe here instead the Neveu-Schwarz-Ramond[18] (NSR) formulation, in which space-time supersymmetry is obscure, but Lorentz covariance can be maintained. (The description will be rather sketchy in any case.)

The action used in the NSR formulation is[37]

$$S(X,\Psi,e,\chi) = -\frac{1}{2\pi}\int d^2\sigma\, e\{g^{\alpha\beta}\partial_\alpha X^\mu \partial_\alpha X_\mu - i\overline{\Psi}^\mu\gamma^\alpha\nabla_\alpha\Psi_\mu + 2\chi_\alpha\gamma^\beta\gamma^\alpha\Psi^\mu\partial_\beta X_\mu\}, \tag{95}$$

where $\mu = 0,1,2,\ldots,D-1$ is a space-time index and $\alpha,\beta = 0,1$ are world-sheet indices as before. This action has a supersymmetry, but it is a *local world-sheet (2d) supersymmetry*,

$$\begin{aligned}
\delta X^\mu &= \overline{\epsilon}\Psi^\mu, \\
\delta\psi^\mu &= -i\gamma^\alpha\epsilon(\partial_\alpha X^\mu - \overline{\Psi}^\mu\chi_\alpha), \\
\delta e^a_\alpha &= -2i\overline{\epsilon}\gamma^a\chi_\alpha, \\
\delta\chi_\alpha &= \nabla_\alpha\epsilon,
\end{aligned} \tag{96}$$

rather than a space-time supersymmetry. Here Ψ^μ are D two-component Majorana fermions (equivalent to D of the spinors Ψ discussed in section III.H), which are the superpartners of X^μ; χ_α is a world-sheet gravitino field, the superpartner of the 2d metric $g_{\alpha\beta}$; and e^α_a is the zweibein for the metric, satisfying $e^\alpha_a e^\beta_a = g^{\alpha\beta}$, $e \equiv \det(e^\alpha_a) = \sqrt{-g}$. In addition to having all the symmetries of the Polyakov action, plus (96), the action (95) has a symmetry transforming only the gravitino field,

$$\begin{aligned}
\delta_\eta\chi_\alpha &= i\gamma_\alpha\eta, \\
\delta_\eta e &= \delta_\eta\Psi^\mu = \delta_\eta X^\mu = 0.
\end{aligned} \tag{97}$$

Both ϵ and η are infinitesimal, anticommuting 2d spinors.

The path integral of interest is now[38]

$$Z = \sum_{\text{genush}=0}^{\infty} (g_s)^{2h-2}\int \mathcal{D}^D X^\mu\, \mathcal{D}^D\Psi^\mu\, \mathcal{D}g_{\alpha\beta}\, \mathcal{D}\chi_\alpha\, e^{-S(X,\Psi,e,\chi)}. \tag{98}$$

We again use the Fadeev-Popov procedure to gauge-fix the local symmetries of the action (95) in the path integral, choosing

$$\chi_\alpha = 0 \tag{99}$$

in addition to the conformal gauge choice (76) for the metric. The integration over metrics $g_{\alpha\beta}$ is traded for an integral $\mathcal{D}\xi\,\mathcal{D}\lambda$ as before, and the integration over the gravitino χ_α is traded for $\mathcal{D}\epsilon\,\mathcal{D}\eta$. Since $\delta\chi_\alpha = \nabla_\alpha \epsilon$, the Jacobian for the change of variables in the Grassmann gravitino path integral is

$$J \;=\; (\det(\delta\chi_\alpha/\delta\epsilon))^{-1} \;\sim\; (\det\nabla_z)^{-1}\cdot(\det\nabla_{\bar z})^{-1}. \tag{100}$$

In this case, the operators $\nabla_{\bar z}$ and ∇_z act on spinors (ϵ) rather than the vectors (ξ) of the bosonic Jacobian. The determinants are therefore represented[6] as path integrals over commuting fields with half-integer dimensions (ghosts), β,γ and $\overline{\beta},\overline{\gamma}$ with $h_\beta = 3/2$, $h_\gamma = -1/2$,

$$\begin{aligned}(\det\nabla_{\bar z})^{-1} \;&=\; \int \mathcal{D}\gamma\,\mathcal{D}\beta\,\exp\left(-\frac{1}{\pi}\int d^2z\;\beta\partial_{\bar z}\gamma\right),\\ (\det\nabla_z)^{-1} \;&=\; \int \mathcal{D}\overline{\gamma}\,\mathcal{D}\overline{\beta}\,\exp\left(-\frac{1}{\pi}\int d^2z\;\overline{\beta}\partial_z\overline{\gamma}\right).\end{aligned} \tag{101}$$

