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Running Couplings in $SU(2)_L$ x U(1)

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ABSTRACT

We prove that the running $*$ couplings of Nuclear Physics 322B (1989) 1, which include certain universal parts of one loop radiative corrections in the standard model, are gauge invariant. The residual corrections (those explicitly excluded from running couplings) must include universal parts of box and vertex diagrams in order that they be gauge invariant; the discussion of non-abelian residual corrections for neutral current four-fermion processes is made complete with an expression for the universal part of the W-W boxes. We clarify the construction of sets of gauge invariant running couplings and give as an example a set which includes all universal parts from self- energies, vertices and boxes. Updated results for high precision electroweak observables are given.

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Running couplings organize a huge class of electroweak radiative corrections into simple expressions closely resembling the Born approximation [1]. Since experimental data is usually (and most easily) analyzed with a Born approximation, they are therefore extremely useful for comparisons of different experimental high precision measurements. An effective Lagrangian was used in reference 1 to prove that 'universal' parts (proportional to two external particle quantum numbers) of one-loop electroweak corrections can be incorporated, along with tree-level diagrams, into-running couplings in four-fermion processes. This result follows the classification of $SU(2)_L$ x $U($ 1) radiative corrections"' and is just an extension of the incorporation of leading ultraviolet logs into running couplings by means of truncated renormalization group equations.[3] The running 'star' $(*)$ couplings (defined in eqns. A.29 - A.32, ref. 1) are gauge invariant; another proof of this fact will be given below. It has been brought to our attention† that insufficient information was given in ref. lto form gauge invariant 'residual' corrections (those radiative corrections specifically not included in running couplings) off Z pole. This paper is meant to remedy the ommision of the non-a-belian part of the W-W boxes and clarify the construction of gauge invariant running couplings. We concern ourselves with neutral current four-fermion processes involving massless fermions on external lines and within vertex and box diagrams. The extension to charged current processes is trivial; the extension to massive fermions is not. We consider here only the 'universal' part of the effective 1 loop matrix element : vector self-energies and the non-abelian parts, involving commutators of SU(2) generators, of vertex corrections and the W-W boxes. All graphs are to be calculated using gauge invariant bare parameters.

The tree-level graphs appear in figure la. We divide the vector self-energy corrections into figures lb and lc with the introduction of some arbitrary functions $f_{ij}(q^2)$ with $i,j = Z,A$. Next, the non-abelian parts of the vertex functions are augmented by an arbitrary function $f(q^2)$ in figures 1d and le. The non-abelian

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part of the W-W boxes appears in figure 1f. The sum of $O(\hbar)$ corrections in figures 1b-f is gauge invariant as are the arbitrary functions in figure 1c and 1e and the tree-level figure 1a.

Our running * couplings are gotten by summing the contributions of figures 1a, 1c and 1e with Dyson's series and the particular choices $f_{ij} = \Pi_{ij}^{\xi=1}$ (self-energies in 't Hooft $R_{\xi=1}$ gauge, eq. B.4-11, A.10, ref.1) and $f = \Gamma'$ (eq. B.16,ref. 1) and are thus gauge invariant by construction. With this choice of f_{ij} only the running * couplings depend on new matter species in 'oblique' loops since the contribution of new matter fields with zero vacuum expectation value to the vector self-energies is by itself gauge invariant. The specific form of Γ' is chosen to rediagonalize the $Z-A$ mass matrix, force residual vertex corrections to vanish at $q^2 = 0$ and force the large q^2 behavior of the running * couplings to be the same as that from renormalization group beta function ultraviolet logs alone. This definition is in agreement with conventions found in all standard texts[3] in the large q^2 limit (see eqns. 4.4 and 4.6, ref. 1); infra-red Sudakov logs are specifically excluded from our definition of running * couplings and are instead included in the residual corrections.

