SLAC–PUB–5027
October 1989
(T)

FERMIONS AND SOLITONS IN THE $O(3)$ NONLINEAR SIGMA MODEL IN $2+1$ SPACE-TIME DIMENSIONS*

PURUSHOTHAM VORUGANTI◇
Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94309
and
Stanford University, Department of Physics
Stanford, CA 94305

SEBASTIAN DONIACH
Stanford University, Department of Applied Physics
Stanford, CA 94305

ABSTRACT

The field theory limit of antiferromagnetism with holes is described using the 0(3) nonlinear sigma model. The minimal coupling to the spin-wave gives a Pauli term that couples the hole charge density to the topological charge density for solitons. This term leads to drastic consequences; an attractive potential for holes and solitons, spin-charge coupling for the holes and zero-momentum modes. The zero-momentum modes are exact nonperturbative solutions to the full coupled equations of motion for spin-waves and holes. The effect of a Chern–Simons term, connections to other mean-field approaches and the quantum boundstate problem are discussed. If the zero-modes are the states of lowest energy, then the holes in this field theory limit are attached to skyrmions.

Submitted to *Physical Review B.*

⋆ Work supported by Department of Energy contract DE–AC03–76SF00515.
◇ Now at University of Toronto, 60 St. George St., Toronto, Ontario, Canada M55 1A7.

I. MOTIVATION

Many of the new high-Tc superconductors exhibit a Néel antiferromagnetic state at zero doping. After the addition of roughly one hole per ten Cu atoms, this ordering becomes short-ranged,[1] and moreover, the system becomes superconducting. The anisotropic magnetic and superconducting properties[2] hint at the importance of dimensionality. If the feature of antiferromagnetism is at all crucial to the superconducting state, an adequate description of the magnetic state becomes necessary. The unique feature of two-dimensional antiferromagnetism, once the connection to the nonlinear sigma model is made in the continuum limit, is the finite energy soliton excitations.[3] If these are to play any role in the transition to the superconducting state, their interaction with holes must be fully addressed. If the holes and the solitons form bound pairs,[4] a natural explanation for the short-range magnetic order emerges, since each soliton presents a small area of disorder. Thus the disordered state could be equivalently viewed as the soliton condensed state.[5] If the soliton could bound two holes and no more, then the other half of the picture, namely Cooper pairs, provide an explanation for the superconductivity. Interestingly enough, many of these notions can be explicitly written as solutions to various equations. Although the solutions are difficult to extract, the classical and quantum regimes will be systematically discussed. Even though our conclusions are that soliton-hole energetics do not favor the above picture, the possibility remains that new methods may demonstrate the above scenario. The attractive feature of this description is its uniqueness to two dimensions and its deep connection to the magnetic properties.

II. FIELD THEORY OF THE 2D HEISENBERG MODEL

With the exception[6] of the 30K Bismuth-oxide superconductors, numerous evidence exists that the most of the high-Tc compounds undergo a transition to the Néel state at low doping. The Néel state is a variational state for the $S = 1/2$ 2D Heisenberg model and is the exact ground state for spin $S \geq 1$, due to a theorem by Dyson, Lieb and Simon.[7] Numerical simulations of the nonlinear sigma model and the Heisenberg model show that it is Néel ordered[8] for $S = 1/2$. Starting with the 2D Heisenberg model on the square lattice and expanding around the Néel State, Haldane[9] was able to show that the relevant operators that appear in the long wavelength continuum limit are just given by the O(3) nonlinear sigma model. The assumption of this particular ground state then leads to a relativistic 2+1 field theory, which, after making the standard CP^1 map,[10] is given by

$$\mathcal{L}_{CP} = f(D_\mu Z)^\dagger(D^\mu Z) - \eta(Z^\dagger Z - 1) \quad , \tag{1}$$

with $\mu = 0,1,2$; $D_\mu = \partial_\mu + iA_\mu$; and Z a complex two-spinor (α, β). The theory has one dimensionful coupling constant f and constraint fields $\eta(x)$ and $A_\mu(x)$. The Néel state corresponds to the field theory ground state of $\alpha = constant$ and $\beta^2 = 1 - \alpha^2$. The field theory description has precisely the same number of massless spinwave modes as the lattice version, but unlike the lattice version, has a space–time Lorentz symmetry.[11]

The Hamiltonian in the presence of holes on the lattice is probably something like the Hubbard Hamiltonian with onsite repulsion and nearest-neighbor hopping.[12] The field theory limit of the Hubbard Model is much more difficult to derive although many mean-field models exist.[13]

In the limit of large onsite repulsion, one can instead start with the $t-J$ model with hopping parameter t and a Heisenberg spin-stiffness term J. At zero doping, the t term is irrelevant since it will lead to double occupancy which in the large U limit is energetically suppressed. The Heisenberg interaction is then mapped onto the sigma model by dividing the two-dimensional lattice into nonintersecting plaquettes of four sites each giving four mean-field variables for each plaquette. The massless modes in the large S limit are just the ϕ and π_ϕ fields which then describe the sigma model. The number of massless modes is two, even though four sites lead to four degrees of freedom in the continuum in agreement with the Goldstone theorem upon expanding around the Néel state. The expansion around the Néel state thus describes the spin-spin interaction portion of the $t-J$ mode.