The β,γ system is another example of the general $\hat b,\hat c$ system discussed in section IV.A, with $j=3/2$ and stress tensor

$$T^{\beta\gamma} \;=\; -\frac{3}{2}\,\beta\partial_z\gamma - \frac{1}{2}(\partial_z\beta)\gamma\,. \tag{102}$$

Since β,γ commute, the central charge computation (85) gives $c^{\beta\gamma} = +11$. The gauge-fixed action for the 'matter fields' X^μ and $\Psi^\mu = (\psi^\mu\;\;\overline{\psi}^\mu)^T$ is

$$S_{\rm gf} \;=\; -\frac{1}{2\pi}\int d^2z\;\{\partial_z X^\mu\partial_{\bar z}X_\mu + \psi^\mu\partial_{\bar z}\psi_\mu + \overline{\psi}^\mu\partial_z\overline{\psi}_\mu\}. \tag{103}$$

The central charge for this system is $D(1+\frac{1}{2})$. Thus the total (ghost plus matter) stress tensor $T^{\rm tot} = T^{bc} + T^{\beta\gamma} + T^X + T^\psi$ has central charge

$$c^{\rm tot} \;=\; c^{bc} + c^{\beta\gamma} + c^X + c^\psi \;=\; -26 + 11 + D(1+\tfrac{1}{2}) \;=\; \tfrac{3}{2}(D-10), \tag{104}$$

which vanishes if $D = 10$. As in the bosonic case, the conformal factor λ decouples only in this dimension.[38] So superstrings pick out $D_{\rm crit} = 10$ as their critical dimension.

The gauge-fixed action (103) has an important residual symmetry besides just conformal invariance, namely *superconformal invariance*,[50] which is the gauge-fixed version of the local supersymmetry (96):

$$\begin{aligned}\delta X^\mu \;&=\; \epsilon\psi^\mu + \overline{\epsilon}\overline{\psi}^\mu,\\ \delta\psi^\mu \;&=\; \epsilon\partial_z X^\mu,\\ \delta\overline{\psi}^\mu \;&=\; \overline{\epsilon}\partial_{\bar z}X^\mu.\end{aligned} \tag{105}$$

In the same way that the fields $T(z)$ and $\overline{T}(\bar z)$ generate conformal transformations, superconformal transformations are generated by a pair of fields $T_F(z)$ and $\overline{T}_F(\bar z)$, which are the superpartners of T and $\overline{T}$. For the $X^\mu, \psi^\mu, \overline{\psi}^\mu$ system of (103), the superconformal generator is

$$T_F^{(X,\psi)}(z) \;=\; -\frac{1}{2}\,\psi^\mu \partial_z X_\mu, \qquad \overline{T}_F^{(X,\overline{\psi})}(\overline{z}) \;=\; -\frac{1}{2}\,\overline{\psi}^\mu \partial_{\overline{z}} X_\mu. \tag{106}$$

The OPEs of $T(z)$, $T_F(z)$ are determined by superconformal symmetry to be

$$\begin{aligned} T(z)\,T(w) &\sim \frac{3\hat{c}/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{z-w} + \ldots, \\ T(z)\,T_F(w) &\sim \frac{(3/2)T_F(w)}{(z-w)^2} + \frac{\partial_w T_F}{z-w} + \ldots, \\ T_F(z)\,T_F(w) &\sim \frac{\hat{c}/4}{(z-w)^2} + \frac{(1/2)T(w)}{z-w} + \ldots. \end{aligned} \tag{107}$$

The first line of (107) is just the usual stress tensor OPE, with $\hat{c}$ defined to be $\hat{c} = \frac{2}{3}c$. (The reason for this definition is so that the system (X, ψ) of a single free boson plus its fermionic superpartner has $\hat{c} = \frac{2}{3}(1+\frac{1}{2}) = 1$.) The second line states that T_F is a dimension 3/2 holomorphic primary field. The third line is required by the fact that superconformal transformations close into conformal transformations. The fields $\overline{T}, \overline{T}_F$ obey the same algebra with $z, w \to \overline{z}, \overline{w}$. As usual, T_F can be expanded into Laurent modes, according to

$$\left\{ \begin{aligned} T_F(z) &= \sum_{r\in\mathbf{Z}+\frac{1}{2}} \tfrac{1}{2} G_r\, z^{-r-3/2}, \\ G_r &= \oint \frac{dz}{2\pi i}\, 2T_F(z)\, z^{n+1/2}. \end{aligned} \right. \tag{108}$$

