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TWO TOPICS IN QUANTUM CHROMODYNAMICS*

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ABSTRACT

The two topics are (1) estimates of perturbation theory coefficients for $R\,(e^+e^- \rightarrow hadrons)$, and (2) the virtual-photon structure function, with emphasis on the analytic behavior in its squared mass.

INTRODUCTION

Quantum chromodynamics (QCD) has reached a level of credibility and maturity which deserves textbook status. Indeed, textbooks exist[1] and others are on the way.[2] Nevertheless, to my mind a textbook treatment of QCD is made much more difficult than that of quantum electrodynamics (QED) because of the confinement problem. Even perturbative QCD—which is all that will really be discussed here—suffers this problem. There is no S-matrix theory of quarks and gluons as there is for QED, as given in the LSZ formalism.[3] The concept of "on-mass-shell" or "asymptotic" quark and/or gluon is highly suspect. And the typical "Feynman diagram" used in perturbative QCD contains internal quark and gluon lines and external hadron lines. What does that really mean? How does one derive and justify Feynman-rules for such amplitudes in the absence of good control over the confinement question?

These issues are more matter-of-principle ones than operational ones. In general, I find no fault with what is being calculated, only that there is need for a more solid logical basis—as opposed to the intuitive, common-sense one—for what is done. The question is perhaps similar to, albeit much less profound than, the early days of quantum theory, where the calculations came fast and the real understanding of what they meant came more slowly.

In the last year, I have been lecturing on QCD at the University of Chicago, with these issues in mind. While I cannot claim much progress, the material which follows is influenced by the above concerns. It has also been most positively influenced by the students who patiently endured my gropings through this difficult subject and provided much help. Some are here at this school; to all I give thanks.

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GUIDELINES FOR SETTING UP PERTURBATIVE QCD

Perturbative QCD is, at best, applicable only at short space-time intervals because of the "asymptotic freedom" property of the running coupling constant. What does this mean? Perturbative field theory is, essentially by definition, based on Feynman diagrams. Can one construct Feynman-diagram amplitudes whose ingredients depend only upon short distances? The answer appears affirmative, provided these diagrams are for Green's functions whose sources are restricted to small, contiguous space time regions. What, in turn, does this mean? I prefer to think of this restriction in terms of actual physical processes in principle localizable to small space-time regions. Tiny sources, of scale small compared to the confinement scale Λ_{QCD}^{-1}, create "beams" of quarks and gluons which interact, making reaction products which may be observed with tiny detectors, again of scale small compared to the confinement scale.[4] All of this should fit into space-time regions within which perturbation theory is really justifiable. Then (and only then?) can such processes be calculable by perturbative techniques alone. A *strictly-perturbative space-time region* can be defined as one which has the property that any straight line segment lying entirely within the region has an invariant length small compared to the confinement scale Λ_{QCD}^{-1} (whether or not the segment is spacelike or timelike). A little reflection should convince one that such strictly-perturbative domains are just the space-time regions adjacent to light cones. (For a light cone the only line-segments satisfying the criterion are null; so that the regions between hyperboloids $x^2 = a^2$ and $x^2 = -a^2$ evidently satisfy, for small enough a, the criterion for a strictly perturbative domain). While I haven't tried to prove it, it seems to me eminently reasonable that the relevant amplitudes and Green's functions within strictly perturbative domains really can be computed reliably using perturbation theory. Conversely, when the space-time region extends beyond such domains, it seems unavoidable that nonperturbative effects enter. It would be nice to sharpen these opinions further, but in these lectures it will only be done by example and not in generality.

What is the nature of convenient sources of quark and gluon "beams?" At large invariant distances the color-fields should be screened. A most economical way to guarantee this is that the *external sources of initial-state quark and gluon beams be local and color singlet.* For example, to obtain a beam of bottom quarks, first build a beam of Ws (of very high energy) and let them decay into $b\bar{c}$. Virtual photons are evidently an alternative. These are what we shall use in our examples, i.e., the "one-photon" and "two-photon" processes which form the lifeblood of e^+e^- collider physics experimentation. But, in general, we may assert the following:

1. Amplitudes for strictly perturbative processes shall be constructed from Green's functions

$$G(x_1, \dots x_n) = < O|T(O_1(x_1) \dots O_n(x_n))|O > \quad , \tag{1}$$

for which all operators $O_i(x_i)$ are local and color singlet.

2. After Fourier transformation, the momentum-space Green's functions $\tilde{G}(p_1 \dots p_n)$ will depend only on short distances, when the p_i^2 are all large and spacelike, and when the $p_i \cdot p_j$ $(i \neq j)$ are suitably restricted. (A sufficient restriction is that all p_i be "Euclidean" momenta, but this may not be necessary).

3. Given the confinement hypothesis, *all* information should be obtainable by analytic continuation of the Green's functions we have introduced. However, this does *not* imply that analytic-continuation of the approximate Green's functions constructed in perturbation theory provides this information.

The Green's functions we shall use in our examples involve only the electromagnetic current operator:

$$j^{\mu} = \sum_i e_i \, \overline{q}_i \gamma^{\mu} q_i \quad . \tag{2}$$

We first consider the two-point function for the vacuum polarization operator, and then the four-point function for the forward scattering amplitude for two virtual photons. These are sufficient for considering the e^+e^- total annihilation cross section into hadrons, and for the structure-functions for deep inelastic scattering of an electron from a virtual photon. Very interesting, but beyond the scope of these lectures is the question of how to describe the "final-state" properties of such processes, which, according to the lore of perturbative QCD, consist of sets of quark and gluon jets. Formally, these may be seen in the absorptive parts of the (appropriately analytically continued) Green's functions we have defined, as calculated in perturbation theory. Less formally they should be described in terms of the "physical" processes we have alluded to: tiny calorimeters placed "near the light-cone" pick up the quarks and gluons before they hadronize and measure the energy-momentum deposited into finite elements of solid angle $\Delta y \Delta \varphi$. How to link the formal description using the absorptive parts of the Green's functions to this "physical" picture is an interesting problem, well beyond the scope of these lectures. Some day I want to understand it better.

But this is more than enough of such general platitudes. The remainder of these lectures will be devoted to the e^+e^- total cross section and the virtual-photon's structure functions for deep-inelastic scattering. These examples will hopefully elucidate somewhat what I am driving at. Throughout this discussion, I assume the reader has some familiarity with a "standard" presentation of perturbative QCD as found in many places; the most immediate place is the fine set of lectures given by James Stirling in these proceedings.

ELECTRON-POSITRON ANNIHILATION INTO HADRONS

The total cross section for $e^+e^- \rightarrow hadrons$ normalized to the lowest-order cross section for $e^+e^- \rightarrow \mu^+\mu^-$, is given by the famous[5] function $R(s)$, which in the naïve lowest-order calculations is a constant equal to the sum of squares of charges of the participating quarks

$$R_{pert} = \sum_i e_i^2 \, [1 + \ldots] \quad , \tag{3}$$

and where the three dots denote perturbative-QCD radiative corrections, to be discussed later. Formally, R is related to the Fourier transform of a Green's function built from two electromagnetic currents

$$(q_\mu q_\nu - g_{\mu\nu} q^2) R(s) \; \alpha \int d^4x e^{iq\cdot x} < O | j_\mu(x) j_\nu(0) | O > \quad , \tag{4}$$

with $s = q^2$ timelike.

