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EMBEDDING HIGHER LEVEL KAC-MOODY ALGEBRAS IN HETEROTIC STRING MODELS*

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ABSTRACT

Heterotic string models in which the space-time gauge symmetry is realized as a level $k \geq 2$ Kac-Moody algebra on the string world-sheet offer phenomenological features not present for the level $k = 1$ case (which includes all heterotic string models seriously considered to date). In particular, string models with N=l space-time supersymmetry and/or chiral matter fields can contain massless scalar fields in the adjoint (or higher dimensional) representation of the gauge group provided that $k \geq 2$. Some explicit examples in four dimensions constructed solely from free, real world-sheet fermions are given, contradicting recent would-be no go theorems, and reopening the possibilities for embedding standard Grand Unified Theories within string models. The possibilities for more general constructions allowing higher level Kac-Moody algebras are also briefly discussed.

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1. Introduction

The possibilities for consistent heterotic string models in fewer than ten space-time dimensions are myriad[1-11] and largely unexplored. Only those classical string vacua whose underlying two-dimensional conformal field theory is relatively simple have been studied at all systematically. It is tempting to extrapolate the results of these cases to determine which general features of space-time physics can arise in string models, given the stringent constraints on the conformal field theories from which they are derived. One should be suspicious of folklore acquired in this fashion, however, as it is often misleading.

The present work was largely motivated by a particularly influential example of such folklore. It has been found within large classes of four-dimensional heterotic string models that N = 1 space-time supersymmetry and/or chiral fermions do not coexist with massless scalar fields in the adjoint or higher dimensional representations of the gauge group Since typical Grand Unified Theories rely on adjoint or higher dimensional Higgs fields to break the unified gauge group to the standard model,[12] this places a major practical constraint on string phenomenology. Either the gauge group must be broken at the compactification scale (effectively the Planck scale) or a non-standard Higgs breaking must be employed. Symmetry breaking via Wilson lines in Calabi-Yau vacua[1] is an example of the first possibility; flipped $SU(5)\otimes U(1)$[13] is one of the few known examples of the second. Standard GUTs, as well as phenomena such as SU(2) weak matter triplets, are apparently excluded from string theories if this empirical result is in fact a general property of string vacua.

However, as we will demonstrate with explicit examples below, this is not the case. String models with N=l space-time supersymmetry, and/or chiral matter fields, which contain massless scalar fields in the adjoint or higher dimensional representations of the gauge group are indeed possible provided that the gauge group is realized as a level $k \geq 2$ Kac-Moody algebra on the string world-sheet. These features are incompatible only in heterotic string models which contain only

$k = 1$ Kac-Moody algebras; however, this class includes all of the string models seriously considered to date. Restricting oneself to free bosons[3,6] or, equivalently, complex fermions,[4,5] on the string world-sheet, only $k = 1$ Kac-Moody algebras can be realized. Likewise, compactifications of the ten-dimensional heterotic string on Calabi-Yau manifolds"' or symmetric orbifolds,[2] or those constructed by employing symmetric combinations of N=2 minimal models:"' naturally give level 1 subgroups of the original level 1 $E_8 \otimes E_8$ group.[14] This is the source of the empirical result given above.

String models with higher level Kac-Moody algebras can be realized, however, with free *real* world-sheet fermions.[5,15] This construction has not been exploited in the past because the gauge group in such models is generally realized in rather intricate fashion. In fact this possibility was completely overlooked in the suggested proof, given in ref. [16], that N=l supersymmetry and chiral fermions are incompatible with massless adjoint scalar fields in heterotic strings constructed from free fermions. The explicit counter-examples given below are constructed within this framework. Despite being difficult to analyze, the possibilities prove to be rick and numerous. Furthermore, if we can overcome the technical problem of finding general modular invariants for collections of minimal models and/or Kac-Moody characters, it will almost certainly prove possible to construct an even larger class of string models containing higher level Kac-Moody algebras which cannot be realized with free fermions.

The format of this paper is the following. In section 2, we review and collect together some necessary facts about Kac-Moody algebras and their appearance in heterotic string theory. In particular we discuss why higher level Kac-Moody algebras are technically difficult to achieve in heterotic string models with our present methods, and demonstrate why massless adjoint scalar fields and chiral fermions cannot coexist if the gauge group is realized purely at level 1. In section 3, explicit examples of heterotic string models in four dimensions are presented which contain higher level Kac-Moody symmetries. These include models with phenomenologically interesting GUT gauge groups, N=l supersymmetry, chiral fermions and massless adjoint scalar fields. The examples are presented starting with the simplest and progressing to ever more sophisticated and realistic constructions. The emphasis is on exploring possibilities; no attempt is made to reproduce detailed phenomenology. The fermionic spin-structure construction employed in generating these examples is briefly reviewed in the appendix, with emphasis given to the particular subtleties involved in using real, rather than complex, fermions. In section 4, the possibilities and prospects for constructing more general classes of heterotic string theories in which the gauge group is realized as a higher level Kac-Moody algebra are briefly surveyed. The chief points and results presented are summarized in section 5.

2. Kac-Moody Algebras in Heterotic String Models

Let us begin by recalling some basic facts about Kac-Moody algebras; for details and derivations see, e.g., ref. [17]. In a heterotic string model, the vertex operator for a gauge boson (for simple group G, momentum p, and polarization ζ) is a primary field of conformal dimension $(1/2,1)$ of the form,

$$V^a = \zeta_\mu \psi^\mu(\bar{z}) J^a(z) e^{ip\cdot X} \; ; \; p^\mu p_\mu = p^\mu \zeta_\mu = 0 \ . \tag{2.1}$$

X^μ is the string coordinate, ψ^μ is a dimension $(1/2,0)$ Ramond-Neveu-Schwarz fermion and J^a is a dimension $(0,1)$ field which necessarily satisfies the OPE of a Kac-Moody generator,

$$J^a(z)J^b(w) = \frac{1}{(z-w)^2} k\delta^{ab} + \frac{1}{(z-w)} i f^{abc} J^c + \dots \tag{2.2}$$

f^{abc} are the structure constants for the group G normalized so that $f^{abc} f^{dbc} = C_A \delta^{ad} = 2\tilde{h}\delta^{ad}$, where C_A is, by definition, the quadratic Casimir of the adjoint representation of the group G, and $\tilde{h}$ is the dual Coxeter number (listed in table 1 for the simple Lie algebras). With this normalization k is a positive integer, the level of the Kac-Moody algebra. Thus space-time gauge fields imply the existence of a Kac-Moody algebra on the string world-sheet and, in particular, all states in the theory necessarily fall into representations of this algebra. For fixed level k only a finite number of these representations are unitary and may appear in a sensible string model; specifically those satisfying,

$$k \geq \sum_{i=1}^{\mathrm{rank}(G)} n_i m_i \quad , \tag{2.3}$$

where n_i are the Dynkin labels of the highest weight of the representation, and m_i are integers (sometimes called co-marks) listed in table 1 for the simple Lie groups. The conformal dimension of the primary field corresponding to the highest weight representation r is,