The algebra obeyed by the modes L_n and G_r is called the *superconformal algebra.*

The full algebra (107), with $c = c^{(X,\psi)} = 15$ or $\hat{c} = \hat{c}^{(X,\psi)} = 10$, is required to show that the path integral over the field η decouples (whereas the first line of (107) suffices to show decoupling of the conformal factor λ). Thus to 'compactify' D_{int} dimensions in a superstring theory, we replace a system $X^\mu, \psi^\mu, \overline{\psi}^\mu$ by an internal *superconformal* field theory with the same central charge, $\hat{c}^{\text{int}} = D_{\text{int}}$.

Physical states in the superstring Hilbert space can again be identified using BRST invariance. There are several additional subtleties having to do with the commuting β, γ ghosts, which I will not go into here.[6] A physical state that is a space-time boson has a (virtually) ghost-free representation (as in the case of the bosonic string). A state $|\phi_0\rangle$ in this representation is required by BRST invariance to be a 'primary state' with respect to the full superconformal algebra, not just the Virasoro algebra; that is, $|\phi_0\rangle$ must be annihilated by the positive-frequency modes of both T and T_F:

$$\begin{aligned} L_n\,|\phi_0\rangle &= 0, \quad n\geq 1, & L_0\,|\phi_0\rangle &= h\,|\phi_0\rangle\,, \\ G_r\,|\phi_0\rangle &= 0, \quad r\geq \tfrac{1}{2}\ \ (r\in \mathbf{Z}+1/2), & G_{-1/2}\,|\phi_0\rangle &= |\phi_1\rangle\,. \end{aligned} \tag{109}$$

The new state $|\phi_1\rangle$ appearing in (109) is the 2d superpartner of $|\phi_0\rangle$, and is primary under the conformal algebra with dimension $h+1/2$. The fields ϕ_0, ϕ_1 that make the states $|\phi_0\rangle\,, |\phi_1\rangle$ form a 2d *primary superfield* of dimension h; ϕ_0 is the lower component of the superfield and ϕ_1 is the upper component. Their OPEs with T and T_F are fixed by eq. (109) and superconformal symmetry to be

$$\begin{aligned} 2T_F(z)\,\phi_0(w,\overline{w}) &\sim \frac{\phi_1(w,\overline{w})}{z-w}+\ldots, \\ 2T_F(z)\,\phi_1(w,\overline{w}) &\sim \frac{2h\phi_0(w,\overline{w})}{(z-w)^2}+\frac{\partial_w\phi_0}{z-w}+\ldots, \\ T(z)\,\phi_0(w,\overline{w}) &\sim \frac{h\phi_0(w,\overline{w})}{(z-w)^2}+\frac{\partial_w\phi_0}{z-w}+\ldots, \\ T(z)\,\phi_1(w,\overline{w}) &\sim \frac{(h+1/2)\phi_1(w,\overline{w})}{(z-w)^2}+\frac{\partial_w\phi_1}{z-w}+\ldots. \end{aligned} \tag{110}$$

A vertex operator for a bosonic superstring state is actually the *upper* component ϕ_1 of a dimension $h=1/2$ (also $\overline{h}=1/2$) primary superfield. Such a field ϕ_1 is primary with respect to T alone, and has conformal dimension $h+1/2=1$ (and $\overline{h}+1/2=1$), so the physical state restrictions for the superstring are a superset of those for the bosonic string. (Of course the initial Hilbert space is larger for the superstring due to the additional fields $\psi^\mu, \overline{\psi}^\mu$.) It is usually easiest to construct the vertex operators by first finding the lower components ϕ_0, then applying T_F to them to obtain ϕ_1 (using eq. (110)):

$$\phi_1(w,\overline{w}) = \lim_{z\to w}(z-w)(2T_F(z))\phi_0(w,\overline{w}). \tag{111}$$

There is one more requirement on ϕ_1 for it to describe a physical state: It must have even fermion number F, where $F=\oint \frac{dz}{2\pi i}\ \psi^\mu(z)\psi_\mu(z)$ counts the number of ψ^μ fields in a vertex operator, and $\overline{F}=\oint \frac{d\overline{z}}{2\pi i}\ \overline{\psi}^\mu(\overline{z})\overline{\psi}_\mu(\overline{z})$ must also be even.