For the problem with nonzero doping, it is simplest to discuss the case of extremely small doping where the experimentally observed Néel state is still a good description of the ground state and its symmetries. The mean-field description of the small number of vacancies with spin-waves has attracted considerable attention.[14,25] Like the two spin-wave modes, the single hole ground states will identify the relevant number of degrees of freedom. Lattice calculations as well as other mean-field descriptions show that there are four minima in the Brillouin zone located[15] at $\vec{k} = (\pm\pi/2, \pm\pi/2)$. These four states should appear in any mean-field theory. Let us introduce two fermion operators $C_\uparrow$, $C_\downarrow$ which live on each plaquette. From these two degrees of freedom, we can construct a Dirac Spinor, $\psi = (C_\uparrow , C_\downarrow)$. The number of low-energy hole states corresponds to two such Dirac fermions but none of our results on soliton-hole physics will be affected if we just concentrate on one species of holes. Interactions of the spin-waves with the fermion degrees of freedom should flip down spins to up spins in the plaquette and this corresponds to the hopping of a hole from the up sublattice to the down sublattice. In order to write the mean-field theory for the Dirac fermion coupled to the spin-waves, we follow closely the symmetries of the spin-wave theory which like the Hamiltonian for the vacancies comes from the $t - J$ model. In the limit of an extremely small number of holes, we expect the mean-field Lagrangian to have the same symmetries as the system without any holes. The surprising symmetry of the Néel state is of course the Lorentz symmetry, a pseudorelativity with the speed of light being the spin-wave velocity. At large doping, the Néel state is no longer the valid ground state and no symmetry arguments are valid in this regime, but luckily the low-density regime may determine the nature of the spin-disordered state. The other symmetry of the nonlinear sigma model is the internal symmetry related to $Z \rightarrow GZ$ where G is an element of $SU(2)$. By restricting our attention to just the one Dirac fermion, we can consistently treat it as a singlet under this internal symmetry. Had we used the two "flavors" of fermions on each plaquette, other rich possibilities exist.[13] After considering various $\psi - Z$ interactions, the simplest Lagrangian that preserves all the symmetries and contains the lowest number of derivatives and includes the spin-flips is given by

$$\mathcal{L} \;=\; f(D_\mu Z)^\dagger(D^\mu Z) + \eta(Z^\dagger Z - 1) + \overline{\psi}\gamma^\mu(i\partial_\mu - gA_\mu + m)\psi \quad , \qquad (2)$$

where we consider holes of any general coupling g and mass m ($\hbar = 1, c = 1$). The ψ is a complex two-spinor and the γ^μ is a 2×2 spin matrix that will be defined below. The coupling g begins as S but receives the usual renormalization. This Lagrangian for holes at low density has a rigorous formulation for the one-dimensional spin chain.[14] Neither the sign of the mass nor charge nor the relativistic version of the fermion Lagrangian is crucial to the behavior around solitons. We consider this coupling for its generality, and the cases of same-sign fermions and nonrelativistic fermions are subcases of what follows. Having discussed the interactions of holes with spin-waves, we next examine the soliton sector of the theory. The solutions, however, come with arbitrary size and thus are not true minima of the energy functional. This property, in the absence of additional physics like holes, makes solitons unstable against quantum or $1/S$ corrections to the nonlinear σ-model. This problem in the presence of fermions is examined in Sec. III.

III. SOLITONS AND SIZE INSTABILITY

Solitons of the O(3) nonlinear sigma model in the CP^1 language can be brought in cylindrical coordinates to the form[14]

$$Z_{sol} = \begin{pmatrix} \frac{r'}{(1+r'^2)^{1/2}} exp(iN\theta) \\ \frac{1}{(1+r'^2)^{1/2}} \end{pmatrix} \quad , \tag{3}$$

where $r' = (r/\lambda)^{|N|}$, with λ being the size and N the winding number. The lowest-energy excitations are the $N = \pm 1$ solutions with energy $2\pi f$. The energy is independent of λ and this is the size instability that was addressed in an earlier paper.[14] Two solutions have been suggested to cure this problem. First, as suggested by Dzyaloshinskii, Polyakov and Wiegmann,[4] topological terms, relevant in the long-distance limit, like the Chern–Simons term or the Hopf term, can arise either from fermions being integrated out of the path integral or from a complete field theory limit of the Hubbard Model in two dimensions. However, another possibility arises for soliton stability through the self-consistent trapping of holes in solitons. The first scenario will be shown easily not to hold true. Our idea on the second possibility, as we demonstrate here, provides a consistent mechanism.

As we saw above, the effect of holes is to introduce two species of fermions interacting with the CP^1 gauge field.[15] If we integrate out the fermion degrees of freedom,[16] we obtain the following effective action for the spin-waves to lowest-order momentum,

$$\mathcal{L}_{eff} = \mathcal{L}_{CP} + k\varepsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \quad , \tag{4}$$

with $k = g^2/4\pi$ since we have two species with the same-sign mass. The same-sign mass has been suggested as a way of realizing the parity-violating Chern–Simons term[17] and our purpose is to study only its effect on the soliton stability. Our results on soliton-hole pairing do not depend upon the signs of the mass terms. Dilation invariance of the energy functional is broken when the Chern–Simons term is present. In an earlier paper,[14] we suggested a parameterization of the Z fields for determining the new soliton solutions. Here, we show that in the presence of this new term, solitons are pushed to infinite size. The energy functional with the Chern–Simons term present becomes

$$\mathcal{E} = f\int d^2x[\frac{k^2}{f^2}(\vec{\nabla}\times\vec{A})^2 + (D_iZ)^\dagger(D_iZ)] \quad , \tag{5}$$

where we consider static configurations $Z(x)$ only and Gauss's Law gives $A_0 = (k/f)(\vec{\nabla}\times\vec{A})$. Suppose an exact solution to the equations of motion in the presence of the Chern–Simons term is known, called $\widetilde{Z}(x)$. By doing an appropriate scale transformation, the energy functional can be brought to the form $\mathcal{E} = f\int d^2x[(\vec{\nabla}\times\vec{A})^2 + (D_iZ)^\dagger(D_iZ)]$. For the solution $\widetilde{Z}(x)$, this becomes $\mathcal{E} = f(\mathcal{E}_1 + \mathcal{E}_2)$. If we then consider a solution $\widetilde{Z}(ax)$ for some scale factor a, the energy is $\mathcal{E} = f(a^2\mathcal{E}_1 + \mathcal{E}_2)$, and we see that the energy is reduced for $a \to 0$. Therefore, the addition of the Chern–Simons term, whatever the reason, to the sigma model Lagrangian gives minimum energy to the solitons for small a or equivalently infinite size. The energy of the minimum in the presence of the Chern–Simons term is still $2\pi f$. The Chern–Simons term breaks the dilatation invariance of the energy functional, but since it is a higher-derivative correction, pushes the solitons to infinite size. Although the Chern–Simons term does not appear in the covariant energy-momentum tensor due to the absence of the metric $g_{\mu\nu}$, it affects the Gauss's law condition through the A_0 equation of motion[18] and changes the form of the canonical momenta. Leading quantum effects—zero-point corrections to the soliton mass[19]— in the presence of the Chern–Simons term were considered elsewhere; again, it was found that large size solitons were preferred.[18] Quantum effects to the first order in k remain to be done and it is not clear whether finite size solitons will arise. Lattice effects may also stabilize the soliton when the Chern–Simons term is present, but the solitons solutions on the lattice even for $k = 0$ are not known. Instead, we consider the motion of a single hole in the background of soliton and see if the eigenspectrum reveals any nontrivial stabilization.