$$h_r = \frac{C_r}{2k + C_A} \quad , \tag{2.4}$$

where C_r is the quadratic Casimir of r, C_A that for the adjoint. These invariants are most extensively tabulated in ref. [18] . Finally, the contribution to the Virasoro central charge from the level k Kac-Moody algebra is,

$$c = \frac{k\, dim(G)}{k + \tilde{h}} \quad . \tag{2.5}$$

In essentially all heterotic string models studied to date, the space-time gauge group is realized as a level 1 Kac-Moody algebra on the string world-sheet. The reasons for this are practical rather than fundamental ones. Generally, the chief technical difficulty involved in constructing any string model is insuring modular invariance of the loop amplitudes. The simplest modular invariants involving Kac-Moody characters, and in many cases the only ones which have been written down, are left-right (i.e., antiholomorphic-holomorphic) symmetric. These are fine for constructing bosonic strings on group manifolds,[19] but inapplicable for constructing heterotic string models, for the same reason that the standard model cannot fit within the type II string[20]: Any component of the world-sheet supersymmetric half of a four-dimensional string model must be world-sheet supersymmetric yet contribute central charge 9 or less, while any super Kac-Moody algebra large enough to contain the standard model contributes at least c = 10.

Systematically constructing asymmetric modular invariants is difficult and has so far proved possible only for collections of free world-sheet fields. These do not lead most naturally to higher level Kac-Moody algebras, however. Consider, for example, representing all of the internal degrees of freedom of the bosonic half of a heterotic string model in D space-time dimensions with 26-D free world-sheet bosons ϕ^i (to saturate the necessary central charge, c =26-D). The dimension 1 holomorphic operators $\partial\phi^i$ each generate a U(1) Kac-Moody algebra, so the rank of the gauge group is at least 26-D. It follows from (2.5) and table 1 that the rank of G cannot exceed the central charge of the corresponding Kac-Moody algebra, and the two are equal only for the simply laced algebras with $k = 1$. Thus heterotic string models constructed from free world-sheet bosons (or, equivalently, complex fermions) [3–7] include only simply laced $k = 1$ Kac-Moody algebras regardless of the momentum lattice chosen to define the theory. This includes the original $E_8 \otimes E_8$ string in ten dimensions.[14] Furthermore, compactifications of this model, whether on Calabi-Yau manifolds:" left-right symmetric orbifolds[2] or using left-right symmetric combinations of N=2 minimal models together with free bosons,[9] all naturally inherit level 1 subgroups of this level 1 $E_8 \otimes E_8$. On the other hand, it is possible to construct asymmetric modular invariants including higher level Kac-Moody algebras by using free ***real*** fermions, as we will see in section 3. There are many subtleties involved, however, and one would not likely arrive at the necessary constructions without having the particular goal of achieving higher level Kac-Moody algebras in mind.

The problem with limiting our attention to heterotic string models with level 1 gauge groups is that the possible unitary highest weight representations of a level 1 Kac-Moody algebra are quite limited in number and size as can be seen from eqn. (2.3) and table 1. Considering likely grand unified groups, only four unitary representations appear at level 1 for SO(4n+2) (the singlet, spinor, conjugate spinor and vector), N representations for SU(N), and three representations for E_6 (the singlet, 27 and $\overline{27}$). These include the necessary representations for matter fields, but not all of the representations for scalars that we might like. In particular the known mechanisms for breaking E_6, SO(10) or SU(5) down to the standard model require Higgses in the adjoint representation (or representations of even higher dimension) ,[12] but the adjoint highest weight representation is not among the unitary ones for any of the level 1 Kac-Moody algebras.

There seems at first to be a way out of this particular difficulty: the Kac-Moody currents themselves transform in the adjoint representation and we might use them to construct vertex operators for massless adjoint scalar fields. Unfortunately this proves to be incompatible with the presence of chiral fermions and, as a practical matter, N=l space-time supersymmetry. The kernel of this argument has appeared in many guises before; here we paraphrase observations originally made in ref. [20] in the context of generalized type II theories. Consider the only possible form for the vertex operator of a massless adjoint scalar field in a four-dimensional heterotic string if the gauge group is level 1,

$$V^a_{scalar} = O(\bar{z})J^a(z) . \tag{2.6}$$

J^a is one of the Kac-Moody currents. For the state represented by this operator to be a massless scalar in space-time, O must be an anti-holomorphic dimension (1/2,0) fieldwhichin its OPE's and behavior under GSO projections is indistinguishable from an additional RNS fermion. Thus the space-time spinor degrees of freedom, which are governed by the structure of the world-sheet supersymmetric half of the heterotic string, fall into representations of the five-dimensional Lorentz group, SO(4,1). Decomposed into SO(3,1)spinors these always give rise to non-chiral pairs. Thus the presence of adjoint scalars in the form (2.6) necessarily precludes the presence of chiral fermions.

This argument does not apriori rule out the presence of N=l supersymmetry, since an **N=l** supersymmetric theory in five dimensions will, upon compactification of one coordinate, produce a four-dimensional theory with N=l supersymmetry and adjoint scalar fields. Within the simple compactifications and free field constructions which have been considered to date, however, it appears impossible to construct a string theory in five dimensions with just N=l supersymmetry. In any case these theories would be of no phenomenological interest, being non-chiral, and such possibilities will not concern us further here.

3. Examples

3.1. REAL FERMION CONSTRUCTION

In the spin-structure construction of heterotic string models[4,5,15] all internal degrees of freedom of the string are represented by free fermions. Three pieces of input are required to define any given model: 1) the form of the fermionic realization of the supercurrent for the supersymmetric half of the heterotic string; **2)** a set of vectors, {Vi}, whose linear combinations define the allowed boundary conditions for all of the fermions; 3) parameters, k_{ij}, which specify in part which world-sheet states appear as physical fields in space-time and which are projected out of the spectrum. These three inputs must be specified consistent with each other and with the constraints of modular invariance and a physically sensible operator interpretation (including correct space-time statistics).

The details and derivation of the solution to the required constraints is amply covered in the original literature. These results, tailored to the specific case of interest here, namely four-dimensional heterotic strings built from real fermions, are summarized in the appendix. Some practical guidelines for constructing models with desirable features are also given. One subtlety of the real fermion case bears special mention here, however, as it figures directly in some of the models we give in the following, and has not been sorted out completely in the published literature. The real fermion case is more intricate than the complex one largely because, in contrast to the latter case, the formal fermion number operators which appear in the analysis need not all commute with one another. In addition, while complex fermion determinants have been explicitly evaluated on multi-loop Riemann surfaces and their behavior under modular transformations determined, the analogous results for real fermions can only be inferred by taking a square root— which results in a sign ambiguity. Both of these subtleties disappear in the real fermion case if we demand that the non-vanishing components common to any *four* boundary condition vectors are even in number. At one time this "quartic" constraint was believed necessary for a consistent theory at both the one loop and multi-loop levels;[5,7] however, a more careful analysis[15] showed that at the tree and one-loop level a lesser constraint- that the number of nonvanishing components common to any *three* vectors is even- is sufficient. Furthermore it has been proved (more recently) that the crossing symmetry of all four-point amplitudes on the plane, and modular invariance of all one-point one-loop amplitudes, are enough to insure multi-loop modular invariance in any conformal field theory.[21] Thus in the real fermion construction the "cubic" constraint on boundary condition vectors is sufficient for consistency. The additional freedom gained in relaxing the quartic constraint will prove useful in constructing some of the models we present here.