This requirement is known as the GSO projection, and it leads to a space-time supersymmetric mass spectrum.[19]

Now let's look at the lowest mass bosonic states. The field $\phi_0 = e^{ik\cdot X}$ is primary under the superconformal algebra with $h = k^2/2$. However, the upper component, obtained using (111) and a similar limit with $\overline{T}_F$, is $ik\cdot\psi\, ik\cdot\overline{\psi}\, e^{ik\cdot X}$ and has odd fermion numbers F and $\overline{F}$, so it is not a physical state. This is just as well, because it would have been a tachyon, with $m^2 = -k^2 = -1$.

The lowest mass physical states are obtained from the field $\zeta_{\mu\nu}\psi^\mu\overline{\psi}^\nu e^{ik\cdot X}$, which is the lowest component of a primary superfield if $\zeta_{\mu\nu}$ satisfies the same transversality condition as in the bosonic string case and if $k^2 = 0$. Applying T_F and $\overline{T}_F$ and checking that $F, \overline{F}$ are even, one gets

$$V_g = \zeta_{\mu\nu}(\partial_z X^\mu + ik\cdot\psi\psi^\mu)(\partial_z X^\nu + ik\cdot\overline{\psi}\overline{\psi}^\nu)\, e^{ik\cdot X}\,. \tag{112}$$

The vertex operators (112) are just the superstring version of (93); they again give rise to a graviton, an antisymmetric tensor and a dilaton in the massless spectrum.

The construction of vertex operators for space-time fermions in the NSR formulation is more subtle,[51,6] and requires a better understanding of the β,γ ghost system than I have given here. The main difference from the bosonic vertices is that the fermionic vertices have a square-root singularity with respect to T_F. That is, if such a vertex operator is located at the origin $z = 0$, then the Laurent expansion of $T_F(z)$ is in terms of integer rather than half-integer modes,

$$T_F(z) = \sum_{n\in\mathbf{Z}} G_n z^{-n-3/2}, \tag{113}$$

and G_n must annihilate the state for $n > 0$. Also, the vertex operators (in one choice of ghost representation) must have dimension 5/8. Such fields exist in the $\psi^\mu, \overline{\psi}^\mu$ conformal field theory; they are called spin fields, are denoted by Σ^α and $\Sigma^{\dot\alpha}$, and have square-root singularities with the world-sheet fermions $\psi^\mu, \overline{\psi}^\mu$. Here α and $\dot\alpha$ are ten-dimensional Lorentz spinor indices. Thus the vertex operators

$$\begin{aligned} V_{\text{gravitino}} &= \zeta_{\mu\alpha}\Sigma^\alpha(z)(\partial_{\bar z} X^\mu + ik\cdot\overline{\psi}\overline{\psi}^\mu)\, e^{ik\cdot X}, \\ V'_{\text{gravitino}} &= \zeta'_{\mu\alpha}\Sigma^\alpha(\bar z)(\partial_z X^\mu + ik\cdot\psi\psi^\mu)\, e^{ik\cdot X}, \end{aligned} \tag{114}$$

carry both vector and spinor indices an give rise to gravitinos (spin-3/2 particles) in the massless spectrum. The presence of gravitinos implies that the theory should be space-time supersymmetric, and indeed it is.[51,6]

The spectrum for a compactified version of the superstring can be found in similarly. As in the bosonic case, fields $e^{ik\cdot X}$, ψ^μ, ∂X^μ, *etc.* from the space-time part of the theory are combined with fields from the internal (super)conformal theory to make vertex operators, with the additional restrictions that the combined vertex operators (for space-time bosons) must be upper components of superfields, and must be even under a generalized GSO projection.