To begin the analysis, we start with the original Lagrangian for spinwaves and fermions. The full equations of motion couple the fermions and the spinwaves through the gauge field A_μ. We will examine the one soliton sector of the spinwave theory and treat the A_μ that arises as the background in which a single fermion is to move. Since there is no Chern–Simons term in the original theory (although quantum effects could generate it at order $\hbar$), the classical equation of motion for

A_0 becomes $A_0 = 0$ for static solitons. Therefore, the soliton presents field configurations $\vec{E} = 0, B \neq 0$. If the background magnetic field has bound eigenstates for fermions, then the requirement that the binding energy E plus the soliton mass $2\pi f$ be less than zero could place a condition on the size. The size of the soliton effects boundstate energies, since the potential well for the fermions becomes narrower as the size decreases. Using the soliton solutions of winding number N we wrote earlier, we are automatically in the gauge $\vec{\nabla} \cdot \vec{A} = 0$. $B(x)$ can thus be written as $B = -\nabla^2\phi$, and A_i becomes $A_i = \epsilon_{ij}\partial_j\phi$ for some scalar field $\phi(x)$. For general soliton winding number N, the scalar potential $\phi(x)$ is

$$\phi(x) = \frac{sgn(N)}{2} ln(1 + r^{2|N|}) \quad . \tag{6}$$

The fermion equation of motion is the usual dirac equation of motion in the above soliton background field A_i or $i\gamma^0\partial_0\psi + \gamma^i(i\partial_i - gA_i)\psi + m\psi = 0$. Using the 2×2 Pauli matrices with $\gamma^0 = \sigma_z, \gamma^1 = i\sigma_x, \gamma^2 = i\sigma_y$, we get the Schröedinger equation

$$\vec{\alpha} \cdot (i\vec{\nabla} - g\vec{A})\psi - \sigma_z m\psi = E\psi \quad , \tag{7}$$

where $\alpha_1 = -\sigma_y, \alpha_2 = \sigma_x$. For fermions $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$, when $m \neq 0$, the two components are coupled. In order to analyze the eigenstate spectrum, we square the equation to get the usual decoupled form,

$$[\partial^2 + 2ig\vec{A} \cdot \vec{\nabla} + ((E^2 - m^2) - g^2A^2 - g\sigma_z B)]\psi = 0 \quad . \tag{8}$$

Had we started with nonrelativistic fermions, essentially the same equation would have arisen if we had added the two-dimensional Pauli term $s_z B$ to the usual

$(p - gA)^2/2m$ term. Although such a term has to be put in my hand in the nonrelativistic picture, the internal spin degree of freedom of the holes requires it in the case of electrons moving in a constant external magnetic field, just as in the Quantum Hall phenomena. The addition of this Pauli term preserves all the gauge symmetries of the nonrelativistic field theory, and written purely in terms of the $Z(x)$ fields, appears as

$$\delta\mathcal{L}_{Pauli} \sim \psi^\dagger \varepsilon_{ij}(\partial_i Z^\dagger \partial_j Z)\psi \quad , \tag{9}$$

where in the nonrelativistic case, ψ is a single component anticommuting field. This interaction represents a coupling of the fermion charge density to the topological charge density where the general topological conserved current is given by $J_\mu^{top} = \varepsilon_{\mu\nu\rho}\partial^\nu Z^\dagger \partial^\rho Z$.[20] This additional term is the origin of the attractive potential for both the relativistic and the nonrelativistic cases. *This higher-derivative interaction between spin-waves and holes has been missed in all other mean-field approaches, but here it emerges naturally.* Having now a wave equation for the holes in a soliton background, we look for eigenstates.

To generalize for a moment, it is known[21] that for $A_0 = 0$, static fermion zero modes exist for any B as long as the $\phi(x)$ corresponding to it is continuous. These zero-momentum modes are, in the gauge $\vec{\nabla} \cdot \vec{A}$,

$$\psi_0 = \exp\left(\mp g\phi(x,y)\right) r^k \exp\left(\pm ik\theta\right) \quad , \qquad k = 0, 1, \ldots \quad , \tag{10}$$

for $\sigma_z\psi = \pm\psi$, where the restriction on k to the nonnegative integers arises from the integrability of the wavefunctions. Therefore, the zero-momentum modes in the soliton background have $E = \pm m$ and are given by

$$\psi_0 \;=\; (1 + r^{2|N|})^{\mp\frac{sgn(N)g}{2}}\, r^k \exp{(\pm ik\theta)} \quad . \tag{11}$$