In general, is $R(s)$ a perturbatively calculable quantity for large s, according to our criteria? If so it should only depend upon the current correlation function at short space-time intervals. In the center-of-mass frame we deal with time intervals only. To test whether R is only sensitive to short time intervals, we may cut off the current correlation function at large times:

$$< O|j_\mu(x)j_\nu(0)|O > \to < O|j_\mu(x)j_\nu(0)|O > \exp\{-t^2/2\tau^2\} \quad , \tag{5}$$

and see the effect on $R(s)$. An easy calculation shows (for $\mu = \nu \neq 0$) that

$$sR(s) \to \int dE \ \exp\{-\tau^2 E^2/2\} \ s' R(s') \ \equiv \ s\overline{R}(s) \quad , \tag{6}$$

with $\sqrt{s'} = \sqrt{s} - E$.

In other words, $R(s)$ must be averaged[5] over an energy interval $\Delta\sqrt{s} \gg \tau^{-1}$ in order to be a strictly perturbative quantity. In particular, quarkonium resonances and sharp features of heavy-flavor thresholds must be smeared out over an energy scale large compared to $\Lambda_{QCD} \sim 200$ MeV. This is evidently just the uncertainty principle at work. It is amusing that, given a top quark mass in excess of the W mass, *physics* does the local averaging. The width of the top-quark decay $t \to Wb$ is large, in excess of 1 GeV. Thus there is no time available for toponium formation or even the formation of $T \equiv t\overline{q}$ mesons. Such processes of hadronization take place on a time scale long compared to the confinement time Λ_{QCD}^{-1}. Thus the threshold structure is already made smooth by the short t-quark lifetime. However, as pointed out by Fadin and Khose,[6] there are significant QCD radiative corrections near threshold which are numerically large, and which can be reliably estimated using a perturbative-QCD calculation, because everything happens within a strictly perturbative space-time domain.

A quantity related to $R(s)$ is the hadronic vacuum-polarization, evaluated at spacelike momenta. This is a sum of Feynman diagrams; one has:

$$(q_\mu q_\nu - g_{\mu\nu} q^2) \ \Pi(Q^2) \ \propto \ \int d^4x e^{iq\cdot x} < O|T(j_\mu(x)j_\nu(0))|O > \quad . \tag{7}$$

For large spacelike $q^2 = -Q^2$, the function $\Pi(Q^2)$ is necessarily smooth so that the averaging procedure is not needed (We will see this explicitly in what follows.) To lowest order in a perturbative calculation, $\Pi(Q^2)$ vanishes at $Q^2 = 0$ (after charge renormalization has been carried out) and grows logarithmically at large Q^2:

$$\Pi(Q^2) \propto \sum_i e_i^2 \ \ell n \frac{Q^2}{m_i^2} \quad . \tag{8}$$

A somewhat more convenient quantity for what will follow is the logarithmic derivative of $\Pi(Q^2)$, which we denote by $D(Q^2)$:

$$D(Q^2) \ = \ Q^2 \frac{d\Pi}{dQ^2} \ = \ \sum_i e_i^2 \ [1 + \ldots] \quad . \tag{9}$$

At the near-trivial parton level of calculation, D and R are, in fact, identical.

It should be clear that knowledge of R, i.e., of $< O|j_\mu(x)j_\nu(0)|O >$ implies knowledge of $\Pi(Q^2)$, hence of D. This is formally expressed in momentum space in the fact that $\Pi(Q^2)$ (or D) is an analytic function of Q^2 in the cut complex Q^2 plane, and that R is obtained from $\Pi(Q^2)$ by analytic continuation. In particular, R is the discontinuity of $\Pi(Q^2)$ across the branch cut. The formula is:

$$\Pi(Q^2) = Q^2 \int_0^\infty \frac{ds\ R(s)}{s(s+Q^2)} \quad . \tag{10}$$

Note that the threshold of the s-integral is at $4m_\pi^2$. Were R to be estimated perturbatively, the threshold would be somewhere else ($\sim 4m_q^2$) and $\sqrt{s}$ smearing is definitely called for.

We are now prepared to discuss the nature of the perturbative-QCD correction to $\overline{R}(s)$ [or $D(Q^2)$]. We again assume some familiarity with the workings of the theory and summarize what happens (in the MS or $\overline{MS}$ renormalization scheme)

1. In order to manage the divergences which appear, one cuts off the Feynman integrals by evaluating them in a space-time dimension D slightly less than four:

$$D = 4 - \epsilon \quad . \tag{11}$$

2. By dimensional analysis, the (bare) coupling constant α_0 multiplying the Feynman integrals acquires a nonvanishing dimensionality proportional to ϵ.
3. A new dimensionless coupling $\alpha_0(\mu^2)$ is defined by introducing an arbitrary scale-factor μ^2:

$$\alpha_0 \equiv \alpha_0(\mu^2)\mu^{4-D} \quad . \tag{12}$$

4. The theory is renormalized, removing the singular dependence on space-time dimension, i.e., on ϵ, of the Feynman-integrals.
5. The renormalized quantity $\overline{R}$ then depends upon three variables:

$\sqrt{s}$: the energy variable;

$\alpha_s(\mu^2)$: the renormalized (dimensionless) coupling constant; and

μ^2 : the arbitrary mass scale.

6. However, since $\overline{R}$ represents physics and the value of μ is an arbitrary choice, $\overline{R}$ cannot depend upon μ. This means:

$$O = \mu^2 \frac{d}{d\mu^2} \overline{R}(s,\ \alpha_s(\mu^2),\ \mu^2) = \mu^2 \frac{\partial\alpha_s}{\partial\mu^2}\frac{\partial\overline{R}}{\partial\alpha_s} + \mu^2 \frac{\partial\overline{R}}{\partial\mu^2} \quad . \tag{13}$$

Defining, somewhat unconventionally,

$$\beta(\alpha_s) = -\frac{\mu^2}{\alpha_s}\frac{\partial\alpha_s}{\partial\mu^2} \quad , \tag{14}$$

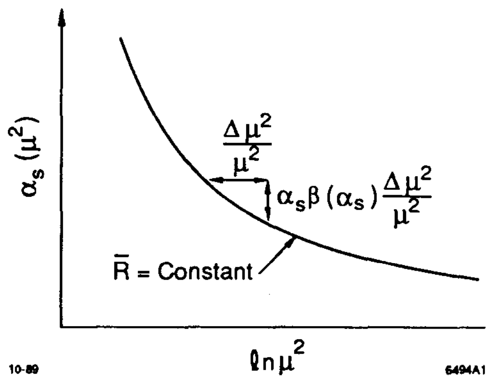

Figure 1: Behavior of the running coupling constant with scale factor μ.

we see that in $\alpha_s - \ell n\, \mu^2$ space (Fig. 1), there are lines along which $\overline{R}$ does not change. The local slope of such lines can be read off Eq. (13):

$$\frac{\mu^2}{\alpha_s}\frac{d\alpha_s}{d\mu^2} = -\beta(\alpha_s) \quad . \tag{15}$$

These curves $\alpha_s = \alpha_s(\mu^2)$, along which $\overline{R}$ remains constant, define the running coupling constant.