B. Heterotic Strings

We have seen how the introduction of world-sheet supersymmetry and the GSO projection has led to a superstring with no tachyon and with massless fermions (indeed, with space-time supersymmetry), which are two steps towards obtaining a spectrum that might include the standard model spectrum. Still missing, however, are gauge bosons. They are in fact rather difficult to obtain from the superstring just described, even when it is compactified.[30,52] So we now briefly describe the one remaining theory of closed superstrings, namely the heterotic string.[23]

The heterotic string is a hybrid of the bosonic string and the superstring just discussed. Essentially, the antiholomorphic, or left-moving, degrees of freedom are taken to be those of the bosonic string, and the holomorphic, or right-moving, degrees of freedom are those of the superstring. That is, the 2d conformal field theory for the heterotic string has a right-moving superconformal invariance generated by a field T_F, but no left-moving superconformal invariance (no $\overline{T}_F$). The same arguments as were applied above to the bosonic string and the superstring would now seem to imply that the critical dimension for the heterotic string is both 26 (the bosonic calculation) and 10 (the superstring calculation)!

This paradox is resolved by 'compactifying' 16 of the dimensions of the bosonic part of the heterotic string; that is, one replaces 16 of the directions X^μ by an internal conformal field theory with central charge $\overline{c} = 16$ and $c = 0$. Since $c = 0$, this internal CFT is trivial in its z-dependence (all fields have right-moving dimension $h = 0$, and so they are independent of z), so it is called an '(anti)holomorphic conformal field theory'. It turns out that there are only two such theories with $\overline{c} = 16$. Both of them turn out to have exactly 498 primary fields $\overline{J}^a(\overline{z})$, $a = 1, 2, \ldots 498$, of dimension $\overline{h} = 1$. The fields $\overline{J}^a$ are important because they give rise to 498 gauge bosons in the massless spectrum of the heterotic string. The gauge boson vertex operators are

$$V_A(z,\overline{z}) \sim \zeta^a_\mu(\partial_z X^\mu + ik\cdot\psi\psi^\mu)\,\overline{J}^a(\overline{z})\,e^{ik\cdot X(z,\overline{z})}, \tag{115}$$

where $k^2 = 0$ and the polarization vector ζ^a_μ is transverse, $k^\mu\zeta^a_\mu = 0$. Note that V_A is the upper component of a dimension $h = 1/2$ superfield $(\psi^\mu e^{ik\cdot X})$ for the right-moving superconformal algebra, and is a dimension $\overline{h} = 1$ conformal field thanks to $\overline{h}_{\overline{J}} = 1$. The gauge group depends on the properties of the $\overline{J}^a$, and turns out to be either $E_8 \times E_8$ or $SO(32)$, depending on which of the two $\overline{c} = 16$ antiholomorphic CFTs is used.

The heterotic string also has in its massless spectrum a graviton, antisymmetric tensor, dilaton, and superpartners of all of these. The corresponding vertex operators are the same as those (112),(114) of the superstring in their z-dependence, and the same as the graviton vertex operator (93) of the bosonic string in their $\overline{z}$-dependence. For example,

$$V_g = \zeta_{\mu\nu}\partial_{\overline{z}}X^\mu(\partial_z X^\nu + ik\cdot\psi\psi^\nu)\,e^{ik\cdot X}, \tag{116}$$

with $k^\mu\zeta_{\mu\nu} = k^\nu\zeta_{\mu\nu} = 0$, $k^2 = 0$. The superpartners of the gauge bosons are gauginos, massless fermions transforming under the adjoint representation of $E_8 \times E_8$ or $SO(32)$. When one tries to make contact between some four-dimensional version of the heterotic string and the standard model, one usually identifies the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ with a subgroup of $E_8 \times E_8$. Then some components of the ten-dimensional gauginos become four-dimensional gauginos (superpartners of the standard model gauge bosons); but other components have the proper gauge quantum numbers to be identified with the fermions (quarks and leptons) of the standard model.[24]

To see whether such an identification is viable, one must first understand how to construct four-dimensional versions of the heterotic string. These compactifications are described by internal superconformal theories with $c = 9$ and $\overline{c} = 22$ (we absorb the $\overline{c} = 16$ system into the internal CFT); only a right-moving superconformal invariance needs to be present, as was true of the uncompactified heterotic string. The additional restrictions on physical vertex operators that were found for the superstring also apply to the heterotic string, but only to the right-moving degrees of freedom. There are now many different schemes for constructing four-dimensional heterotic strings,[24,25,26,27] which all fall under the general approach sketched here, and detailed massless spectra have been computed for numerous individual examples. Some of the spectra appear to have phenomenological promise, but there are many uncertainties in extrapolating from the Planck scale to the weak scale. In particular, one needs to know a good deal about the interactions of particles, especially the massless particles, as given by string theory.