Normalizability of the zero-modes therefore requires $|gN| - |k| > 1$. Given a soliton of winding number $N = +|N|$, normalizable zero-modes exist only for $sgn(s_z g) > 0$ where s_z is the eigenvalue of σ_z. Hence, for the fermion species with $g > 0$, spin up electrons for normalizable zero-modes and vice versa for $g < 0$ and exactly the opposite holds in the background of an antisoliton with $N = -|N|$. This coupling between soliton number, charge and spin also extends to the angular momentum of the fermion. For general N, there will be $|gN|$ zero-modes for $k = 0, 1, \ldots\ (|gN| - 1)$. When g and N are fixed, all these modes will be of one chirality only. The integrated flux for a soliton of winding number N is $\int d^2xB = -2\pi N$. If it were not for the dependence of the number of zero-modes on g, there would be an index theorem relating the difference of normalizable opposite chirality zero-modes to the winding number of the gauge field. In the case of solitons of winding numbers one and two, the potential and the zero-modes for the s-states is plotted in Fig. 1. Had we instead started with the full nonrelativistic form for the holes,

$$\frac{1}{2m}[(-i\vec{\nabla} + g\vec{A})^2 + g s_z B]\psi \;=\; E\psi \quad , \tag{12}$$

then the final zero-mode equation would be exactly as before, and similar restrictions on $sgn(gNs_z)$ would arise for $s_z = \pm 1$. For higher angular momentum states, we write $\psi(r,\theta) = e^{ik\theta}\chi(r)$. The Schroëdinger equation for $\chi(r)$ becomes

$$\chi'' + \frac{\chi}{r} + [(E^2 - m^2) - V_{eff}(r)]\chi \;=\; 0 \quad , \tag{13}$$

where the effective potential is determined by summing all the relevant terms, giving

$$V_{eff}(r) \;=\; \frac{k^2}{r^2} + \frac{2gNkr^{2|N|-2}}{1+r^{2|N|}} + \frac{r^{2|N|-2}}{(1+r^{2|N|})^2}((gN)^2 r^{2|N|} - 2g|N|Ns_z) \quad . \tag{14}$$

Previously, we analyzed the restrictions on g, s_z and N arising from normalizability. The k quantum number for the zero-modes corresponds precisely to the angular momentum quantum number, and integrability of $r{\psi_0}^2(r)$ for the zero-modes requires $|gN| - |k| > 1$ and $k > -1/2$. The effective potential shows attraction in the s-channel only when $sgn(gNs_z) < 0$ and the long-distance behavior is like $+1/r^{2|N|}$. The potential is thus short-ranged but attractive at small distances (see Fig. 1). For $k > 0$, attraction arises whenever $sgn(gkN) < 0$ and $sgn(gNs_z) > 0$. Requiring both conditions simultaneously thus couples k to s_z. Had we dropped the Pauli term, the effective potential could be written as

$$V_{eff} \;=\; \frac{r^{4|N|}(gN+k)^2 + k^2}{r^2(1+r^{2|N|})^2} \quad , \tag{15}$$

which is clearly positive for all r. The Pauli term therefore gives an attractive piece to the effective potential in all angular momentum channels and is crucial for the existence of the zero-modes. $Sgn(gks_z)$ determines whether the potential is attractive or repulsive. Having discussed the hole-soliton wave equation and the existence of an attractive potential and unexpected zero-modes, we turn now to the size stability of the solitons.

Classically, the hole would drop into the minimum of the potential and the quantum zero-point motion would be the first correction to the energy. The zero-point motion can be estimated by making a saddle point expansion of the potential around the minimum and thus determine the relevant oscillator frequency ω. For simplicity, we consider the case $k = 0$ and $N = 1$, for which, upon restoring the soliton size λ, the effective potential has the form

$$V_{eff} = \left(\frac{gr}{\lambda}\right)^2 \frac{(\frac{r}{\lambda})^2}{(1+r^2)^2} - \frac{2gs_z}{\lambda^2}\frac{1}{(1+r^2)^2} \quad . \tag{16}$$

There is an attractive minimum for any $sgn(gs_z) > 0$, and we pick $s_z = +1$ and $g > 0$. The classical binding energy becomes

$$E_{class} = -\frac{2|gs_z|}{\lambda^2} \quad .$$

Solitons would be spontaneously produced only if the binding energy gain is enough to offset the cost of creating the soliton, which we computed to be $2\pi f$. This restriction constrains λ to

$$\lambda \leq \frac{g}{\pi f} \quad . \tag{17}$$

In physical units $\lambda < g/3$ Å, which, if we use values of g in the range of one to ten, gives an upper limit to soliton size of the order of one lattice spacing. Since the sigma model represents the long wavelength fluctuations, details at the level of one-lattice spacing are more difficult to accept. In any case, the interaction of holes and subsequent binding, at least at the classical level, break the dilation invariance and establish an upper bound on soliton size. Although the classical result favors $\lambda \to 0$, the exact quantum problem, in principle, fixes the soliton size to a finite range, as we will see below.

IV. QUANTUM EFFECTS

The Schröedinger equation for $k = 0, N = 1, s_z = 1$ becomes, after rescaling by λ,

$$\chi'' + \frac{\chi'}{r} + [-\epsilon + \frac{(g^2r^2 - 2g)}{(1+r^2)^2}]\chi \;=\; 0 \quad , \tag{18}$$

where $-\epsilon \equiv (E^2 - m^2)\lambda^2$. For boundstates, the eigenvalues satisfy $\epsilon > 0$. If ϵ_0 is the lowest eigenvalue, then the energy of the hole becomes $E = m(1-\epsilon_0/m^2\lambda^2)^{1/2}$. Requiring $E^2 > 0$ gives the lower bound $\lambda > \sqrt{\epsilon_0}/m$. The binding energy in the large m limit is given by $-(\epsilon_0/2m\lambda^2) + \mathcal{O}(m^{-3})$. Since this must be enough to offset the soliton rest energy, an upper bound like the classical regime results in $\lambda \leq \sqrt{\epsilon_0/4\pi f m}$. Therefore, positivity of the energy-squared for stationary states and the energetics of soliton-hole binding gives a finite range for the soliton size λ,

$$\frac{\sqrt{\epsilon_0}}{m} \;<\; \lambda \;\leq\; \sqrt{\frac{\epsilon_0}{4\pi f m}} \quad . \tag{19}$$

Solutions for eigenvalues $\epsilon = 0$ were the zero-modes we explicitly constructed above. It is necessary for soliton-hole binding to demonstrate that solutions for positive ϵ also exist. We turn now to the search for boundstates.

Saddle-Point Approximation. Corrections to the classical trajectory can be done by a saddle-point expansion around the minimum of the potential. In the harmonic approximation to the potential $V(r) = V(0) + (1/2)V''(0)r^2$, the new energy is $E = E_{class} + (1/2)\hbar\omega$. We expand about zero, since the minimum is at $r = 0$ for $k = 0,\; N = 1,\; s_z = 1,\; g > 0$. In the case of a minimum at zero, it is necessary to analytically extend $V(r)$ to the unphysical region of $-r$, since the problem in radial coordinates can also be thought as a one-dimensional problem.