7. Only one of the curves will be consistent with experiment: for a given choice of μ^2, $\alpha_s(\mu^2)$ has to be chosen to agree with the data.
8. Dimensional analysis demands that $\overline{R}$ be a function only of α_s and of s/μ^2. Putting this constraint together with the argument that R be independent of μ^2 leads to the conclusion that R is a function of only a *single* variable, which must be $\alpha_s(s)$.
9. Because perturbative QCD allows a formal power series expansion in $\alpha_s(\mu^2)$, this implies the existence of a formal power series expansion in $\alpha_s(s)$.

We have left out details; the student is urged to consult Stirling's lectures and standard sources to fill them out. But the main point is that this line of argument is based on the renormalizability of the theory and is quite general. It therefore applies equally well to the vacuum polarization $\Pi(Q^2)$ or, better, $D(Q^2)$. We therefore have for the quantity experimentalists measure

$$\overline{R}(s) = \sum_i e_i^2 \left[1 + \sum_{m=1}^{\infty} r_m \left(\frac{\alpha_s(s)}{\pi}\right)^m\right] \quad , \tag{16}$$

and for the quantity theorists calculate

$$D(Q^2) = \sum_i e_i^2 \left[1 + \sum_{m=1}^{\infty} d_m \left(\frac{\alpha_s(Q^2)}{\pi}\right)^m\right] \quad . \tag{17}$$

These two quantities are related by a dispersion relation following directly from Eq. (10).

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds R(s)}{(s+Q^2)^2} = \int_0^\infty \frac{dx R(xQ^2)}{(1+x)^2} \cong \int_0^\infty \frac{dx \overline{R}(xQ^2)}{(1+x)^2} \quad . \tag{18}$$

It turns out that interesting information emerges just from the fact that both D and $\overline{R}$ admit power-series expansions in α_s. We see from Eq. (18) that D is just a local average of $\overline{R}$, so that if $\overline{R}$ is slowly varying, D and $\overline{R}$ are essentially the same. But if we expand $\overline{R}$ as a power series in $\alpha_s(xQ^2)$, then it is possible to use the equation for the running coupling constant to express $\alpha_s(xQ^2)$ (perturbatively) in terms of $\alpha_s(Q^2)$ and thereby construct the power series for $D(Q^2)$. Evidently, D and $\overline{R}$ are not *identically* the same, so that the series are not the same. Indeed, in what follows we shall find evidence that the series expansion of D–$\overline{R}$ is almost certainly asymptotic, not absolutely convergent. The same statement probably applies for D and $\overline{R}$ separately, as well.

In order to make the connection one must know how the coupling constant runs. With our definition, the β-function admits a power-series expansion in α_s which begins in first order. Keeping only that contribution leads to the well-known expression:

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + \frac{b}{\pi}\,\ell\text{n}\,\frac{Q^2}{\mu^2} \equiv \frac{b}{\pi}\,\ell n\,\frac{Q^2}{\Lambda^2_{QCD}} \quad , \tag{19}$$

with $b = (33-2n_f)/12 = 2.08\pm0.17$ for the effective number of flavors 4 ± 1. Expressing $\alpha_s(xQ^2)$ in terms of $\alpha_s(Q^2)$ will therefore lead to a power-series expansion in $\alpha_s(xQ^2)$ with the nth coefficient being (at most) an nth-order polynomial in $\ell\text{n}\,x$. (This is still true when the higher-order corrections to the β-function are included.) Thus the generic term in the convolution integral is of the form (note the symmetry under $x \leftrightarrow x^{-1}$):

$$\int_0^\infty \frac{dx(\ell\text{n}\,x)^n}{(1+x)^2} = \begin{cases} 0 & n \text{ odd} \quad , \\ 2\zeta(n)[1-2^{1-n}]n! & n \text{ even} \quad . \end{cases} \tag{20}$$

(It is gratifying to see a ζ-function appearing here, signalling perhaps some connection to conformal symmetry. It would be a terrible thing not to be at least slightly *au courant.* Long ago, Feynman observed[7] that there were then only two options open to theorists: to either form a group or to disperse. There seems to be even less choice nowadays: one must conform.)

With knowledge of how to do the integrals, it is only algebra to figure out the series for D–$\overline{R}$. Evidently, the order α_s/π term vanishes. Because the integral over one power of $\ell\text{n}\,x$ vanishes, the $(\alpha_s/\pi)^2$ term also vanishes. Only in third order do the two series begin to differ, and one easily finds

$$D - \overline{R} = \left(\frac{\pi^2 r_1 b^2}{3}\right)\left(\frac{\alpha_s}{\pi}\right)^3 + \ldots \approx 14\left(\frac{\alpha_s}{\pi}\right)^3 \quad , \tag{21}$$

where we use the known result that the first radiative correction to $\overline{R}$ is

$$r_1 = 1 \quad . \tag{22}$$

The perturbative series for $\overline{R}$ has been calculated through order $(\alpha_s/\pi)^3$, and the results for the higher order coefficients are

$$\begin{aligned} r_2 &= 1.53 \pm 0.12 \quad , \\ r_3 &= 66.10 \pm 1.24 \quad , \end{aligned} \tag{23}$$

where, again, we let $n_f = 4 \pm 1$ for this purpose. The large value of the $(\alpha_s/\pi)^3$ term, calculated recently[8] by Gorishny, Kataev, and Larin (hereafter GKL), has surprised many people. However, from the point of view of Eq. (21) for the difference D–$\overline{R}$ (which, by the way, is explicitly presented in the GKL paper; they, after all, calculate D and quote $\overline{R}$), this should not be surprising. There seems no particular reason why D should converge better than $\overline{R}$ (or vice versa), so that the estimate for the difference should reflect the behavior of the individual quantities.

But this is not the end of the story. The coefficient of the $(\alpha_s/\pi)^3$ term for the difference of D–$\overline{R}$ was very simple to calculate and used only lowest-order QCD calculations. Not only have higher order corrections to $\overline{R}$ been computed, but also higher-order corrections to the β-function; in our notation, we write

$$-\frac{\mu^2}{\alpha_s}\frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = b\left(\frac{\alpha_s}{\pi}\right)\left[1+\sum_{n=1}^{\infty} b_n \left(\frac{\alpha_s}{\pi}\right)^n\right] \quad , \tag{24}$$

with

$$\begin{aligned} b &= \frac{33-2n_f}{12} = 2.08 \pm 0.17 \\ b_1 &= 1.51 \pm 0.27 \\ b_2 &= 2.97 \pm 1.50 \quad . \end{aligned} \tag{25}$$