The non-abelian residual corrections, now the sum of figures 1b, 1d and 1f, are also gauge invariant. An expression for Γ^{nAb} in $R_{\xi=1}$ gauge is found in eq. B.13-14, ref. 1. This includes Sudakov logs from the infra-red sector as the W mass goes to zero (see B.20, ref. 1). Although numerically much smaller than expected experimental errors in $\xi = 1$ gauge, the W-W boxes do have a non-abelian part (figure 1f) necessary in principle for the calculation of gauge invariant residual corrections‡

‡ **It is necessary to make minor changes in the wording of ref. 1 in order that it be strictly correct. Substitute the words "vertex and box" for the word "vertex" (singular or plural as appropriate) in (page; line of prose text) as follows: (11;4,9), (35;17,18,22,24),(36;5,16). Note that all formulae, graphs, tables, numbers and results quoted are correct and therefore unchanged. The contribution of boxes is completely negligible $\ll 10^{-4}$ on Z pole so that results for SLC/LEP physics[4] are also unchanged. Nevertheless, the non-abelian boxes at $u = 0$ have now been included in the numerical program EXPOSTAR[5] to ensure gauge invariance of the results for all q^2.**

$$M^{non-abelian}_{W-Wboxes} = 2I_3 I'_3 g_0^4 \ Box \quad (1)$$

Note that the gauge-dependent piece is independent of the angle between incoming and out going fermions. Defining the usual Mandelstam variables $q^2 = -s = (p_1+p_2)^2, -t = (p_2+p_4)^2, -u = (p_1+p_4)^2$ with $p_1 + p_2 + p_3 + p_4 = 0$ we have in $\xi = 1$ gauge

$$\begin{aligned} Box^{\xi=1}(u=0) = \frac{1}{16\pi^2 M^2(2+x)}\Big[& -\frac{5}{2} + \frac{x ln(-x-i\varepsilon)}{2} - \frac{7x}{6} + \frac{x^2}{12} \\ & + \frac{2-2x-x^2}{2(1-2x)}(B_0(q^2,M,M) - B_0(0,M,M) + \frac{x}{6}) \\ & + \frac{-1+2x+x^2}{2(1-2x)}(\Lambda(M) + \frac{x}{2}) \\ & - \frac{1+x}{x^2}[(Sp(-x-i\varepsilon) + x - x^2/4) \\ & + ln(-x-i\varepsilon)(ln(1+x+i\varepsilon) - x + x^2/2)]\Big] \end{aligned} \quad (2)$$

Here $x = \frac{q^2}{M^2}$, Sp is the Spence function and $M = M_W$. B_0 and A are defined in eqns. B.l and B.14 of ref. lrespectively. This result[6] is easily obtained using the program of simplification of loop form-factors of R.G.Stuart and co-workers!"

The fact that the functions f_{ij} and f are completely arbitrary means that effective running couplings may be defined in an infinite number of different possible ways (which will of course give rise to a proliferation of slightly different definitions); the price is that the residual corrections will differ. For example, one might incorporate all of the universal one-loop contributions into running couplings. The sum of figures lb-f naturally organizes itself into the gauge invariant combinations

$$\begin{aligned} \Pi^{**}_{QQ} &= \Pi_{QQ} + 2q^2\Gamma^{nAb} + 2q^4 Box \\ \Pi^{**}_{3Q} &= \Pi_{3Q} + 2q^2\Gamma^{nAb} + 2q^4 Box + g_0^2\langle 2I_3^2\rangle_0(\Gamma^{nAb} + 2q^2 Box) \\ \Pi^{**}_{33} &= \Pi_{33} + 2q^2\Gamma^{nAb} + 2q^4 Box \\ &\quad + 2g_0^2\langle 2I_3^2\rangle_0(\Gamma^{nAb} + (2q^2 + g_0^2\langle 2I_3^2\rangle_0)Box) \end{aligned} \quad (3)$$