Using the effective potential we wrote earlier, $V''(0) = 2g^2 + 8g \equiv 2\alpha^2$. Therefore, in the harmonic approximation, the $\chi(r)$ equation becomes

$$\chi'' + \frac{\chi'}{r} + (\epsilon' - \alpha^2 r^2)\chi = 0 \quad , \tag{20}$$

where $\epsilon' = -\epsilon - V(0)$ and $V(0) = -2g$. This equation has a spectrum $\epsilon' = 4\alpha(n + 1/2)$. The eigenvalue equation then becomes

$$-\epsilon = -2g + 4\sqrt{g^2 + 4g}\left(n + \frac{1}{2}\right) \quad . \tag{21}$$

Even for $n = 0$, we find no positive, *i.e.*, boundstate, solutions to ϵ. The case for $k \neq 0$, and separately $N \neq 1$, cannot be done so easily; but even here, we find that the quantum fluctuations push the hole out of the minimum into the continuum, *i.e.*, ϵ is always negative. The saddle-point approximation is sensitive to anharmonic r^3 and higher-order terms; especially so here, since our potential goes to zero quickly for $r > 1$. Since these higher-order effects make the approximate differential equation harder to solve, perhaps Rayleigh–Ritz variational methods are more efficient.

Variational Methods. The asymptotic behavior of $\chi(r)$ is obtained by keeping the leading powers in $V(r)$ for the two limits $r \to 0$ and $r \to \infty$. This gives, for $\chi(r)$, the Bessel functions

$$\begin{aligned} \chi(r) &\to J_0(r\sqrt{2g - \epsilon}) \quad as\ r \to 0 \\ &\to K_g(r\sqrt{\epsilon}) \ as\ r \to \infty \quad . \end{aligned} \tag{22}$$

This is the asymptotic behavior for wavefunctions in a box of finite width a with $V = -2g$ for $r \leq a$ and $V = 0$ for $r > a$. The eigenvalues in this case are obtained by matching the wavefunctions and their derivatives at $r = a$. By varying the parameter a, an entire set of wavefunctions can be obtained which, in turn, can be used as variational functions for our potential. For a variational function ϕ, we compute the variational energy by inverting the Schröedinger equation,

$$E_{var} = \frac{\int d^2x[(\frac{d\phi}{dr})^2 + \phi V \phi]}{\int d^2x \phi^2} \quad . \tag{23}$$

In the case of the wavefunctions from the box, we have $\phi = J_0(\sqrt{2g - \epsilon r}) + cK_0(\sqrt{\epsilon} r)$, where both ϵ and c are determined by the boundary matching conditions. In the entire range of a and the accompanying wavefunctions, we find $E_{var} \geq 0$. Similarly, for Coulomb functions like $r^\beta e^{-\alpha r}$, Gaussian functions like $r^\beta e^{-\alpha r^2}$, hyperbolic functions $sech(\alpha r)$, and rational functions like $r^\alpha/(1 + (\beta r)^\gamma)^\delta$, we always find $E_{var} > 0$. It is peculiar that the system admits zero-energy modes but apparently no negative-energy quantum states. The above variational functions were able to get arbitrarily close to the zero-mode state through a suitable choice of the parameters. Variational methods have the drawback that a small change in the parameter effects the function significantly for all r. A linear combination of a complete set is desirable and minimization with respect to the coefficients is often tractable. Alternatively, exact differential integrators in numerical packages handle linear differential equations like this one well. Using an ODE solver based on the Gear-stiff method, by varying the eigenvalue in increments of a thousandth, we examined the large r behavior of the solutions. An unphysical eigenvalue makes the function diverge at large r. Although the program identified the zero-mode correctly, no positive ϵ solution was found.

Therefore, using three different methods, we are unable to find any boundstates for the soliton-hole problem. Starting from either the relativistic or nonrelativistic form for a general coupling g, the essential boundstate equation is the same. A soliton is simply a small region of nonzero magnetic flux $B(r,\theta)$. Spinless particles, *i.e.*, dropping the Pauli term, feel no attractive potential. In the presence of spin-coupling, an attractive portion arises which is very sensitive to the coupling g. It would seem that the extreme limit of constant magnetic flux everywhere should also have boundstates if the soliton problem does. The constant field problem gives the well-known Landau levels which start at zero energy, yet have an attractive piece for V_{eff} in the presence of the Pauli term. This argument is overly simplistic and still does not rule out boundstates for fermions in magnetic fields. We can write the s-wave equation for $\chi(r)$ for $s_z = 1$ in the Coulomb gauge as before, using the scalar field $\phi(x)$ as

$$\chi'' + \frac{\chi'}{r} + (-\epsilon - g^2(\vec{\nabla}\phi)^2 + g\nabla^2\phi)\chi = 0 \quad . \tag{24}$$

The Coulomb potential in two dimensions is an instructive example for boundstates. For $V_c = -\alpha/r$, the system has an infinite number of boundstates with energy $E = -\alpha/(1+2n)$ (Appendix). If the class of potentials that have boundstates in the s-channel is called $[\widetilde{V}]$, the statement that the two-dimensional magnetic field cannot form boundstates implies that for any $\widetilde{V}$, the nonlinear, inhomogenous second-order differential equation

$$g^2(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) - g\nabla^2\phi = \widetilde{V} \quad , \tag{25}$$

has no real solutions.