C. String Interactions

The interactions of particles in either a string theory or a field theory are summarized by a set of scattering amplitudes, known as the S-matrix. The external states in a scattering amplitude are physical particles on their mass-shell. In a field theory one generally obtains the scattering amplitudes from Green's functions that are also defined off-shell. If the coupling constant(s) are small, the Green's functions have a perturbative expansion, in terms of Feynman diagrams with increasing numbers of loops, and there may also be nonperturbative contributions. In the first-quantized approach to string theory that we have followed here, only the *perturbative* contributions to scattering amplitudes can be calculated at present.

The prescription for computing perturbative amplitudes via the Polyakov path integral is simply to insert one vertex operator $V(z_i, \overline{z}_i)$ into the gauge-fixed path integral for each external particle.* The vertex operator provides the appropriate path-integral boundary condition to make an on-shell, physical particle at the point $(z_i, \overline{z}_i)$ on the world-sheet. Since the particle can be emitted from any point on the world-sheet during the scattering process, the point $(z_i, \overline{z}_i)$ should be integrated over the world-sheet. The 2d path integral with vertex operators inserted is just a correlation function in the appropriate conformal field theory. Note that the integrated correlation function is conformally invariant (and hence well-defined) if the external particles are on-shell, because then the vertex operator $V(z_i, \overline{z}_i)$ has dimension (1,1) and the integral $\int d^2 z_i\, V(z_i, \overline{z}_i)$ is conformally invariant. If one is evaluating the correlation function on a 2d surface with genus $h \geq 1$, then one should also integrate over the moduli $(\tau_j, \overline{\tau}_j)$ characterizing the surface. Thus scattering amplitudes for particles in string theory are calculated as

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⋆ See ref. 5 for many more details.

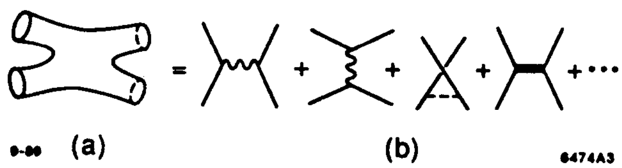

FIGURE 9

(a) A four-string scattering process at tree level. (b) The field theory Feynman diagrams that represent the contributions of individual particles to the amplitude. Thick lines denote massive particles; all other lines denote massless particles.

integrated CFT correlation functions,

$$S(k_1,\ldots,k_N)=\sum_{h=0}^{\infty}(g_s)^{2h+N-2}\int\prod_{i=1}^{N}d^2z_i\prod_{j=1}^{3h-3}d^2\tau_j\,\langle V_{k_1}(z_1,\overline{z}_1)\ldots V_{k_N}(z_N,\overline{z}_N)\rangle. \tag{117}$$

The factors of the string coupling constant g_s in a given term in eq. (117) account for the number of times a string has to split in two or join together to make the world-sheet of that topology. For the sphere (genus zero), three of the N points $(z_i,\overline{z}_i)$ should not be integrated (due to an invariance of the correlation function under transformations by L_{-1},L_0,L_1 and $\overline{L}_{-1},\overline{L}_0,\overline{L}_1$); for the torus one such point should not be integrated.† The remaining integration variables z_i,τ_j are the analogs of the Feynman parameters x_i used to represent the integrals occurring in field-theory Feynman diagrams.

Indeed, for each genus h the string-theory amplitude (117) can (heuristically) be decomposed into a sum of an infinite number of field-theory Feynman diagrams, representing the contributions of an infinite number of particles — both massless and massive — that can appear as intermediate states in the scattering process. This decomposition for a four-point scattering amplitude at tree-level (genus zero) is shown in figure 9, and the one-loop decomposition is shown in figure 10.

It is remarkable that at a given order in perturbation theory any amplitude

† Superstring and heterotic string scattering amplitudes have many additional subtleties at higher genus.