3.2. Simple SU(2) Level 2 Examples

The simplest example of a higher level Kac-Moody algebra is SU(2) level 2, which can be realized with three real fermions, ψ_i. The currents, necessarily with conformal dimension 1, are the three non-vanishing bilinears, $\psi_i\psi_j$. The SU(2) $\approx$ SO(3) symmetry is just that of rotations in the three-dimensional fermion space. That the algebra has $k = 2$ can be determined by explicitly computing the OPE, eqn.(2.2), or, more easily, by noting that $c = 3/2$. SU(2) level 2 has three unitary representations (c.f. eqn. (2.3)) the singlet, doublet, and triplet (which is the adjoint). The corresponding primary fields in the three fermion theory are the identity, $\sigma_1\sigma_2\sigma_3$ (the product of the three spin fields), and ψ_i, respectively, with conformal dimensions 0, $3/16$, and $1/2$ in agreement with eqn. (2.4). Mixed operators such as $\sigma_1\psi_2$ are not present as they would break the symmetry of the ψ_i's.

While SU(2) level 2 is a rather trivial example, it already proves sufficient to illustrate some of the chief differences between level $k = 1$ and $k > 1$ Kac-Moody algebras in string theory. In particular there are two distinct realizations for the adjoint in SU(2) level 2 but not for level 1 (for which only the singlet and doublet are unitary highest weight representations). In a string model we can use the fermion bilinears to construct the gauge boson vertex operators and the individual ψ_i to construct vertex operators for adjoint scalar fields, thereby side stepping the arguments given in the previous section that adjoint scalar fields whose vertex operators are constructed from the currents are incompatible with N=1 supersymmetry and/or chiral fermions. Simple counterexamples within the spin-structure construction of string models are easily written down. For example consider the model generated by the set of vectors,

$$\begin{aligned}
\mathbf{V}_0 &=((\tfrac{1}{2})^2(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^6 \mid (\tfrac{1}{2})^{44})\\
\mathbf{V}_1 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^6 \mid (0)^{44})\\
\mathbf{V}_2 &=((0)^2(\tfrac{1}{2}\tfrac{1}{2}0)^4(0\tfrac{1}{2}\tfrac{1}{2})^2 \mid (\tfrac{1}{2})^4(0)^{40})\\
\mathbf{V}_3 &=((0)^2(\tfrac{1}{2}\tfrac{1}{2}0)(\tfrac{1}{2}0\tfrac{1}{2})(0\tfrac{1}{2}\tfrac{1}{2})^2(\tfrac{1}{2}\tfrac{1}{2}0)^2 \mid (\tfrac{1}{2})^3 0 \tfrac{1}{2}(0)^{39}) \quad .
\end{aligned} \tag{3.1}$$

With the $[SU(2)]^6$ form of the supercurrent, $\mathbf{V}_1$ is the unique choice (up to re-ordering components) which can contribute gravitinos and hence space-time supersymmetry. $\mathbf{V}_2$ serves the dual purpose of cutting the supersymmetry from N=4 to N=2, and the gauge group from SO(44) to SO(4)$\otimes$SO(40). $\mathbf{V}_3$ breaks the symmetry down finally to N=1 supersymmetry and SU(2)$\otimes$SO(39), the SU(2) realized, as desired, at level 2. The choice of parameters k_{ij} fixing the physical state projections has no real effect on the massless states in this model so we leave them unspecified. The underlying conformal field theory for the bosonic half of this heterotic string model can be viewed as a correlated combination of SU(2) level 2 and SO(39) level 1 Kac-Moody representations together with a pair of Ising models.

All of the massless bosons in the space-time spectrum in this simple model arise from the **0** sector (i.e., the vector with all components 0 corresponding to the fully Neveu-Schwarz sector), while the $\mathbf{V_1}$ sector contributes all the accompanying super-partners. In addition to the SU(2)$\otimes$SO(39) gauge bosons, the massless bosonic spectrum includes two sets of scalars transforming as (1,1) and (3,39) of SU(2)$\otimes$SO(39) and four sets each of scalars in the representations (1,39) and (3,1). As desired, massless scalars in the adjoint of SU(2) level 2 peacefully coexist with the N=1 space-time supersymmetry. The vertex operators for these states are built, as required, from the highest weight adjoint primary field of SU(2) level 2 (the ψ_i and not the currents $\psi_i\psi_j$); the additional conformal dimension of 1/2 required to make the overall vertex operator for a massless state dimension 1 is supplied by either the primary field for the vector representation of SO(39), or the fermion in one of the Ising models. This is typical of such constructions: the adjoint highest weight representation for higher level Kac-Moody algebras has conformal dimension less than 1 (unlike the currents) and so something must be added to it to construct the vertex operator for a massless state. This additional piece may transform non-trivially under some other factor of the gauge group, but this doesn't have to be the case.

Neither SU(2) nor SO(39) admits chiral representations. It is easy enough, however, to construct a model which does contain chiral matter along with a level 2 SU(2) algebra. Consider the vectors,

$$\begin{aligned}
\mathbf{V}_0 &=((\tfrac{1}{2})^2(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^6 \mid (\tfrac{1}{2})^{44}) \\
\mathbf{V}_1 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^6 \mid (0)^{44}) \\
\mathbf{V}_2 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^2(0\tfrac{1}{2}0)^4 \mid (\tfrac{1}{2})^9(0)^3(\tfrac{1}{2})^7(0)^{25}) \\
\mathbf{V}_3 &=((\tfrac{1}{2})^2(0\tfrac{1}{2}0)^2(\tfrac{1}{2}00)^2(00\tfrac{1}{2})^2 \mid (\tfrac{1}{2})^6(0)^3(\tfrac{1}{2})^3(0)^7(\tfrac{1}{2})^7(0)^{18}) \\
\mathbf{V}_4 &=((\tfrac{1}{2})^2(00\tfrac{1}{2})^2(00\tfrac{1}{2})^2(\tfrac{1}{2}00)(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}) \mid (0)^{12}(\tfrac{1}{2})^{18}(0)^{14})
\end{aligned} \tag{3.2}$$

$\mathbf{V_0}$ and $\mathbf{V_1}$ are as before. The other three vectors were written down with the following goals in mind: to reduce the space-time supersymmetry from N=4 to

N=1, produce a gauge group including SU(4)⊗SU(2)⊗SU(2) with at least one SU (2) realized at level 2, introduce chiral fermions (under the SU(4)), and introduce massless scalar fields in the adjoint of SU(2)*. In fact, the gauge bosons arise entirely from the 0 sector and generate the group SU(4) $\otimes$ $SU(2) \otimes SU(2) \otimes [SO(7)]^2 \otimes SO(4) \otimes SO(14)$; V_1 contributes the single gravitino required for N=l supersymmetry provided we choose $k_{01} = k_{21} + k_{31} + k_{41}$ modulo 1 (otherwise N=0); the sectors $\mathbf{V_2}$ and $\mathbf{V_3}$ contribute two sets each of Pati-Salam like "quarks" and I'anti-quarks" in the group representations (4,2,1,8,1,1,1) and ($\bar{4}$,1,2,1,8,1,1) respectively; and the 0 sector includes a host of massless scalar fields including ones transforming as (1,1,3,1,1,4,1), (1,3,1,1,1,4,1) and (1,3,3,1,1,1,1), i.e., in the adjoint of the SU(2)'s.