This allows computation of the next two orders of D–$\overline{R}$, out through the $(\alpha_s/\pi)^5$ contribution! One finds, after slightly more arduous algebra,

$$\begin{aligned} d_4 - r_4 &= \pi^2 b^2 \left(r_2 + \frac{5}{6} r_1 b_1\right) = 43(1.5+1.3) = 119 \pm 17 \\ d_5 - r_5 &= \pi^2 b^2 \left(2r_3 + \frac{7}{3} r_2 b_1 + r_1 b_2 + \frac{1}{2} r_1 b_1^2\right) + \frac{7\pi^4}{15} r_1 b^4 \\ &\cong 43(132+5+3+1) + 855 \\ &= 6920 \pm 620 \quad . \end{aligned} \tag{26}$$

The largeness of these terms mostly reflects the largeness of b (note that for QED, the corresponding quantity is about seven times smaller). However, one also sees the beginnings of an asymptotic series emerging in the factors of $n!$ in numerators from the integrals of Eq. (20). To go to still higher orders is, in general, tedious, and requires unknown input. However, one observes that the contributions from higher order corrections to the β-function proportional to b_1 and b_2 were not especially significant numerically. This invites considering the approximation of neglecting all but the leading term for the β-function. It turns out that in this limit one can easily estimate the nth-order coefficient in the expansion of D–$\overline{R}$. To do this most efficiently, it is convenient to introduce the Borel transform of the perturbation series. One assumes that the functions D, $\overline{R}$, etc., which we generically call $F(\alpha_s)$, are obtainable as Laplace transforms in α_s^{-1} of another function $\tilde{F}(z)$ (the Borel transform):

$$F(\alpha_s) = \int_0^\infty dz e^{-z/\alpha_s}\, \tilde{F}(z) \quad . \tag{27}$$

Why this representation? If $\tilde{F}(z)$ admits a power series expansion in z (more or less),

$$\tilde{F}(z) = F_0\delta(z) + \sum_{n=1}^{\infty} \tilde{f}_n \, z^{n-1} \quad , \tag{28}$$

then one immediately finds:

$$F(\alpha_s) \equiv \sum_{n=1}^{\infty} f_n \alpha_s^n = F_0 + \sum_{n=1}^{\infty} (n-1)! \, \tilde{f}_n \, \alpha_s^n \quad . \tag{29}$$

In other words, the power series for $\tilde{F}$ is related to that of F and converges much better:

$$\tilde{f}_n = \frac{f_n}{(n-1)!} \quad . \tag{30}$$

For this reason, this Borel transform has been used by theorists[9] to investigate the convergence of the perturbation series. It is believed that $\tilde{F}(z)$ has a finite radius of convergence, with branch-point singularities on the real z-axis known in the trade as "renormalons." If this is truly the case, then the radius of convergence of the usual perturbation series for $F(\alpha_s)$ is zero with the nth term in the series eventually growing roughly as $n!$

The utility of the Borel transform, in our case, comes from the fact that the convolution relating D and $\overline{R}$, Eq. (18), factorizes. Introducing the Borel transform for $\overline{R}$ and using Eq. (19) for the running coupling constant yields:

$$\begin{aligned} D(\alpha_s) &= \int_0^{\infty} \frac{dx}{(1+x)^2} \int_0^{\infty} dz \tilde{R}(z) \, \exp\{-z/[\alpha_s(xQ^2)]\} \quad , \\ &= \int_0^{\infty} dz \, \exp\{-z/[\alpha_s(Q^2)]\} \, \tilde{R}(z) \int_0^{\infty} \frac{dx \, \exp\{-bz\ell n \, x/\pi\}}{(1+x)^2} \quad . \end{aligned} \tag{31}$$

The x-integral is a beta-function, and the remainder is in the Borel-transform format. Therefore, the result is simply:

$$\tilde{D}(z) = \tilde{R}(z) \cdot \left(\frac{bz}{\sin bz}\right) \quad . \tag{32}$$

One sees singularities appearing on the real z axis; these occur at the positions of the renormalons to which we alluded.

From this recursion relation, one may easily construct the power series expansion for $\tilde{D}$, hence of D, from that of $\overline{R}$. The easiest way to write this is:

$$\tilde{d}_n = \tilde{r}_n + \sum_{k=2,4\ldots} 2\zeta(k) \left(1 - 2^{1-k}\right) \tilde{r}_{n-k} \left(\frac{b}{\pi}\right)^k \quad , \qquad (n \geq 3) \quad . \tag{33}$$

Note there are no large coefficients in this expansion-nor small ones. However, upon returning from the Borel expansion coefficients to the original ones, we pick up a factor $(n-1)!$, which *is* large.

I have made some estimates of the nth coefficient, assuming:

$$r_n \gtrsim |d_n - r_n| \quad . \tag{34}$$

This leads to the values of $\delta_n = d_n - r_n$ quoted in Table I. One sees a remarkable growth in the coefficients. In Table II, the actual values for the nth term of the series expansion for δ_n and/or r_n are tabulated for a variety of choices of α_s/π. The entries above the line are secure, while the entries below depend on the guesswork we have introduced, namely, the approximate validity of the leading order expression, Eq. (19), for the running coupling constant, as well as the estimate for r_n in Eq. (34).

TABLE I Estimated coefficients for $\delta_n = d_n - r_n$.

$$\begin{aligned}
\delta_3 &= 15 \\
\delta_4 &= 120 \\
\delta_5 &= 6900 \\
\delta_6 &= 23{,}000 \\
\delta_7 &= 2{,}400{,}000 \\
\delta_8 &= 12{,}000{,}000 \\
\delta_9 &= 1{,}600{,}000{,}000
\end{aligned}$$

TABLE II N^{th} order contributions to $D - \overline{R}$ and $\overline{R}$.

		$\frac{\alpha_s}{\pi} = 0.3$		$\frac{\alpha_s}{\pi} = 0.1$		$\frac{\alpha_s}{\pi} = 0.05$ (PEP/PETRA)		$\frac{\alpha_s}{\pi} = 0.03$ (SLC/LEP)		$\frac{\alpha_s}{\pi} = 0.01$ (GUT)	
1		1.00		1.000		1.000		1.000		1.00	
$r_1(\frac{\alpha}{\pi})$		0.30		0.100		0.050		0.030		10^{-2}	
$r_2(\frac{\alpha}{\pi})^2$		0.13		0.015		0.004		0.001		1.5×10^{-4}	
$r_3(\frac{\alpha}{\pi})^3$	$\delta_3(\frac{\alpha}{\pi})^3$	1.8	0.4	0.069	0.015	0.008	0.002	0.002	4×10^{-4}	6.9×10^{-5}	1.5×10^{-5}
	$\delta_4(\frac{\alpha}{\pi})^4$		1.0		0.012		0.0008		1×10^{-4}		1.2×10^{-6}
	$\delta_5(\frac{\alpha}{\pi})^5$		17		0.069		0.002		1.7×10^{-4}		6.9×10^{-7}
?	$\delta_6(\frac{\alpha}{\pi})^6$		17		0.023		4×10^{-4}		1.7×10^{-5}		2×10^{-8}
?	$\delta_7(\frac{\alpha}{\pi})^7$		500		0.24		2×10^{-3}		5×10^{-5}		2×10^{-8}
?	$\delta_8(\frac{\alpha}{\pi})^8$		800		0.12		5×10^{-4}		8×10^{-6}		1.2×10^{-9}
?	$\delta_9(\frac{\alpha}{\pi})^9$		30,000		1.6		3×10^{-3}		3×10^{-5}		1.6×10^{-9}

I think the main lesson to be learned from this is something recognized for a long time by the experts: the perturbation series *is* asymptotic. The exercise we have done helps to give some feeling as to when the trouble appears. According to Table II, one should truncate the series at order $n \sim 1.5\alpha_s^{-1}$. For most "practical" energies (PEP/PETRA and above), this is still well beyond the calculations. However, there is perhaps a message for those who work on perturbative QCD at the interface with nonperturbative effects, i.e., at small mass scales, e.g., where $\alpha_s(\mu^2) \sim 1$. It is simply that one should probably *stop* at leading order and settle for roughly 30 to 50 percent agreement with experiment: attempts to "improve" the situation by calculating higher orders most likely only create confusion and worsen the situation. I think that perturbative-QCD theorists should find this welcome. It is some justification for laziness.