For the symmetric case $\widetilde{V} = V(r)$, we can write $\rho(r) = gd\phi/dr$ giving the first order equation, $\rho' + \rho/r - \rho^2 = -V$. Defining $\rho = y + 1/2r$, we get $y' - y^2 = -V + 1/4r^2 \equiv p(r)$. This is Riccati's equation and making the substitution $u = e^{-\int^r y(t)dt}$, we obtain the linear second-order equation

$$\frac{d^2u}{dr^2} + pu = 0 \quad . \tag{26}$$

The solution of this equation is the zero-mode of the one-dimensional quantum problem $u'' + (E - \overline{V})u = 0$, where the attractive potential $\overline{V} = -p = V - 1/4r^2$. If we call the zero-mode u_0, the corresponding solution ϕ for the original problem then becomes

$$\phi(r) = \frac{1}{g}\int^r dr'\Big(-\frac{du_0}{dr'}\frac{1}{u_0} + \frac{1}{2r'}\Big) \quad . \tag{27}$$

This is readily integrable and gives $\phi(r) = (1/g)\ln(\sqrt{r}/u_0(r))$. It is not unreasonable to expect zero-modes for appropriate choices of V, since by assumption V belongs to the class of attractive potentials with boundstates. The solutions to this equation for V, a member of $[\widetilde{V}]$, is very interesting and implies boundstates for fermions moving in static magnetic fields in two dimensions. Even if solutions are found for some V, our soliton problem would still not be directly addressed. The conventional methods do not favor the existence of boundstates, but a proof of existence may lie in considering the differential equation for ϕ. The above analysis made simple assumptions of the continuum field theory description of holes and solitons. It leads to a rich coupling of spin, charge, angular momentum, and soliton quantum numbers; we explicitly demonstrated zero-energy states for the original soliton-hole interaction. The prospects for binding beyond the above treatment are summarized in Sec. V.

V. CONCLUSIONS

A long wavelength description of spin-waves and fermions is the 2 + 1 QED-like Lagrangian with nondynamical A_μ. Aside from the signs of masses, and whether one works in the relativistic or nonrelativistic regime or how many species of fermions there are, one always finds attraction of holes to solitons in all channels when the Pauli term is present and in the approximation of the soliton as a stationary background. The attraction places precise constraints on the signs of the fermion charge, its spin, its angular momentum and the soliton winding number. Curiously, it is straightforward to prove that zero-modes exist in any background where $A_0 = 0, A_i = \varepsilon_{ij}\partial_j\phi$. Had boundstates existed, *two opposite charge holes of spin up and spin down* would tumble to the lowest eigenstate, provided the binding energy thus gained is sufficient to offset the soliton rest energy. Many other rich possibilities exist. The boundstate problem was analyzed in the limit of one soliton and one hole. Since both the hole and the soliton carry A_μ charge, the two equations of motion become coupled through $A_\mu = (-i/2)(Z^\dagger\partial_\mu Z) + g\overline{\psi}\gamma_\mu\psi$. We considered the limit where A_μ is entirely given by the soliton background. To this order, boundstates could not be found. Perhaps self-consistent corrections of A_μ through ψ will modify the potential sufficient to find boundstates. This path was not pursued, since the same consistency requires we resolve the soliton equations of motion, etc. We showed that the very existence of boundstates in the relativistic regime breaks the dilation invariance of the soliton very nicely and gives sizes in the range of a few lattice spacings. With the apparent lack of boundstates for the soliton background, we considered a general magnetic field parameterized by $\phi(r)$. The existence of s-wave boundstates in this case depends on zero-mode solutions of a new equation with a purely attractive potential energy $V(r)$. Deriving the boundstates after finding the zero-modes for interesting choices of $V(r)$ is currently under progress. The relation of these new configurations to the soliton one is not clear but it may shed light on what the self-consistent potential may be like. The importance of the Pauli term for binding was discovered in another context by Ya. Kogan[22] in a system described by charged fermions of mass m interacting with a gauge field that obtains a mass M through a Chern–Simons term. Even though the two fermions have equal sign charge, the magnetic attraction through the Pauli term is stronger than the electric repulsion if $M/m > 1$. The relevance of this to our own soliton problem requires further work.

It is possible that other indications of a soliton condensate may emerge using different methods. The field theory approach taken here was the most convenient extension of the nonlinear sigma model to include fermions. Other mean field theory approaches have been used to describe hole motion in the antiferromagnetic background.[23] Some other approaches begin with the Hubbard $t-J$ model itself and replace bilinear-fermion operators with mean-field c-numbers. The expectation value of the operator $\chi_{ijk} =< \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k) >$ for three noncolinear sites on the lattice can often be computed in the mean-field approaches.[24] The path integral turns $\vec{S}$ into c-number fields and the above expectation value corresponds to $< \varepsilon_{abc} n^a(x_i) n^b(x_j) n^c(x_k) >$ using fields $n^a(\vec{x}) = (-)^{x_1+x_2} < \vec{S} >_{\vec{x}}$. Going to the continuum, we obtain $\chi_{ijk} \sim \pm(a^2/2) < \varepsilon^{ij} \varepsilon_{abc} n^a \partial_i n^b \partial_j n^c >$ evaluated at $\vec{x}_i$ where a is the lattice constant. The sign comes from the orientation of the triangle as shown in Fig. 2. This last quantity is just expectation value of the soliton number density. The precise relationship of these chiral spin states and flux phases which emerge in the mean-field limits of the Hubbard model to the soliton condensed state is only now being studied. For example, the stability of these phases with respect of quantum corrections would seem to imply that self-consistent boundstate solutions must exist. The presence of a soliton in this lattice picture implies a sum rule $\sum \eta \ \chi_{i,j,k} \sim integer$, with $\eta = \pm 1$, summed over a neighborhood of sites.

The zero-modes themselves are of great interest and perhaps the boundstate problem is a red herring. These modes are localized wavefunctions near the center of the soliton. Unfortunately, they do not break the dilation invariance of soliton size. However, they represent self-consistent solutions to all the equations of motion, since for the zero-modes the current $\overline{\psi}\gamma_i\psi$ vanishes for $i = 1, 2$, as only one component of the two-spinor is nonzero. The A_0 modifications through $\psi^+\psi$, in turn, do not affect the Z equations of motion. The energy of the state is just the mass of the soliton $2\pi f$. If other variational states give lower energy for the lattice with one hole,[25] the zero-mode states will not be energetically favorable. There exist many perturbative calculations of single holes in an antiferromagnet. The zero-momentum modes presented here are exact classical solutions in the long wavelength limit. Like the work of Shraiman and Siggia, the state of lowest energy has a surrounding topological texture. The zero-modes found in this work and their relation to the lattice zero-modes found Wen and Zee also deserves careful reexamination.[26] In the above work, we have demonstrated the relevance of the long wavelength field theories to understanding the spin-liquid state that could be described as the soliton-condensed state.