(see eq. B.4-11, ref. 1) multiplied by various bare couplings, bare propagators and quantum numbers. Now define the gauge invariant 'two-star' (**) running couplings

$$\begin{aligned}\frac{1}{e_{**}^2} &= \frac{1}{e_0^2} - Re(\frac{\Pi_{QQ}^{**}}{q^2})\\ s_{**}^2 &= Re(\frac{\frac{1}{g_0^2} - \frac{\Pi_{3Q}^{**}}{q^2}}{\frac{1}{e_0^2} - \frac{\Pi_{QQ}^{**}}{q^2}})\\ \frac{1}{4\sqrt{2}G_\mu^{**}\rho^{**}} &= \langle 2I_3^2\rangle_0 - Re(\Pi_{33}^{**} - \Pi_{3Q}^{**})\end{aligned} \tag{4}$$

and $c_{**}^2 = 1 - s_{**}^2$. The non-abelian part of the W-W boxes is evaluated at u = 0 in order that the resultant running couplings be independent of the scattering angle. Therefore, the non-abelian residual corrections (consisting now of the difference between the non- abelian part of the W-W boxes evaluated at the scattering angle and at u = 0) are zero at u = 0 in the ** scheme.

The positions of poles in gauge boson Green's functions are unaffected by residue contributions like vertices and boxes (LSZ theorem) so that

$$M_Z^2 = [\frac{e_*^2}{s_*^2 c_*^2 4\sqrt{2}G_\mu^* \rho_*}]_{q^2=-M_Z^2} = [\frac{e_{**}^2}{s_{**}^2 c_{**}^2 4\sqrt{2}G_\mu^{**}\rho_{**}}]_{q^2=-M_Z^2} \tag{5}$$

After resuming the Dyson's equations, the effective neutral current four- fermion matrix element including (apart from small terms proportional to squares of imaginary parts of 1 loop graphs neglected here, see eq. A.38,A.40, ref. 1) all contributions-from figures la-e and figure 1f at u = 0 may be written

$$\begin{aligned}\mathcal{M}_{N.C.}^{**} = &\frac{e_{**}^2 QQ'}{q^2(1 - iIm\Pi_{AA}'^{**})} + \frac{e_{**}^2}{s_{**}^2 c_{**}^2}\\ &\times \frac{(I_3 - (s_{**}^2 - is_{**}c_{**}Im\Pi_{ZA}'^{**})Q)(I_3' - (s_{**}^2 - is_{**}c_{**}Im\Pi_{ZA}'^{**})Q')}{q^2 + \frac{e_{**}^2}{s_{**}^2 c_{**}^2 4\sqrt{2}G_\mu^{**}\rho_{**}} - iIm\Pi_{ZZ}^{**}}\end{aligned} \tag{6}$$

This result (like eqns. 3.8, 3.9 in ref. 1) so resembles the Born approximation that it is extremely useful in analyzing data for comparison of different high precision experimental measurements. Imaginary parts are computed with running ** couplings (in analogy with eqns. A.33-35, ref. 1); e.g. $Im\Pi_{ZZ}^{**}$ =

$(e^2_{**}/s^2_{**}c^2_{**}) Im(\Pi^{**}_{33} - 2s^2_{**}\Pi^{**}_{3Q} + s^4_{**}\Pi^{**}_{QQ})$ with $Im\Pi^{**}_{ZZ} \rightarrow \sqrt{s}\Gamma^{**}_Z$ as $q^2 \rightarrow - M^2_Z$. The obvious result (see eq. C.10, ref. 1, β_f is the **fermion** velocity)

$$\frac{\Gamma^{**}_Z}{\sqrt{s}} = \frac{e^2_{**}}{12\pi s^2_{**}c^2_{**}} \sum [(\frac{I_3}{2} - Qs^2_{**})^2 \frac{3-\beta^2_f}{2} + \frac{I_3^2}{4}\beta^2_f] \beta_f C_{QCD} \qquad (7)$$

is recovered with cancellation of dependence on $Im\Gamma^{nAb}$ and $Im(\ Box)$.