The reader may find it surprising that the apparently innocent relationship of D to an average of $\overline{R}$, as expressed in Eq. (18), should lead to such evidence for the asymptotic nature of the perturbation series. What happened? It is true that, for any α_s reasonably small, D and $\overline{R}$ are nearly equal. But because of the ubiquity of QCD logarithms, to obtain the difference to high accuracy, one needs to sample R over a dynamic range in $\log x$ which grows linearly with the perturbation order n. Thus when $n \gtrsim \ell n\, s/\Lambda^2_{QCD}$, one must sample the infrared region, where $\overline{R}$ is poorly defined by the perturbation series. But this condition is the same as the estimate $n \sim 1.5\alpha_s^{-1}$ quoted earlier.

As for the interpretation of renormalon poles, one may see from the defining equation for the Borel transform that a pole at $z = n$ is related to a power-law contribution to R (to leading order in the series expansion of the β-function):

$$\begin{aligned} F(\alpha_s) &= \sum_n \int_0^\infty \frac{dz \exp\{-z/\alpha_s\}}{bz - n\pi} \to \sum_n (Residues) \cdot \exp\{-n\pi/[b\alpha_s(Q^2)]\} \\ &\sim \sum_n (Residues) \cdot (Q^2/\Lambda^2_{QCD})^{-n} \quad . \end{aligned} \tag{35}$$

Thus one finds linkage between power-law contributions to $\overline{R}$, as computed using operator-product expansion techniques, and the location and structure of the renormalon singularities.

THE STRUCTURE FUNCTION OF A VIRTUAL PHOTON

Our second example of a strictly perturbative process is deep inelastic scattering of an electron from a virtual (spacelike) photon. The classic deep-inelastic process of electron-proton scattering does not qualify because the proton is evidently too big to fit into a strictly perturbative space-time domain near the light-cone. The virtual spacelike photon with squared mass $p^2 = -P^2$ *is* small, with transverse extension of order P^{-1}. It is produced from an electron or positron in the familiar two-photon process studied at e^+e^- colliders.

While the electromagnetic structure of the spacelike photon is amenable to a perturbative analysis, it would, of course, be nice to consider the real photon, not to mention the timelike photon as well; especially the extrapolation to the vector-boson ρ, ω, ϕ states. Indeed, the case of the real photon is a very interesting subject with a rich history. There was at one time considerable optimism that this process was an excellent test-bed for perturbative QCD, and might provide an accurate measurement of α_s. But there arose complications, to be described in more detail in what follows. By now, the optimism has waned considerably.[10] Nevertheless, the process is most interesting theoretically.

For a spacelike virtual photon with four-momentum p, there are several structure functions to consider. We shall restrict our attention to transverse photons only, and

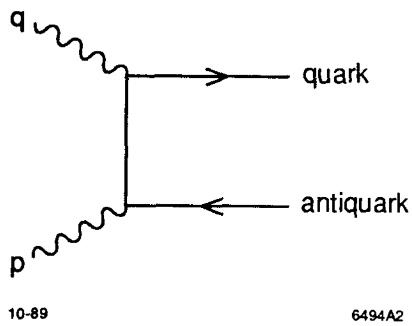

Figure 2: Lowest-order graph for "photon fusion" into quark–antiquark pairs.

only the helicity-independent contribution as well. The kinematic formulae can be found in many places[11] and will not be reproduced here.

Letting $Q^2 = -q^2$ be the (spacelike) squared mass of the probing photon, it should be clear that when both Q^2 and $P^2 = -p^2$ are large, the starting-point is the simple "photon-photon fusion" of quark pairs (Fig. 2), which gives what naïvely would be the leading contribution to the cross section. This was calculated long ago,[12] even before the advent of QCD. For $Q^2 \gg P^2 \gg \Lambda^2_{QCD}$, the relevant structure function (analogous to F_2 for nucleons) is easily computed to be:

$$F_2(x) \approx \left(\sum_i e_i^4 \right) x \left[x^2 + (1-x)^2 \right] \ell n \, Q^2/P^2 \quad . \tag{36}$$

For $P^2 \gg Q^2 \gg \Lambda^2_{QCD}$, the structure function vanishes rapidly, because the probing photon sees a small dipole of size P^{-1}, but only has resolving power Q^{-1}. Hence, there is a power-law scaling violation proportional to Q^2/P^2 (up to logarithmic factors).

The factors in Eq. (36) are reasonably easy to understand:

1. The amplitude is proportional to the square of the quark charge; hence the cross section to the fourth power.
2. The small x behavior follows the Regge-pole rule: exchange of spin J in the amplitude leads to an s^{2J-2} or x^{2-2J} behavior in the cross section. In this case, $J = 1/2$, the exchanged quark being treated as "elementary."
3. The factor $[x^2 + (1-x)^2]$ is the probability of finding a quark of momentum xp_μ in the photon of momentum p_μ (as $p_0 \to \infty$); it is just an Altarelli–Parisi "splitting function." (We assume the reader to be familiar with the basics of the Altarelli–Parisi formalism).[13]
4. The logarithm appears from a "collinear" singularity in the angular distribution of the quark pair relative to the photon direction, and is of the same nature that generates the leading logarithms in QCD.

It is of special interest that scaling in x is violated logarithmically (for $Q^2 \to \infty$; P^2 fixed), and that at large x the structure function is big: it approaches a constant as $x \to 1$. This is in sharp contrast to what is expected from vector-meson dominance, where the hadronic part of the photon is assumed to be a mixture of ρ, ω, $\phi \ldots$. The structure functions of such mesons are believed to vanish as $x \to 1$ in a manner similar to that of the pion or kaon, where the behavior is $(1-x)^n$, with n between 1 and 2. Therefore, observation of the photon structure function at large x ("large" meaning $x > 0.4$) is a good method of revealing its "pointlike" components.

The theoretical situation became considerably more interesting when Witten[14] considered the QCD radiative corrections to this parton-model calculation. He showed that at sufficiently large Q^2 even a real photon should exhibit the pointlike behavior, and that its structure function should violate scaling by one power of log Q^2 just as in Eq. (36). However, the *shape* of the function of the scaling variable x multiplying the log Q^2 term is changed in a calculable way by the higher-order QCD corrections. Thus it appeared that a measurement of this predicted shape would be a good test of QCD.