ACKNOWLEDGMENTS

P. V. would like to thank Dieter Issler, Paul Wiegmann, Ganapathy Baskaran, Vipul Periwal, and Xiao Wen for numerous discussions.

APPENDIX: TWO-DIMENSIONAL COULOMB PROBLEM

Here, we solve the Schröedinger equation for a particle moving in the Coulomb potential $V = -\alpha/r$. Both the relativistic and nonrelativistic form can be reduced to

$$\nabla^2\psi + (-\epsilon^2 + \frac{\alpha}{r})\psi = 0 \quad ,$$

for boundstate energies $-\epsilon^2$. If we expand $\psi(r,\theta) = e^{im\theta}\chi(r)$, the effective potential becomes $V_{eff} = m^2/r^2 - \alpha/r$. Now, writing $\chi(r) = r^{|m|}e^{-\epsilon r}f(r)$, we get

$$\xi\frac{d^2 f}{d\xi^2} + (2|m| + 1 - \xi)\frac{df}{d\xi} - \frac{\epsilon + 2\epsilon|m| - \alpha}{2\epsilon}f = 0 \quad ,$$

where $\xi = 2\epsilon r$. This is the equation for a confluent hypergeometric function and convergence for large r requires the coefficient of the f term to be a nonnegative integer n. Therefore, the eigenvalues $-\epsilon^2$ become

$$-\epsilon^2 = -(\frac{\alpha}{1 + 2|m| + 2n})^2 \quad .$$

REFERENCES

[1] G. Shirane *et al.,* Phys. Rev. Lett. **59**, 1613 (1987);
Y. J. Uemura *et al.,* Phys. Rev. Lett. **59**, 1045 (1987);
S. Mitsuda *et al.,* Phys. Rev. **B36**, 822 (1987).

[2] D. M. Ginsberg, *Physical Properties of High-Temperature Superconductors I* (World Scientific, Singapore, 1989);
W. Pickett, Rev. Mod. Phys. **61**, No. 2 (April 1989).

[3] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975).

[4] I. Dzyaloshinskii, A. Polyakov, and P. Wiegmann, Phys. Lett. **A127**, 112 (1988);
P. W. Anderson *et al.,* "Fermions and Topology in the Two-Dimensional Quantūm Antiferromagnet," Princeton preprint (February 1988).

[5] P. Wiegman, *Nobel Symposium Proceedings 1988*;
G. Baskaran and P. W. Anderson, Phys. Rev. **B37**, 580 (1988);
P. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988);
X. G. Wen and A. Zee, Phys. Rev. Lett. **62**, 1937 (1989).

[6] R. J. Cava *et al.,* Nature **332**, 814 (1988).

[7] F. J. Dyson, E. H. Lieb, and B. Simon, J. Stat. Phys., **18**, 335 (1978); E. Jordao Neves and J. Fernando Peres, Phys. Lett. **A114**, 331 (1986).

[8] E. Manousakis and Román Salvador, Phys. Rev. Lett. **62**, 1310 (1989);
D. Scalapino *et al.,* UCSB–ITP preprint (1989).

[9] F. D. M. Haldane, Phys. Lett. **93A**, 464 (1983);

Phys. Rev. Lett. **50**, 1153 (1983);
J. Appl. Phys. **57**, 3359 (1985);
Phys. Rev. Lett. **61**, 1029 (1988).

[10] R. Rajaraman, *Solitons and Instantons in Quantum Field Theory,* (North-Holland, 1982) p. 120.

[11] I. Affleck, *Theories of High-Temperature Superconductivity* (Addison-Wesley, 1988).

[12] P. W. Anderson, Science **235**, 1196 (1987).

[13] P. B. Wiegmann, *High-Temperature Superconductors and Materials and Mechanism of Superconductivity* (Interlaken, 1988);
X. G. Wen and A. Zee, NSF–ITP–88–35, NSF–ITP–88–115;
X. G. Wen, NSF–ITP–88–125;
I. Ichinose and T. Matsui, UT–Komaba 89–8;
Dietrich Foerster, PRINT–88–0926 (1988).

[14] P. B. Wiegmann, Physica C **153–155**, 103 (1988);
R. Shankar, NSF–ITP–89–39 (1989);
Patrick A. Lee, MIT preprint (1989).

[15] S. Trugman, Phys. Rev. **B37**, 1597 (1988).

[16] P. Voruganti, SLAC–PUB–4807 (to appear in Phys. Lett. B) (1989).

[17] X. G. Wen, "Symmetries of the Superconducting Order Parameters in the Doped Spin-Liquid State," ITP preprint (1989).

[18] A. N. Redlich, Phys. Rev. **D29**, 2366 (1984);

S. Deser and A. N. Redlich, Phys. Rev. Lett. **61**, 1541 (1988);

I. J. R. Aitchison and N. E. Mavromatos, OUTP–89–12P (1988).

[19] X. G. Wen, Frank Wilczek, A. Zee, "Chiral Spin States and Superconductivity," NSF–ITP–88–179;

Z. Zou, "Broken Symmetry and Half-Statistics of Spinons in 2+1 Dimensional Antiferromagnets," Stanford preprint.

[20] P. Voruganti, Phys. Rev. **D39**, 1179 (1989).

[21] J. P. Rodriguez, Phys. Rev. **B39**, 2906 (1989).

[22] P. Voruganti, Phys. Lett. **B214**, 413 (1988).

[23] R. Jackiw, Phys. Rev. **D29**, 2375 (1984).

[24] Ya. I. Kogan, "Bound Fermion and Superconducting Ground State in (2+1) Gauge Theory with Topological Chern–Simons Term," ITEP preprint 89–28 (1989).