Sudakov infra-red logs are now incorporated into the running ** couplings rather than-only in the residual corrections. The reader is warned that the large q^2 behavior of the running ** couplings cannot therefore be understood from ultraviolet renormalization group beta functions alone, in disagreement with conventions found in standard texts. Therefore, the ** scheme is not 'universal' for processes involving external bosons; e.g. the infra-red structure of $e^+e^- \rightarrow W^+W^-$ is not the same as $e^+e^- \rightarrow f\bar{f}$. In contrast, the UV gauge coupling beta functions are universal for all processes with external fermions and bosons; this is why we prefer the running * couplings.

A renormalization scheme specifies the running ** couplings at certain q^2. If we choose the set [a, G,, M_Z, belief that only Higgs' doublets get vacuum expectation values] as input parameters, they are easily calculated; the results for M_Z = 91.2 GeV[8] m_t = 60GeV, m_H = 100 GeV (yielding $1 - M^2_W/M^2_Z$ = .2355 and M_W = 79.74 GeV) are given in Table I; we have used an updated dispersion relation for $e^+e^- \rightarrow hadrons$[9] . Comparison may also be made there with the more standard running * couplings; the large difference as $q^2 \rightarrow \pm\infty$ (stored in residual corrections in the * scheme) is infra-red in origin. We note with amusement that a sophisticated 1 loop calculation of the left-right polarization asymmetry"' $A_{LR}(q^2 = - M^2_Z)$ for $e^+e^- \rightarrow \mu\bar{\mu}$ using EXPOSTAR (5 x 10^5 events, no **endcap** or acolinearity cuts, E_μ, $E_{\bar{\mu}} \geq 10$ GeV) gives .1198 while the simple formula $A_{LR} \approx (2 - 8s^2_{**})/(1 + (1 - 4s^2_{**})^2)$ = .1218 , a difference of only 0.002. The dependence of the running ** couplings on m_t, m_H and new oblique matter fields from SUSY, Technicolor, etc.[2][10] is the same as that of the running * couplings.

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TABLE I

$\sqrt{-q^2}$	$4\pi/e_{**}^2$	s_{**}^2	M_Z^*
1000	125.5	.2315	94.33
91.20	128.9	.2347	91.20
3	133.8	.2366	89.39
.2	136.1	.2405	88.14
3i	133.6	.2366	89.43
100i	128.5	.2335	91.67
1000i	125.4	.2340	95.41

$\sqrt{-q^2}$	$4\pi/e_*^2$	s_*^2	M_Z^*
1000	126.5	.2406	91.84
91.20	128.7	.2335	91.20
3	133.8	.2389	88.80
.2	136.1	.2416	87.75
3i	133.6	.2390	88.83
100i	128.7	.2349	91.12
1000i	126.6	.2411	91.79

Running $**$ and $*$ couplings. The first four entries are timelike while the last three entries are spacelike q^2 . Here $(M_Z^{**})^2 = e_{**}^2/(4\sqrt{2}s_{**}^2c_{**}^2G_\mu^{**}\rho_{**})$ with an analogous definition for M_Z^* , M_Z = 91.20, $m_t = 60, m_H$ = 100. All masses and energies are in GeV.

Figure 1: Universal tree and 1 loop contributions to neutral current four-fermion processes $\psi\bar{\psi} \rightarrow \psi'\bar{\psi}'$ (in either s or t channel) are proportional to one ψ quantum number (either I_3, Q) times one ψ' quantum number (either I'_3, Q'). Wavy lines in figs. 1a-e are either Z or A (photon); in fig. 1f they are charged W's. The shaded blob in fig. 1b represents insertions of $\Pi_{ij} - f_{ij}$ with $i, j = Z, A$; the unshaded blob in fig. 1c represents insertions of f_{ij}. The shaded blobs in fig. 1d represent insertions of $g_0^3 I_3(\Gamma^{nAb} - f)c_0$ for Z exchange and $g_0^3 I_3(\Gamma^{nAb} - f)s_0$ for photon exchange. I_3 is the isospin along the appropriate external line, $s_0 = e_0/g_0$, $c_0^2 + s_0^2 = 1$ and Γ^{nAb} is the non-abelian universal part of the vertex corrections. The unshaded blobs in fig. 1e represent insertions of $g_0^3 I_3 f c_0$ for Z exchange and $g_0^3 I_3 f s_0$ for photon exchange. $f_{ij}(q^2)$ and $f(q^2)$ are arbitrary gauge invariant functions. Fig. 1f includes only the non-abelian universal part of the W-W boxes. All graphs are calculated with gauge invariant bare couplings and bare propagators.