The complications began when next-to-leading order corrections to Witten's leading-order calculation were considered.[15] At small x, these had a very large effect—so large that the computed structure function went *negative,* a clearly inadmissible result. Subsequent work which traced down the origin of this phenomenon showed that one needed to include terms nonleading in $(\ell n\, Q^2)$ and link them carefully to the leading term in order to resolve this problem. The importance of the nonleading contributions, which have a Q^2 dependence typical of hadron structure functions instead of the $(\ell n\, Q^2)^1$ scaling-violation present in leading order, makes the phenomenology more complicated, and thus far has dampened the original optimism that this process was a good quantitative testing ground for perturtubative QCD.

It is our intention to review this somewhat confusing situation. We shall start with consideration of the structure function when $p^2 = -P^2$ is large and spacelike[16] so that by our criterion the process is strictly perturbative. (Historically, Witten considered real photons only, and while there is nothing technically wrong with his analysis, this choice has been a source of some of the confusion.) Once this case is worked out, we consider what happens when P^2 becomes null or timelike. The main tool to be used here is analytic continuation in P^2. The variable P^2 will here play a role quite similar to the momentum variable Q^2 used in the discussion of R and vacuum polarization. We will be able to see in a controlled way how the hadronic, nonperturbative aspects of the problem enter when P^2 is allowed to become small.

Within the QCD ideology, there are two major lines of attack on the structure-function problem. One uses the operator-product expansion plus renormalization group considerations to calculate the scaling violations of moments of the structure functions.[17] The other uses the Altarelli–Parisi evolution-equations for the structure-functions themselves. The former method is more rigorous, but also more abstract. The latter method allows some physical insight at the parton-model level into what is going on, but is harder to justify theoretically, especially when nonleading contributions are to be included. Indeed, the best justification for the Altarelli–Parisi approach is that it gives the same answers as the operator-product-expansion methodology.

In this discussion we shall use both methods, but begin with the Altarelli–Parisi approach. Their equation is schematically written as:

$$\begin{aligned} Q^2\,\frac{dF_2\,(x,Q^2)}{dQ^2} &= P_{qq}\otimes F_2 + P_{qG}\otimes G \quad , \\ Q^2\,\frac{dG}{dQ^2} &= P_{Gq}\otimes F_2 + P_{GG}\otimes G \quad , \end{aligned} \tag{37}$$

with the convolution being given by a ratio-kernel:

$$P \otimes F \equiv x \int_x^1 \frac{dy}{y} P(x/y, Q^2)\, F(y, Q^2) \quad , \tag{38}$$

and with the "splitting-functions" $P(z, Q^2)$ simple, known quantities.

The physics is that: (i) the importance of the QCD radiative corrections increases with Q^2 because of the increase in available phase-space; and (ii) the important contributions to leading logarithmic accuracy come from approximately collinear configurations of the initial-state and final-state quarks and/or gluons. Item (i) implies the integro-differential nature of the equation: the *change* with $\ell n\, Q^2$ in the parton distribution is given by the convolution. Item (ii) assures the survival of the parton-model interpretation despite the increase in transverse momentum of relevant constituents. The essential dynamics remains collinear, as required by parton-model ideology.

The standard Altarelli–Parisi equations, as written, are homogeneous. But for the photon structure-function there is an inhomogeneous driving team because the "bare" process in Fig. 2 has the linear dependence on log Q^2 as given by Eq. (36). For simplicity in what follows, we write down the modified Altarelli–Parisi equation omitting the gluon contributions; i.e., we consider a "nonsinglet" structure function. (Nothing essential is lost in this simplification, and in what follows we shall indicate the necessary modifications at the appropriate places). The equation then becomes:

$$Q^2 \frac{dF_2}{dQ^2} = P \otimes F_2 + f(x) \quad , \tag{39}$$

with

$$f(x) = \left(\sum_i e_i^4 \right) x\, [x^2 + (1-x)^2] \quad . \tag{40}$$

The solution of integral equations with ratio kernels is found with the aid of the Mellin transform. One defines moments of the structure-function and splitting function:

$$\begin{aligned} \tilde{F}(n, Q^2) &= \int_0^1 dx x^{n-2}\, F_2(x, Q^2) \quad , \\ \tilde{P}(n, Q^2) &= \int_0^1 dz z^{n-1}\, P(z, Q^2) \quad . \end{aligned} \tag{41}$$

Applying this to the integral equation unravels the convolution:

$$Q^2 \frac{d\tilde{F}\,(n, Q^2)}{dQ^2} = \tilde{P}(n, Q^2) \tilde{F}(n, Q^2) + \tilde{f}(n) \quad . \tag{42}$$

The general solution of this simple differential equation is a sum of a homogeneous solution and a particular solution. The homogeneous solution is the (hopefully) familiar one used in hadron structure-function analysis:

$$\tilde{F}(n, Q^2) = \tilde{F}(n, Q_0^2) \exp\left\{ \int_{Q_0^2}^{Q^2} (d\sigma^2/\sigma^2)\tilde{P}(n,\sigma^2) \right\} . \tag{43}$$

The "anomalous dimension" $\tilde{P}(n,\sigma^2)$ is, to leading order, proportional to the running coupling constant $\alpha_s(\sigma^2)$; hence, proportional to $(\ell n\, \sigma^2/\Lambda_{QCD}^2)^{-1}$. When this dependence is introduced into Eq. (42), a short calculation leads to the behavior:

$$\frac{\tilde{F}(n,Q^2)}{\tilde{F}(n,Q_0^2)} = \left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{d(n)} . \tag{44}$$

The exponent $d(n)$ is obtained from the appropriate moment of the splitting function and Eq. (19) for α_s. It is positive for large n (large x) and negative for small n. This behavior leads to the familiar pattern of scaling violations for the structure function F_2.

(If one were to consider the coupled equations, Eq. (39) would include the gluon structure functions, and one would find after taking moments a 2×2 matrix problem to solve. The answer involves a sum of two pieces behaving as $[\alpha_s(Q^2)]^{d^{\pm}(n)}$ with $d^{\pm}(n)$ the eigenvalues of a 2×2 "anomalous dimension" matrix. This modification will not alter in a significant way the discussion to follow.)