Other vectors can be added to the set (3.2) to break the large horizontal symmetry and reduce the number of chiral matter families to make the spectrum look more realistic, but models such as this one generally suffer from a more fundamental problem if we try to treat them seriously for phenomenology. The relation of the gauge couplings to the fundamental string-coupling at the Planck scale depends on the level of the Kac-Moody algebra!"] Thus if different pieces of the standard model gauge group are realized at much different levels, k, as in the model above, it will almost certainly prove impossible to get the correct value for $\sin^2\Theta_W$ from the running of the gauge coupling constants. Nonetheless, the simple model defined by (3.2) provides an existence proof that features such as N=l supersymmetry, chiral fermions, massless adjoint scalars and weak matter triplets can be present simultaneously in a heterotic string model if (and only if) some of the gauge group is realized as a higher level Kac-Moody algebra.

* This model was constructed in response to a query of T. Banks.

3.3. E_8 Level 2 and Subgroups

SU(2) (being isomorphic to SO(3)) is a rather special algebra, and affine SU(2) at level 2 represents the only higher level Kac-Moody algebra whose currents may be realized so trivially in terms of fermion bilinears. One might well worry that the success of the constructions given above relies entirely on this fact and can't be extended to more interesting groups, but this is not the case. One interesting and non-trivial example of a heterotic string model whose gauge group is realized via a higher level Kac-Moody algebra has, in fact, already been studied in the literature (and is the only such example to the best of our knowledge). Of the eight known consistent string models in ten flat space-time dimensions, seven have rank 16 gauge groups and can be realized solely in terms of free world-sheet bosons.[23] The eighth has gauge group a single E_8 realized at level $k = 2$, and was first explicitly constructed from free real world-sheet fermions.[24] We will adapt this construction here to produce four-dimensional models with various subgroups of E_8, all at level 2, appearing as the gauge groups.

Consulting eqns. (2.3), (2.4), and (2.5) it is not surprising that E_8 level 2 can be fermionized. Three unitary representations appear, the singlet, adjoint and dimension 3875 with conformal dimensions 0, $15/16$, and $3/2$ respectively, and the central charge is $31/2$. In the ten-dimensional model the E_8 level 2 appears together with a single Ising model ($c = 1/2$) to make up the $c = 16$ of the bosonic half of the heterotic string. A different grouping of the E_8 level 2 and Ising characters leads to an enhanced symmetry — $E_8 \otimes E_8$ level 1. Saying the same thing in reverse, the Ising model can be obtained from the coset construction $E_8^{(1)} \otimes E_8^{(1)}/E_8^{(2)}$. The ten-dimensional model can thus also be obtained by modding the $E_8 \otimes E_8$ model by the symmetry interchanging the two E_8's, leaving behind only the diagonal subgroup, E_8 level 2, and the single Ising model from the coset construction.[23,25]

Consider now the following vectors generating a four-dimensional heterotic string,

$$
\begin{aligned}
\mathbf{V}_0 &=((\tfrac{1}{2})^2(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^6 \mid (\tfrac{1}{2})^{44})\\
\mathbf{V}_1 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^6 \mid (0)^{44})\\
\mathbf{V}_2 &=((0)^{20} \mid (\tfrac{1}{2})^{16}(0)^{15}(0)^{13})\\
\mathbf{V}_3 &=((0)^{20} \mid (\tfrac{1}{2})^{8}(0)^{8}(\tfrac{1}{2})^{8}(0)^{7}(0)^{13})\\
\mathbf{V}_4 &=((0)^{20} \mid (\tfrac{1}{2})^{4}(0)^{4}(\tfrac{1}{2})^{4}(0)^{4}(\tfrac{1}{2})^{4}(0)^{4}(\tfrac{1}{2})^{4}(0)^{3}(0)^{13})\\
\mathbf{V}_5 &=((0)^{20} \mid (\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}(0)^{2}(\tfrac{1}{2})^{2}0(0)^{13})\\
\mathbf{V}_6 &=((0)^{20} \mid \tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}(0)^{13})\\
\mathbf{V}_7 &=((0)^2(000)^2(\tfrac{1}{2}\tfrac{1}{2}0)^4 \mid (\tfrac{1}{2})^{32}(0)^{12})\\
\mathbf{V}_8 &=((0)^2(\tfrac{1}{2}\tfrac{1}{2}0)^2(\tfrac{1}{2}0\tfrac{1}{2})^2(0\tfrac{1}{2}\tfrac{1}{2})^2 \mid (0)^{32}(\tfrac{1}{2})^{4}(0)^{8})
\end{aligned}
\tag{3.3}
$$

$\mathbf{V_0}$ and $\mathbf{V_1}$ take their familiar values. $\mathbf{V_2}$ through $\mathbf{V_6}$, apart from some appended zeros, are borrowed directly from the ten-dimensional construction and give rise to the E_8 level 2 gauge bosons. $\mathbf{V_7}$ and $\mathbf{V_8}$ serve to break the space-time supersymmetry to N=1, and the former, combined with $\mathbf{V}_2$ through $\mathbf{V}_6$, provide massless scalars which will be in the adjoint of E_8.

As we noted previously in section 2, systems of complex fermions can be bosonized and only give rise to level 1 Kac-Moody algebras. A necessary requirement, then, to achieve a higher level algebra within the present construction is to choose sets of boundary conditions which prevent the real fermions from being consistently paired up in any way to form complex ones. This is accomplished by $\mathbf{V_2}$ through $\mathbf{V_6}$ in (3.3) in quite symmetric fashion for the first 32 fermions on the right hand side. The first 31 of these give the fermionized E_8 level 2 (recall c = 31/2).