[25] Boris I. Shraiman and Eric D. Siggia, Phys. Rev. Lett. **62**, 1564 (1989); Phys. Rev. Lett. **61**, 467 (1988);

B. Doucot and X. G. Wen, NSF–ITP–89–22 (1989);

P. W. Anderson, B. S. Shastry, and D. Hristopulos, *A Class of Variational Singlet Wave Functions for the Hubbard Model Away from Half-Filling* (1989).

[26] G. Baskaran, NSF–ITP–89–111 (1989);

X. G. Wen, Frank Wilczek and A. Zee, NSF–ITP–88–179 (1989).

[27] T. Barnes *et al.,* NSF–ITP–89–103 (1989);

E. Dagotto *et al.,* NSF–ITP–89–74 (1989);

S. Trugman, Phys. Rev. **B37**, 1597 (1988).

[28] X. G. Wen and A. Zee, "Winding Number, Family Index Theorem, and Electron Hopping in Magnetic Field," NSF–ITP–88–115 (1988).

FIGURE CAPTIONS

Figure 1. The potentials V_N and s-channel zero-modes ψ_N are plotted for soliton winding number N with $g = 2$.

Figure 2. The two sublattices are distinguished by crosses and dots. $\chi_{ijk} \sim \mp < \varepsilon^{ij} \varepsilon_{abc} n^a \partial_i n^b \partial_j n^c >$ in triangles A and B, respectively.

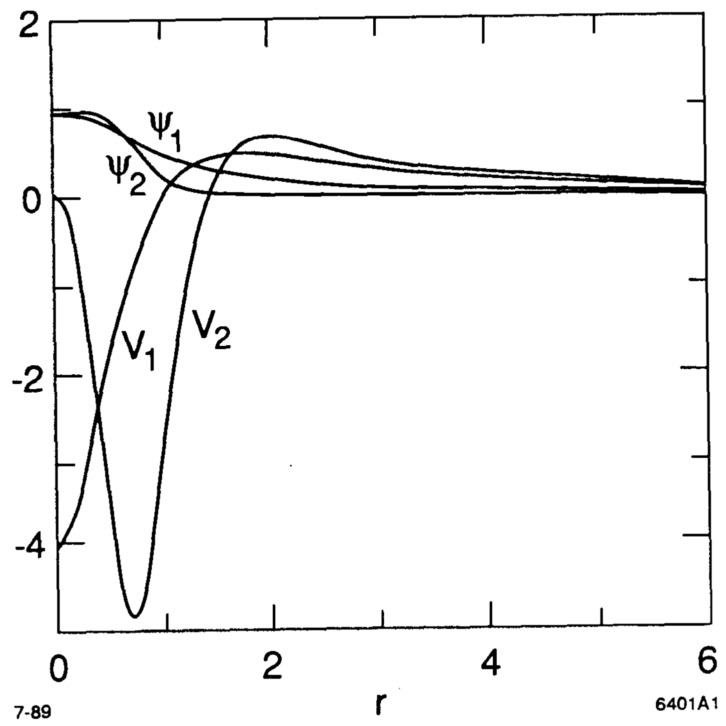
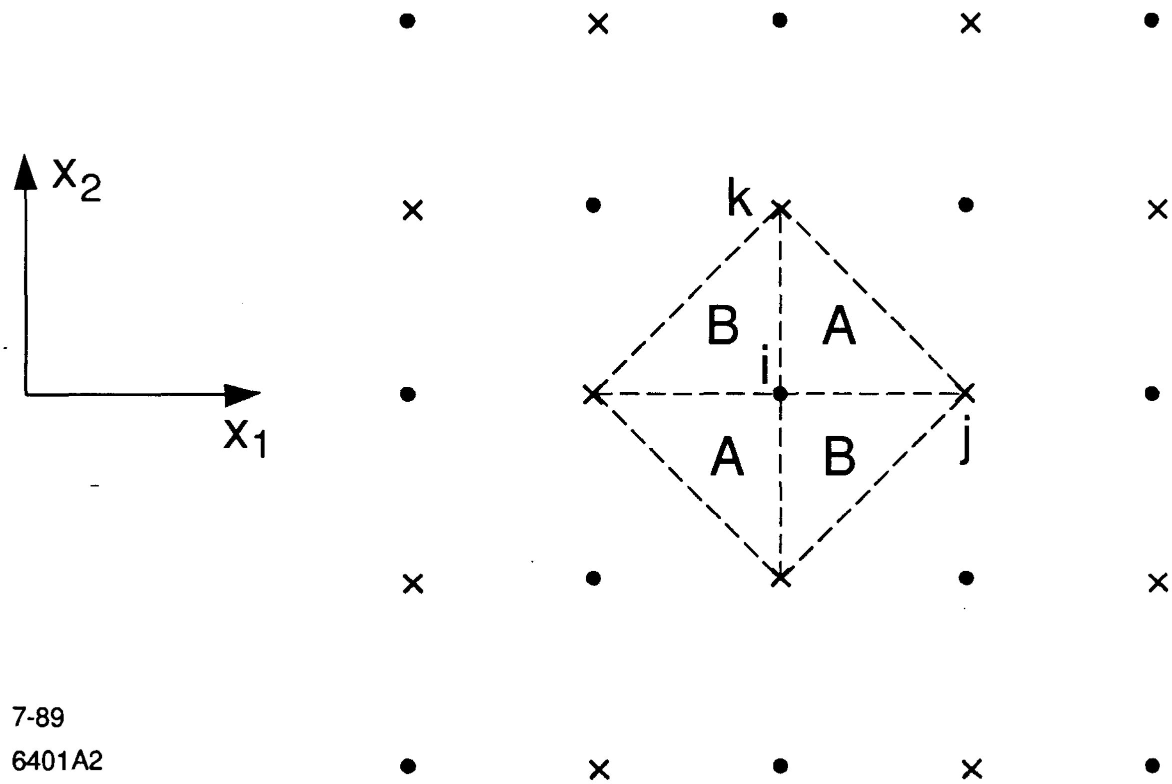


Fig. 1



7-89
6401A2

Fig. 2