Inspection of Eq. (42) shows that a particular solution is, as already advertised, proportional to log Q^2, i.e., to α_s^{-1}.

$$\tilde{F}(n,Q^2) = [constant]\cdot[\alpha_s(Q^2)]^{d(n)} + \frac{\tilde{f}(n)\ell n\, Q^2/\Lambda_{QCD}^2}{1+d(n)} . \tag{45}$$

This follows because of the behavior of the α_s multiplying the splitting function $\tilde{P}$:

$$\tilde{P}(n,Q^2) \equiv \frac{-d(n)}{\ell n\, Q^2/\Lambda_{QCD}^2} . \tag{46}$$

The presence of the inhomogeneous term requires a special procedure for obtaining the normalization constant for the homogeneous term. It is found by observing that for virtual P^2 near Q^2 there is no QCD evolution at all, and that the lowest order "bare quark" term should represent, to this accuracy, the whole contribution. Also, we have been suppressing the P^2 dependence of $\tilde{F}$. Putting that in and applying the boundary condition produces the desired result:

$$\begin{aligned}\tilde{F}(n,Q^2,P^2) &= \frac{-\tilde{f}(n)(\ell n\, P^2/\Lambda_{QCD}^2)}{1+d(n)}\left[\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right]^{d(n)} + \frac{\tilde{f}(n)(\ell n\, Q^2/\Lambda_{QCD}^2)}{1+d(n)} \\ &= \frac{\tilde{f}(n)(\ell n\, Q^2/\Lambda_{QCD}^2)}{1+d(n)}\left\{1-\left[\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right]^{d(n)}\right\} .\end{aligned} \tag{47}$$

Thus the boundary condition for simple behavior when $Q^2 \sim P^2$ has provided linkage between the homogeneous ("hadronic") term, with its typical scaling-violation

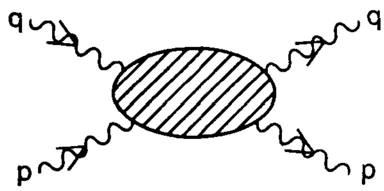

Figure 3: Photon-photon forward scattering amplitude.

behavior, and the inhomogeneous ("pointlike") term with single-log "Witten" behavior $\alpha_s(Q^2)^{-1}$.

Evidently, as $d(n)$ goes negative for small n, it is essential to have *both* terms present; keeping only the "leading" term would lead to an unphysical singularity. This is essentially what happened in the early days of confusion. An additional problem is evident if one wishes to let P^2 go to small or timelike values. If $[1 + d(n)] > 0$, it appears that the limit is stable, and only the leading "Witten" term survives. If $[1 + d(n)] < 0$, the conventional"hadronic" term looms up in importance. And in both cases, one must ask whether *nonperturbative* effects can infiltrate the results. To cope with these issues it is convenient to use analytic continuation in P^2, the probed photon squared mass, for fixed Q^2.

To understand that this is possible, it is convenient to review briefly the formal operator-product approach to deep-inelastic processes. We shall provide only a sketch, omitting almost all technical detail. One starts with the forward scattering amplitude $T(q,p)$ of current q from current p; diagrammatically shown in Fig. 3. It is a sum of Feynman diagrams, the Fourier transform of a four-point function:

$$T(q,p) \;=\; \int d^4x d^4y d^4z \; e^{iq\cdot(x-y)} \; e^{-ip\cdot z} \;\; < O|T(j(x)j(y)j(z)j(0))|O > \quad , \quad (48)$$

where all spin indices, etc., have been omitted. For fixed $q^2 = -Q^2$ and $p^2 = -P^2$ spacelike, T is an analytic function of the energy variable $\nu = q \cdot p$, except for branch cuts on the real axis beginning at $(p \pm q)^2 = 4m_q^2$ or $4m_\pi^2$, depending upon whether one considers the perturbative approximant $\overline{T}$ or the exact amplitude T (for which there are also poles at $(p \pm q)^2 = m_\pi^2$). In the ν plane, the cuts therefore occur at:

$$\nu \;=\; \nu_0 \;\approx\; \pm\frac{P^2 + Q^2}{2} \quad . \quad (49)$$

In terms of the scaling variable,

$$\omega \;=\; x^{-1} \;=\; \frac{2q\cdot p}{Q^2} \quad . \quad (50)$$

T (or $\overline{T}$) satisfies a dispersion relation in the cut ω-plane with threshold

$$\omega_0 \;=\; 1 + \frac{P^2}{Q^2} \;\gtrsim\; 1 \quad . \quad (51)$$

One has (for a "nonsinglet" structure function):

$$T(\omega, Q^2, P^2) = \int_{\omega_0}^{\infty} d\omega' \left[\frac{W(\omega', Q^2, P^2)}{\omega' - \omega} + \frac{\overline{W}(\omega', Q^2, P^2)}{\omega' + \omega} \right] \quad . \tag{52}$$

The discontinuities of T across the branch cuts (i.e., the absorptive part), which we call W or $\overline{W}$, are, up to suppressed kinematic factors, the cross sections for the photon-photon collision. These are essentially just the structure functions F_2.

The operator-product expansion expresses the product of currents $j(x)j(y)$ in Eq. (48) by a sum of local operators $O_n(x)$ multiplied by c-number coefficients $C_n(x-y)$, depending only on the coordinate differences, which are to be regarded as "small". In this application, it turns out to be enough to have the invariant distance $(x-y)^2$ small, i.e., near the light-cone.

Upon doing the y-integration in Eq. (48), one sees that all the dependence on q in the nth term of the expansion goes into the coefficient function. The remaining contribution depends only on p. This, however does not imply a complete factorization. Spin indices, brutally suppressed here, must be restored. Typically, the operator "O_n" has n tensor indices (give or take one or two) which are contracted into similar tensor indices possessed by the coefficient function C_n. This gives a structure for the nth term in the operator product expansion as follows:

$$n\text{th term} = A_n(Q^2)\; B_n(P^2)\; P_n(q \cdot p) \quad , \tag{53}$$

where the P_n is a *polynomial* in $(q \cdot p)$ of order n (give or take one or two). What this implies is that a series expansion of T in $q \cdot p$ (for fixed Q^2 and P^2) essentially allows identification of its nth term with the nth contribution to the operator-product expansion. Because of nonvanishing photon spin, gauge-invariance, etc., things are not quite so simple as sketched above. But the complications are essentially of a technical nature.

Because at large spacelike Q^2 and P^2 the amplitude T is analytic in a large neighborhood around $q \cdot p = 0$, the series-expansion is convergent. The coefficients are given in terms of the moments of the structure functions W and $\overline{W}$ already encountered in the Altarelli–Parisi approach. This can be seen from the dispersion-relation, Eq. (52). In the scaling limit, one finds (for $P^2 \ll Q^2$):

$$T = \sum_n T_n \,(2q \cdot p/Q^2)^n \quad , \tag{54}$$

with

$$T_n = \int_{\omega_0}^{\infty} \frac{d\omega'}{(\omega')^{n+1}} \left[W - (-)^n\, \overline{W}\right] = \int_0^1 dx x^{n-1} \left[W - (-)^n\, \overline{W}\right] \quad . \tag{55}$$

Here, the usual deep-inelastic scaling variable is given by:

$$x \equiv (\omega')^{-1} \quad . \tag{56}$$

Our main reason for going through all this is to emphasize that the moments of the structure functions W and $\overline{W}$ are related to a Green's function (sum of Feynman diagrams), with that Green's function being a finite number of derivatives with respect to $q \cdot p$ of T about $q \cdot p = 0$. Furthermore, $T(Q^2, P^2, 0)$ (and the derivatives thereof) has good analytic properties in P^2 at fixed spacelike Q^2 (or *vice versa*). This is most easily seen by examining, for any Feynman diagram, the expression obtained after "combining denominators" using Feynman's famous formula and doing the momentum-space integrations.[18] The resultant expression has the generic form.

$$I = \int \frac{d\alpha_1 \cdots d\alpha_n N(\alpha_1 \cdots \alpha_n)}{[\zeta_1(\alpha)Q^2 + \zeta_2(\alpha)P^2 + M^2(\alpha)]^m} \quad , \tag{57}$$

where ζ_1, ζ_2, and M^2 can be shown to be positive semidefinite irrespective of choice of $\alpha' s$ or of Feynman-diagram. Thus one concludes that the moments of the structure functions are, for fixed spacelike q^2, analytic in the cut p^2 plane, with a branch cut with normal thresholds at positive p^2. Thus one has, at long last, the desired result.[19]

$$\tilde{F}(n, Q^2, P^2) \equiv \int_0^1 dx x^{n-2}\, F_2(x, Q^2, P^2) = \int_0^\infty \frac{d\sigma^2 \rho(n, Q^2, \sigma^2)}{(P^2 + \sigma^2)} \quad . \tag{58}$$

The dispersion integral should be convergent, since for large $P^2 (P^2 \gg Q^2)$, $\tilde{F}$ tends to zero, according to the discussion following Eq. (36).