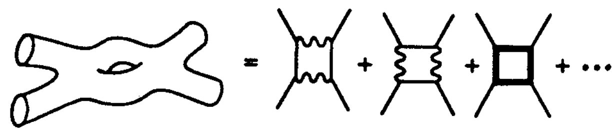

FIGURE 10

A four-string scattering process at the one-loop level. (b) The Feynman diagrams contributing to it.

is given by just one string-scattering diagram. The diagram includes exchanges of many different types of particles, and in all possible momentum channels. For instance, the diagram in figure 9(a) can be viewed as representing either s-, t-, or u-channel scattering, by stretching it either horizontally, vertically, or out of the page. (This property of string scattering amplitudes is called *duality*, and is the reason why string theories were termed 'dual models' early in their history.)

At large distances, where strings no longer appear to be extended objects, we expect that a conventional field-theory should adequately describe the interactions of particles in string theory. To be specific, if the external states in a string scattering process are massless particles, and if the energy of the collision is much less than M_{Pl}, then the amplitude for the process can be reproduced using an *effective Lagrangian* $\mathcal{L}_{\mathrm{eff}}$ which only involves the massless fields. (See ref. 53 for a discussion of how to extract effective Lagrangians for strings.)

To reproduce the amplitudes in figure 9, for example, one needs three-particle couplings of the type shown in figure 11(a). In addition, an infinite set of *non-renormalizable terms* (terms in $\mathcal{L}_{\mathrm{eff}}$ with dimension larger than four, whose coefficients contain inverse powers of M_{Pl}) results from exchanges of massive particles (figure 11(b)). The latter terms are completely analogous to the four-Fermi interaction terms that reproduce the low-energy effects of W-exchange in the standard model. Similar considerations apply to loop- as well as tree-level string scattering amplitudes.

Thus, once one has used a 2d conformal field theory to specify a construction of a string theory, whether in 26, 10 or 4 dimensions, then not only the spectrum

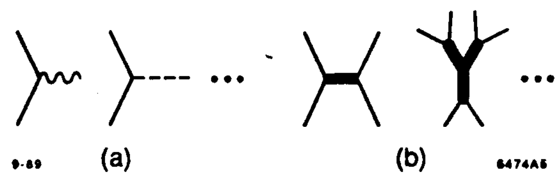

FIGURE 11

Three-particle couplings needed to reproduce the four-string scattering amplitude at tree level. (b) Additional non-renormalizable interactions that are needed, due to exchanges of massive particles in the string amplitude.

of particles but also all of their interactions are completely fixed. The interactions of the massless particles at low energy are summarized by a conventional field theory — supplemented by non-renormalizable terms — which can be extracted from string scattering amplitudes (in principle, and often in practice).

VI. CONCLUSIONS:

In these lectures I have described at a very elementary level the first-quantized approach to string theory, and the role played by two-dimensional conformal field theory. It is certainly possible to go into much more detail in both of these areas, and the reviews listed in the introduction may be useful to the reader in this regard. On the other hand, there is still much that is not understood about string theory, that prevents one from making quantitative predictions regarding physics at low energies (say at the electroweak scale). A major problem is the large number of constructions of four-dimensional string theories, using various types of CFT. At present there is no dynamical reason for selecting one of the constructions over the others. It is hoped that nonperturbative effects could lift this degeneracy; thus it is crucial to develop some kind of nonperturbative framework for string theory.

Even if one selects a CFT by hand, for which the spectrum and low-energy interactions are in principle computable, it is not trivial to extrapolate to electroweak energies. The effective Lagrangian $\mathcal{L}_{\text{eff}}$ that one typically extracts from the CFT describes particle interactions at energies just below the Planck scale; hence $\mathcal{L}_{\text{eff}}$ must still be renormalized through some 17 orders of magnitude in energy, and possibly through several intermediate regimes of symmetry-breaking. Of the many uncertainties in this process, perhaps the greatest is in the mechanism of space-time supersymmetry breaking, which is (in the usual scenarios) ultimately responsible for the enormous hierarchy between the Planck scale and the electroweak scale, and which also can feed into predictions for other measured parameters of the standard model (quark and lepton masses, *etc.*). It seems that supersymmetry breaking should be nonperturbative, in order to generate the observed hierarchy. One might hope that it is a long-distance effect that could be calculated using an effective Lagrangian, for which nonperturbative tools are already available. To date, however, no proposed supersymmetry-breaking mechanism has led to a particularly realistic outcome.

In summary, while string theory is currently the leading candidate for a consistent theory of quantum gravity, the question of whether or not it can also generate definite predictions of low-energy phenomena is still open, and awaits the development of further calculation tools, or conceptual understanding, or both.

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