In the simplest string models constructed from free world-sheet fermions, all of the gauge bosons appear in the 0 (that is the fully Neveu-Schwarz) sector. In contrast, in the model generated by (3.3) the only gauge bosons arising from the 0 sector are those for $O(4)\otimes O(8)$ from the last 12 fermions, which play no role in the present discussion. All of the other massless vector boson states in this sector are projected out by the constraints generated by $\mathbf{V}_2$ through $\mathbf{V}_6$. On the other hand, the operators creating the ground state of any linear combination of $\mathbf{V}_2$ through $\mathbf{V}_6$ from the Neveu-Schwarz vacuum have conformal dimensions $(0,1)$ (i.e., those of Kac-Moody currents) and so can give rise to gauge bosons. Consider first the sector $\mathbf{V}_2$. The product of 16 individual Ramond ground states is 256 fold degenerate, but the vectors $\mathbf{V}_0, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5$, and $\mathbf{V}_6$ generate 5 independent constraints, each cutting this space in half, leaving 8 states surviving these projections. Finally, checking the constraints generated by $\mathbf{V}_1$, $\mathbf{V}_7$ and $\mathbf{V}_8$, we find that the sector $\mathbf{V}_2$ contributes 8 gauge bosons to the spectrumif and only if $k_{02} = k_{72}$ and $k_{12} = k_{82} = 0$. The other 30 nonzero sectors generated by $\mathbf{V}_2$ through $\mathbf{V}_6$ are entirely analogous. Assume for the moment that $k_{ij} = 0$ for any i if $j = 2,3,4,5,6$ (note that $k_{1j} = 0$ are the necessary and sufficient choices for N=l supersymmetry in this model). Then we have a total of 31 sectors, each contributing 8 gauge bosons, to make a total of 248 — just the number required for E_8.

Generally, knowing the number of gauge bosons is insufficient to determine the gauge group. In the present case, however, we also know that the Kac-Moody theory underlying the gauge group contributes central charge $31/2$ or less. Consulting eqn. (2.5), table 1 and the dimensions of the simple Lie algebras, one finds, after a little effort, that E_8 level 1 or level 2 are the only possibilities. To determine at which level the E_8 is realized we need only check the other E_8 representations appearing in the model. At level 1 only the E_8 currents and their descendents appear; at level 2 we have in addition the adjoint and dimension 3875 representations. It is easy to see that the 31 sectors obtained from adding $\mathbf{V}_7$ to all nonzero combinations of $\mathbf{V}_2$ through $\mathbf{V}_6$ are (for the bosonic half of the string) mirror images of the 31 sectors considered above, and thus contribute states in the adjoint of E_8, realized at level 2. These states have conformal dimensions $15/16$ coming from the first 31 fermions (which form the E_8), and $1/16$ from the spin field of the 32^{nd} fermion. Given the form of the left hand side of $\mathbf{V}_7$ these states are massless space-time scalars (in fact there are four families of them); if the left hand side of $\mathbf{V}_7$ were 0 these would provide instead an additional 248 gauge bosons which, mixed with the 248 above, would generate $E_8 \otimes E_8$ level 1.

What other operators transforming non-trivially under E_8 level 2 are included in the model we have constructed? In the 0 sector all states with either 1 or 2 excitations of the 31 fermions forming the E_8 level 2 are projected out of the spectrum; however, some fermion trilinears are allowed. A careful examination of the constraints shows that we may pick any two fermions and then the third is fixed, for a total of $31{\cdot}30/6{=}155$ dimension $3/2$ operators. In any of the 31 sectors from the nonzero combinations of $\mathbf{V}_2$ through $\mathbf{V}_6$ we may excite any of the 15 fermions in the first 31 with Neveu-Schwarz boundary conditions, to again get dimension $3/2$ operators, in this case a total of $31{\cdot}15{\cdot}8{=}3720$. Adding to those above gives the expected multiplet of 3875 dimension $3/2$ operators. Note that the analogous operators in the sectors giving rise to the adjoint scalars are projected out of the spectrum — and rightly so for they would represent states of E_8 level 2 Kac-Moody algebra with dimension $23/16$, which do not exist.

E_8 is not a terribly interesting group for low energy phenomenology, but many of its subgroups are. We could try to construct these groups, realized as higher level Kac-Moody algebras, directly within the spin structure construction. It is in many ways easier, however, to generate them as subgroups of E_8 by modifying the construction just given. One simple possibility has already suggested itself. Namely, if we take, say, $k_{86} = 1/2$ then the sector $\mathbf{V}_6$ (and in fact all of the sectors $\mathbf{V}_6+$ any combination of $\mathbf{V}_2$ through $\mathbf{V}_5$) will no longer contribute any gauge bosons. Only a subset of the previous gauge bosons are still present, numbering $15{\cdot}8{=}120$. These must form the adjoint representation of some subgroup of E_8. The unique E_8 subgroup of dimension 120 is SO(16). Furthermore, it must be realized at level 2, since a subalgebra of a level k Kac-Moody algebra always has level k or greater and the available central charge is not consistent with SO(16) at level $k > 2$.

The same effect of reducing E_8 to SO(16) is also achieved by taking k_{76} =

k_{06} + $1/2$, or by modifying $\mathbf{V}_6$ (e.g., by changing the last 8 zeros on the right hand side to $1/2$'s) so that the associated conformal dimension is no longer $(0,1)$. In any of these approaches the sectors $\mathbf{V}_6+\ldots$ no longer contribute gauge bosons, while the projections in the sectors represented by any combination of $\mathbf{V}_2$ through $\mathbf{V}_5$ remain unchanged. Applying the same trick twice ,e.g., to $\mathbf{V}_5$ and $\mathbf{V}_6$, only 7 sectors still contribute gauge bosons, breaking SO(16) to a subgroup of dimension $7\cdot 8=56$. Given the symmetry of the vectors in (3.3) it seems likely that this group is $SO(8)\otimes SO(8)$. Unlike the case above, however, simple counting arguments (and the knowledge that we are dealing with a subgroup of SO(16)) are no longer sufficient to rule out other possibilities (such as $SO(11)\otimes U(1)$) and uniquely specify the group. In principle this model is completely specified; we could calculate all of the three point couplings and thereby determine the gauge group. In practice, however, the difficulty in determining the gauge group and group quantum numbers of other physical states is a severe limitation of the real fermion construction. We will present one way around this difficulty in the following section.

There is another distinct way to break the E_8 gauge group of (3.3). We can add additional vectors which project out some, but not all, of the gauge bosons in any of the sectors generated by $\mathbf{V}_2$ through $\mathbf{V}_6$. The new sectors added in this process can also contribute massless states transforming non-trivially under the new gauge group. As an example let us replace $\mathbf{V}_7$ and $\mathbf{V}_8$ in (3.3) with the three vectors,

$$\begin{aligned}
\mathbf{V}'_7 &=((0)^2(000)^2(0\tfrac{1}{2}\tfrac{1}{2})^4 \mid (\tfrac{1}{2})^{32}(0)^{12}) \\
\mathbf{V}'_8 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^2(00\tfrac{1}{2})^4 \mid (0)^{16}(\tfrac{1}{2})^8(0)^{20}) \\
\mathbf{V}_9 &=((\tfrac{1}{2})^2(0\tfrac{1}{2}0)^2(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})(00\tfrac{1}{2})(\tfrac{1}{2}00)(0\tfrac{1}{2}0) \mid (0)^{20}(\tfrac{1}{2})^8(0)^4(\tfrac{1}{2})^2(0)^{10}) \ .
\end{aligned} \tag{3.4}$$