What can one say about the weight function $\rho(\sigma^2)$? [We suppress for a while the Q^2 and n dependences]. At the level of the uncorrected $\gamma\gamma \to q\bar{q}$ "fusion" diagram, one has a log Q^2/P^2 behavior which is reproduced by:

$$\rho \approx \begin{cases} constant & \sigma^2 < Q^2 \quad , \\ 0 & \sigma^2 > Q^2 \quad . \end{cases} \tag{59}$$

The QCD modifications to ρ need to be inferred from Eq. (47), which expresses the desired answer. What to do is perfectly clear[20]; one simply writes:

$$\rho \approx \begin{cases} \rho_0 \left[\dfrac{\alpha_s(Q^2)}{\alpha_s(\sigma^2)} \right]^{d(n)} & \sigma^2 < Q^2 \quad , \\ 0 & \sigma^2 > Q^2 \quad . \end{cases} \tag{60}$$

The *absorptive* part in P^2 of the moment of the structure function posses the scaling-violation pattern typical of ordinary "hadronic" structure functions. In this sense, it seems inappropriate to describe the dispersion integral over it as the difference of a "pointlike" part (contribution from the upper limit) and "hadronic" part (contribution from the lower limit). Just as for the case of vacuum-polarization and/or R ($e^+e^- \to$ *hadrons*), the separation into short-distance and long-distance pieces (aside from contributions of heavy-flavor thresholds) has to do with the large σ^2 and small σ^2 contributions of the absorptive part.

The importance of the low-σ^2 part of $\rho(n, \sigma^2, Q^2)$ evidently depends on n (or, if one likes, the "conjugate" variable x). For large n (large x), the perturbative-QCD correction *suppresses* the small-σ^2 contribution. However, for small n (small x), the opposite occurs and the low-σ^2, infrared region is enhanced. In this region in particular,

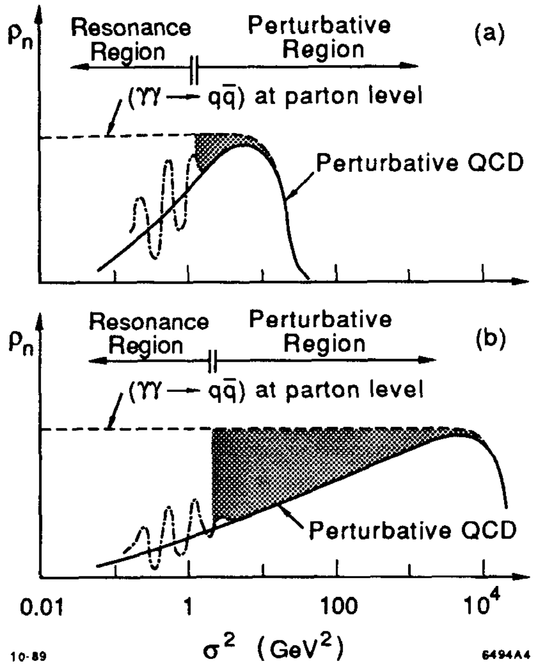

Figure 4: Sketch of the weight-function $\rho_n(\tau^2, Q^2)$ for $n \sim 4$ and (a) $Q^2 \sim 20$ GeV, and (b) $Q^2 \sim 20,000$ GeV.

there will be nonperturbative contributions from Regge-poles. Since Regge-poles are associated with the presence of *bound* $q\overline{q}$ states, they are indeed nonperturbative. And the perturbative calculations simply should not be trusted quantitatively at all when extended into this region. What they say is just that for small n and small x, there is a big infrared contribution, while at large n and large x, the infrared contributions are probably suppressed.

Where does this leave the matter experimentally? For $\Lambda^2_{QCD} \ll P^2 \ll Q^2$, the calculations are quite solid, but the sensitivity to the QCD corrections is poor until very large Q^2 (much larger than is now available) is attained. The real-photon structure-function at large x, according to our argument, can, in principle, be well estimated by the perturbative-QCD expression when Q^2 is sufficiently large. Note that the moments of the structure function are just (for $P^2 = 0$):

$$M(n, Q^2) = \int \frac{d\sigma^2}{\sigma^2} \rho(n, \sigma^2, Q^2) \quad ; \tag{61}$$

i.e., just the area under the curve when ρ is plotted versus $\log \sigma^2$. This is done in Fig. 4 for $Q^2 = 20$ GeV^2 and $Q^2 = 20,000$ GeV^2. The region below, say, $\sigma^2 = M^2 \sim 2$ GeV^2 is no doubt *nonperturbative*. There are double-poles at ρ, ω, ϕ masses in the moments, leading to (approximate) $\delta'(\sigma^2 - m^2)$ singularities in ρ. So the perturbative estimate of the area below $\sigma^2 = M^2$ should not be trusted to a factor of two, if even that much. This leaves the "reliable" perturbative effect the shaded area as shown. For $Q^2 = 20$ GeV^2, the effect is very small; for $Q^2 = 20,000$ GeV^2, there seems to be some hope. It appears that if the *real-photon* structure function could be measured from, say, $Q^2 \sim 100$ GeV^2 to $Q^2 \gtrsim 10^4$ GeV^2, there might be some chance of getting a quantitative measure of the QCD correction. But, at present, what has been accomplished experimentally is observation of the presence of the large-x "pointlike" contribution from the $\gamma\gamma \rightarrow q\overline{q}$

process. This comes both from the shape of F_2 at large x and from its logarithmic Q^2 dependence.

CONCLUDING REMARKS

These discussions of R and of photon structure functions have been based on a conservative view of perturbative QCD. As expressed in the introduction, I feel that there is a need for developing this point of view further. I am encouraged by the fact that the first two examples attacked this way yielded something which at least I have found not uninteresting. In both cases, the use of Green's functions built from local color singlet operators, and especially their analyticity properties, was a central feature. While there are more things to do in generalizations along this line, I think there are also interesting questions of how to deal with the interpretation of the perturbative intermediate states which build the absorptive parts of the Green's functions, and how to relate them to the extant QCD jet calculus used in phenomenology. I hope to work on this in the future.

ACKNOWLEDGMENTS

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