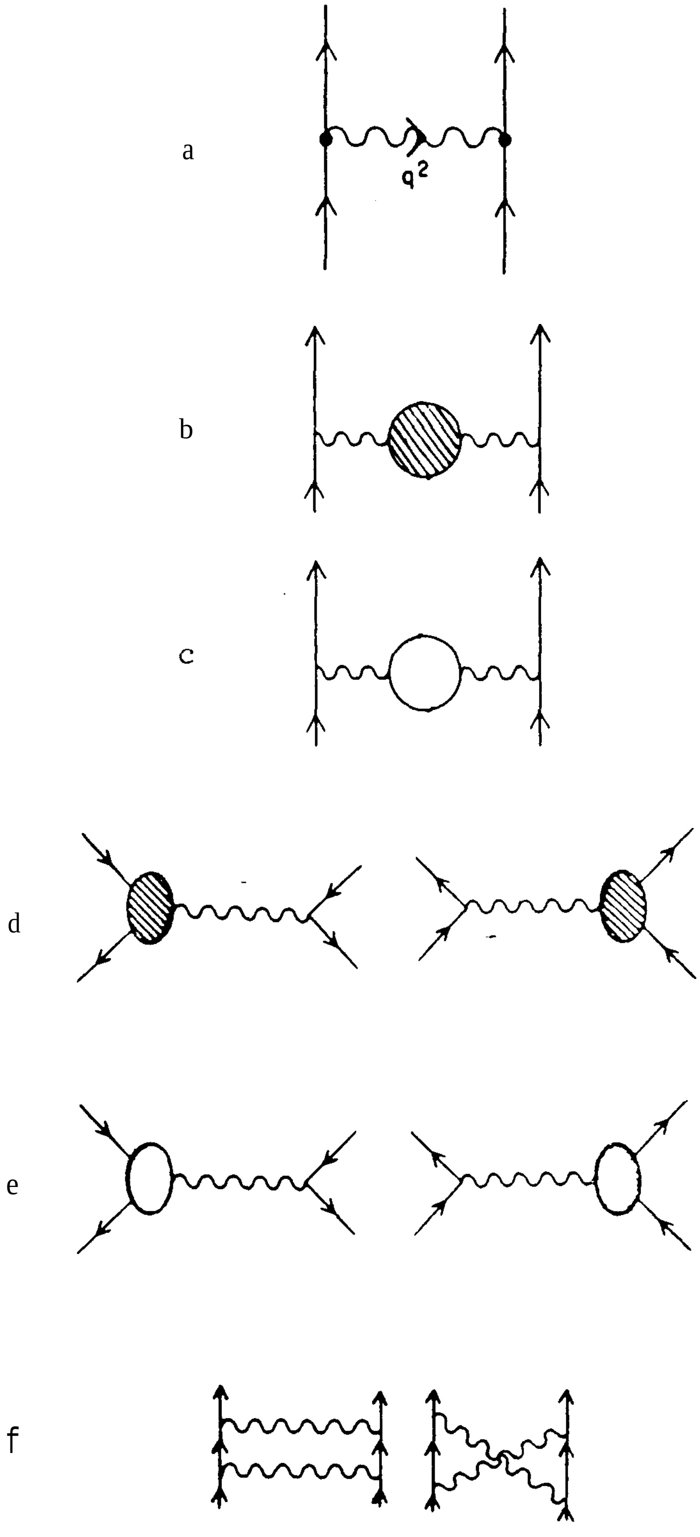

Fig. 1