$\mathbf{V}'_7$ differs from $\mathbf{V}_7$ only by its left hand side, and again, combined with $\mathbf{V}_2$ through $\mathbf{V}_6$ will provide massless adjoint scalar fields. The left hand sides of $\mathbf{V}'_8$ and $\mathbf{V}_9$ serve to break the supersymmetry to N=1 which survives provided that k_{71} = 0. The right hand sides of $\mathbf{V}'_8$ and $\mathbf{V}_9$ break the E_8 gauge group. The result depends on the k_{ij}'s and one must be careful about the precise definition of the projection operators (see the appendix) as the results in different sectors are correlated with each other. We only give the briefest sketch of the results here. Consider first the effect of $\mathbf{V}_8'$ without including $\mathbf{V}_9$. In each of the sectors $\mathbf{V}_1$,$\mathbf{V}_2$, and $\mathbf{V}_1+\mathbf{V}_2$ the new constraint either permits the 8 gauge bosons present before-or projects them all out. Depending on the choices for k_{ij} there are two distinct possibilities. These three sectors together contribute either 24 gauge bosons or only 8. In each of the 28 remaining sectors contributing gauge bosons the new constraint is independent of those present before and so cuts the number of massless vector states in half. The two possible totals, then, are 120 or 136 corresponding to the E_8 subgroups SO(16) or $E_7\otimes$SU(2) both realized as level 2 Kac-Moody algebras.

Now include the constraints generated by **Vg.** In 24 of the sectors the number of gauge bosons is again cut in half to 2 each. Six of the remaining sectors can contribute either 0 or 4 gauge bosons, and $\mathbf{V}_1$ contributes 0 or 8. The results from the different sectors are correlated so particular attention to the relative minus signs and operator orderings are required. The distinct possibilities are $E_7\otimes$SU(2) breaking to groups of dimensions 80 ($E_6 \otimes U(1) \otimes$ U(1)) and 72 (SO(12)$\otimes$SU(2)$\otimes$SU(2)) and SO(16) to dimensions 72 (SO(12)$\otimes$SU(2)$\otimes$SU(2)), 64 (U(8)), and 56 (SO(8)$\otimes$SO(8) and/or SO(11)$\otimes$U(1)). These groups can be further broken down by the first mechanism given above, e.g., choosing $k_{76} = k_{06} + 1/2$ removes all the gauge bosons from the sectors involving $\mathbf{V}_6$. This reduces the 80 gauge bosons of $E_6\otimes$U(1)$\otimes$U(1), for example, to 48, which can only be the subgroup SO(10)$\otimes$[U(1)]^3, again at level 2.

We could proceed to examine further paths for breaking our original E_8 level 2 model to ever smaller groups, e.g., by modifying $\mathbf{V}_8'$ and **Vg** or some of $\mathbf{V}_2$ through $\mathbf{V}_6$ or by adding other vectors to this set. We have taken this exercise far enough already, however, for our present purposes. Namely we have seen that many gauge groups, including phenomenologically interesting ones such as E_6 and SO(10), can be realized as higher level Kac-Moody algebras in heterotic string models even within the narrow confines of models constructed solely from world-sheet fermions. As promised, features such as N=l supersymmetry and massless adjoint scalar fields peacefully coexist within these models. To progress much further we need to address the shortcomings of the constructions given so far. In ten dimensions the E_8 level 2 model is the unique heterotic string (at least within the known examples) with gauge group realized at higher level. In four dimensions this is not the case. We borrowed the very symmetric and tight construction required in ten dimensions for the four-dimensional example (3.3) mainly as a trick to identify the gauge groups by virtue of their being E_8 subgroups. For larger groups such as E_6, however, this construction is much too confining. (With some additional effort one can show, for example, that 27's and $\overline{27}$'s of E_6 always pair up in these constructions.) On the other hand, for smaller groups the possibilities open up, but our ability to identify the gauge group and representations of the various states in the spectrum decreases.

3.4. Towards Realistic Models: an SO(10) Level 2 Example

In a string model constructed from complex fermions there is never any difficulty in unambiguously determining the gauge group and gauge quantum numbers. of all of the states in the spectrum. In this case the rank of the group is equal to the number of complex fermions making up the bosonic half of the heterotic string and we can choose the operators $\psi_i\psi_i^\dagger$ as the generators of the Cartan subalgebra of the group. The set of fermion charges for the physical states are then in direct correspondence with the weight lattice of the group so that all the group representations may be systematically read off. It is this fact that has allowed practical generation and analysis of such models by computer.[11]

To obtain higher level Kac-Moody algebras we cannot restrict ourselves to complex fermions. In fact we must practically avoid them, since if part of the gauge group is realized with complex fermions then it is at level 1 and a level 1 subgroup implies that the entire group is at level 1. There is an exception to this general rule, however. In the abelian group U(1) there are no structure constants to set the normalization of the current (c.f. eqn. (2.2)) and so the level of a U(1) Mac-Moody algebra has no meaning. Thus the possibility exists that even for a higher level Kac-Moody algebra we can represent the generators of the Cartan subalgebra (which is just a product of U(1)'s) by complex fermion bilinears $\psi_i\psi_i^\dagger$. If this is the case the group representations of physical states can be read off from the collection of fermion charges as before, without any ambiguity.

For ease in comparing with the level 1 realization we will consider higher level SO(10) as an explicit example. Of the popular grand unified groups, SO(10) is the simplest to realize (at level 1) in terms of free fermions; the currents are just the non-vanishing fermion bilinears which can be built from a set of 5 complex fermions. In the space of fermion U(1) charges (normalized so that fermions and antifermions have charges 1 and -1 and the two Ramond ground states have charges $\pm 1/2$), the root vectors are permutations of $(\pm 1, \pm 1, 0, 0, 0)$. In particular a set of simple roots is,

$$(1,-1,0,0,0),(0,1,-1,0,0),(0,0,1,-1,0),(0,0,0,1,1),(0,0,0,1,-1) \qquad (3.5)$$

Our immediate goal is to find an alternative embedding of the root vectors of SO(10), in particular one which does not arise from complex fermion bilinears.

We can take arbitrary boundary conditions for complex fermions, but for simplicity let us confine ourselves to periodic and antiperiodic ones. In this case we can uniformly employ the real fermion formulation summarized in the appendix for all of the degrees of freedom of the string. First note the following. In the level 1 realization above the root vectors all have length squared 2 (corresponding to conformal dimension 1). Suppose we construct a representation in terms of the fermions such that the root vectors (with fermion charges normalized as above) have length squared 1. We could rescale the U(1) generators by $\sqrt{2}$ to make the root vectors of length squared 2 as before, but at a cost of doubling the most singular term in the OPE of any of these U(1) generators with itself. In other words, the Kac-Moody algebra would be realized at $k = 2$ (c.f. (2.2)).

With this in mind, promising choices for the root vectors are some permutations of $(\pm 1,0,0,0,0)$ and $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},0)$. It is not difficult to see, however, that within this five-dimensional space these vectors by themselves are insufficient to reproduce the SO(10) root space (scaled down by a factor of $1/2$ from that given above). Adding a sixth complex fermion makes the construction possible, though. A suitable basis of simple roots, ordered the same as those in (3.5), is given by,

$$
\begin{gathered}
(0,0,0,1,0,0), \quad (\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},0,0), \quad (0,0,1,0,0,0), \\
(0,\tfrac{1}{2},-\tfrac{1}{2},0,-\tfrac{1}{2},\tfrac{1}{2}), \quad (0,\tfrac{1}{2},-\tfrac{1}{2},0,\tfrac{1}{2},-\tfrac{1}{2}) \quad .
\end{gathered}
\tag{3.6}
$$

What we are representing is in fact $SO(10)\otimes U(1)$. The sum of the last two U(1) factors is the extra U(1); the difference is the fifth element of the Cartan subalgebra of SO(10).

In order to embed this construction within a four-dimensional string model we must insure that the roots are part of dimension (0,1) operators and that no other (0,1) operators appear which mix with these, especially ones which would extend the Cartan subalgebra. It is perhaps easiest for explanatory purposes to write down a set of boundary condition vectors which does the required job and then motivate this particular choice. Consider then,

$$
\begin{aligned}
\mathbf{V}_0 &=((\tfrac{1}{2})^2(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^6 \mid (\tfrac{1}{2})^{44}) \\
\mathbf{V}_1 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^6 \mid (0)^{44}) \\
\mathbf{V}_2 &=((0)^{20} \mid (\tfrac{1}{2})^8(0)^4(\tfrac{1}{2})^8(0)^{24}) \\
\mathbf{V}_3 &=((0)^{20} \mid (0)^{16}(\tfrac{1}{2})^8(0)^{20}) \\
\mathbf{V}_4 &=((0)^{20} \mid (\tfrac{1}{2})^2(0)^4(\tfrac{1}{2})^8(0)^2(\tfrac{1}{2})^2(0)^2(\tfrac{1}{2})^2(0)^2(\tfrac{1}{2})^2(0)^{18}) \\
\mathbf{V}_5 &=((\tfrac{1}{2})^2(\tfrac{1}{2}00)^2(0\tfrac{1}{2}0)^4 \mid (\tfrac{1}{2})^4(0)^4(\tfrac{1}{2})^2(0)^2\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0\tfrac{1}{2}0(\tfrac{1}{2})^3(0)^{15}) \\
\mathbf{V}_6 &=((\tfrac{1}{2})^2(0\tfrac{1}{2}0)^2(\tfrac{1}{2}00)^2(00\tfrac{1}{2})^2 \mid (0)^8(\tfrac{1}{2})^4(0)^{17}(\tfrac{1}{2})^4(0)^{11}) \\
\mathbf{V}_7 &=((0)^2(000)^2(\tfrac{1}{2}0\tfrac{1}{2})^4 \mid (\tfrac{1}{2})^8(0)^{36}) \ .
\end{aligned}
\tag{3.7}
$$

The root space spanned by (3.6) is meant to be embedded within the first 12 real fermions of the right hand side of (3.7). The boundary conditions are such that they may be grouped in pairs and so these fermions could equally well be replaced by 6 complex ones. Of the original 66 fermion bilinears formed from these fermions (which generate SO(12)) all but 6, generating $[U(1)]^6$, are projected out of the physical spectrum. The boundary conditions for these first 12 fermions are chosen to reproduce the desired SO(10) group quantum numbers. For example all of the boundary conditions which are required to attempt to reproduce the root space spanned by (3.6) are provided by linear combinations of $\mathbf{V}_2$,$\mathbf{V}_3$, and $\mathbf{V}_4$; the components other than the first 12 on the right are chosen so that the states with-these desired quantum numbers will have conformal dimensions (0,1). The vertex operators for these states will consist of either a fermion or 8 spin operators from the first 12 fermions (giving conformal dimension 1/2) together with 8 spin operators from the next 14 fermions (giving the remaining dimension 1/2). We must also eliminate any dimension (0,1) states from these sectors which are not in the SO(10)$\otimes$U(1) root space. In the sector $\mathbf{V}_3$, for example, we only want states with one of the first 8 fermions on the right excited so $\mathbf{V}_7$ is added for the purpose of singling these components out. The boundary conditions for the 14 fermions after the first 12 insure that they may not be consistently paired up in any way, thus eliminating undesired fermion bilinears which could mix with the SO(10)$\otimes$U(1) gauge bosons.

The specific forms of $\mathbf{V}_5$, $\mathbf{V}_6$, and $\mathbf{V}_7$ were chosen with four other goals in mind. First we want massless fermions transforming as 16's or $\overline{16}$'s of SO(10). T-he highest weight for these representations have Dynkin labels (that is twice the inner product with the simple roots as normalized in (3.6)) of (0,0,0,1,0) and (0,0,0,0,1). These correspond to weight vectors in our chosen basis of simple roots of the form $(\frac{1}{2},\frac{1}{2},0,0,\pm\frac{1}{2},0)$. $\mathbf{V}_5$ provides just the required boundary conditions to embed this weight vector within a massless (i.e., dimension (1/2,1)) fermion state. In addition we would like these states to be chiral. $\mathbf{V}_6$ is chosen so that its nonzero components overlap with those of $\mathbf{V}_5$ only for the two space-time components on the left and the ninth and tenth fermions (corresponding to the U(1) charge in SO(10) distinguishing 16's from $\overline{16}$'s) on the right. Thus the constraint from $\mathbf{V}_6$ on the states in sector $\mathbf{V}_5$ will correlate the space-time helicity with the group quantum numbers making chiral states possible. A massless scalar field in the adjoint of SO(10) is another feature on our list. The Dynkin label is (0,1,0,0,0) corresponding to the highest weight vector (1,0,0,0,0,0) in the basis (3.6). This can arise from the sector $\mathbf{V}_2+\mathbf{V}_7$. Finally, the left hand sides of $\mathbf{V}_5$, $\mathbf{V}_6$, and $\mathbf{V}_7$ are such that the supersymmetry is reduced to N=1.

After making a list of desired features and finding embeddings for the roots and weights which achieve them, it was relatively quick and straight forward to write down the set of vectors (3.7) which plausibly realize these goals. It is considerably more lengthy and tedious (but ultimately necessary) to examine the model in some detail to insure that choices for the k_{ij}'s exist which make all of the desired projections compatible. In addition we must check that no linear combinations of $\mathbf{V}_0$ through $\mathbf{V}_7$ provide massless states which are unpleasant surprises. We will sketch the results for (3.7), which prove to be quite satisfactory.

The sector 0 contributes the 6 U(1) gauge bosons of the Cartan subalgebra of SO(10)⊗U(1) and in addition generators of SU(2)⊗SO(4)⊗SO(11). The SU(2) is realized at level 2. In the summary of the model below we will ignore those states which are singlets under SO(10). Checking the $\mathbf{V}_1$ sector we have N=1 space-time supersymmetry if $k_{21} = k_{31} = k_{41} = 0$ and $k_{51} + k_{71} = k_{01}$, otherwise N=0. The sector $\mathbf{V}_3$ contributes 8 gauge bosons corresponding to exciting one of the first 8 fermions on the right hand side if $k_{73} = 1/2$ and $k_{63} = 0$. Provided in addition that $k_{62} = 0$ the sectors $\mathbf{V}_2$ and $\mathbf{V}_2+\mathbf{V}_3$ contribute 8 gauge bosons each. Together with the 8 above and the first 4 U(1)'s these generate an SO(8) subgroup of our eventual SO(10), as can be seen explicitly from their U(1)charges. The sectors $\mathbf{V}_4$, $\mathbf{V}_4+\mathbf{V}_2$, $\mathbf{V}_4+\mathbf{V}_3$, and $\mathbf{V}_4+\mathbf{V}_2+\mathbf{V}_3$ each contribute 4 more gauge bosons to complete the adjoint of SO(10)⊗U(1). If $k_{64} = 1/2$ then the simple roots are precisely in the basis given in (3.6). If $k_{64} = 0$ then the group is the same but now the sum of the final two U(1)'s is in the Cartan subalgebra of SO(10) and the difference is the left over U(1).

The 8 sectors $\mathbf{V}_5+\{\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\}$ i.e., $\mathbf{V}_5$ plus any combinations of $\mathbf{V}_2$, $\mathbf{V}_3$ and

$\mathbf{V}_4$, contribute four families of massless chiral fermions in the SO(10)⊗SU(2)⊗ SO(4)⊗SO(11) representation (16,2,1,1). Whether the space-time left handed fermions are 16's or $\overline{16}$'s depends on the value of k_{65}. The 8 sectors $\mathbf{V}_7+\{\mathbf{V}_2,\mathbf{V}_3, \mathbf{V}_4\}$ contribute four families of massless scalar fields in the adjoint of SO(10)⊗U(1), and some additional scalars in the representation (10,1,1,1). The $\mathbf{0}+\{\mathbf{V}_2,\mathbf{V}_3, \mathbf{V}_4\}$ sectors together also contribute scalars in these representations. In both cases the adjoint is realized in a different way from the SO(10) currents. For example in the latter case, the states in the Cartan subalgebra of the adjoint representation arise from the $\mathbf{V}_3$ sector for the scalars but the 0 sector for the Kac-Moody currents. The 8 sectors $\mathbf{V}_6+\{\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\}$ and the 8 sectors obtained by adding $\mathbf{V}_3$ to these, both contribute massless fermions in the representations (10,1,2,1). The only other sectors contributing massless states are obtained by adding $\mathbf{V}_1$ to those considered above. Not surprisingly, these give the super-partners of the states from those sectors.

The vectors in (3.7) can be modified somewhat or new vectors added without destroying the features we have built into the-model. We will effect only one small improvement here, breaking the "horizontal" symmetry group SU(2) and reducing the number of chiral families of SO(10). Add the vector,

$$\mathbf{V}_8 = ((0)^2(\tfrac{1}{2}\tfrac{1}{2}0)(\tfrac{1}{2}0\tfrac{1}{2})^3(000)^2 \mid (0)^{27}(\tfrac{1}{2})^5 0(\tfrac{1}{2})^3(0)^8) \tag{3.8}$$

to the set (3.7). This has three significant effects on the model discussed above. First-the SU(2)⊗SO(4)⊗SO(11) factor in the gauge group is broken to U(1)⊗SU(2) ⊗SU(2)⊗SO(8). Second the number of chiral fermion states found in the sectors considered above is cut in half to four families of (16,1,1,1) of SO(10)⊗SU(2)⊗SU(2) ⊗ SO(8). Finally, the adjoint scalars from the $\mathbf{0}+\{\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\}$ sectors are projected out, while two of the four copies of adjoint scalars from the $\mathbf{V}_7+\{\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4\}$ sectors remain.

To summarize: The model given by (3.7) and (3.8) includes gauge group SO(10) realized as a level 2 Kac-Moody algebra, N=l space-time supersymmetry, 4 chiral families of SO(10) and massless scalar fields in the adjoint of SO(10).

4. Further Possibilities

The examples of the previous section represent first attempts at embedding higher level Kac-Moody algebras within heterotic string models. The approach taken was neither exhaustive nor very systematic so the possibilities for such models remain largely unexplored. A brief mention of the approaches which might prove useful in this endeavor is in order.

First consider the fermionic constructions of the previous section. At this stage even the question of which gauge groups may be realized and at what levels remains unanswered except that the possibilities are many. Note that the theorem of ref. [26] concerning which Kac-Moody currents can be realized in terms of fermion bilinears is not relevant for our purposes here. First, because the currents need not be realized as bilinears (as we have seen) and second because finding a representation of the currents is a far cry from finding a complete, modular invariant theory. It would be extremely useful if we had powerful and systematic methods for discerning the group structure of the most general real fermion constructions. This would greatly enlarge the practical possibilities for model building. In the meantime much more can be done with the trick used for the SO(10) model above, namely representing the Cartan subalgebra with complex fermions. In particular, arbitrary boundary conditions can be chosen for the complex fermions, allowing for a broader range of roots and weights to be realized. Note that while we focused on the spinor and adjoint representations in the examples above, massless states in other representations (e.g., 54 or 126 of SO(10)) should be equally possible to realize.

We chose the fermionic construction for the examples of section 3 principally because the explicit operator representation available for free fermions allows us to systematically construct asymmetric modular invariants and successfully address the issues of crossing symmetry of amplitudes and multi-loop modular invariance. Another possibility which shares some of these advantages is constructing heterotic string models as asymmetric orbifolds.[8] Higher level Kac-Moody algebras can certainly be achieved in this fashion. For example, modding out a level 1 group $G \otimes G$ by the outer automorphism interchanging the two G's leaves the diagonal subgroup, G at level 2. Presumably these include some models which can't be realized with fermions. On the other hand multi-loop modular invariance is not guaranteed and is difficult to analyze within this construction. In addition, insuring that an asymmetric twist can be made consistent with one-loop modular invariance in general requires a detailed knowledge of the effect of the twist on the particular lattice realization of the model we start with. For phenomenologically interesting cases the models we wish to twist asymmetrically are already quite intricate, which makes this construction awkward to apply in practice. Perhaps the "twisted boson" formalism of ref. [10] will prove a more systematic way of constructing these models.*

Ideally the most general and in some sense transparent possibility would be to construct heterotic string models directly from tensor products of Kac-Moody characters at different levels, perhaps together with tensor products of assorted minimal models. We know from experience that an enormous and varied collection of consistent models can be obtained from tensor products of Ising models (i.e., free fermions) of which only a tiny and rather uninteresting subset are based on left-right symmetric modular invariants. It is reasonable to expect proportionally richer possibilities for left-right asymmetric tensor products of Kac-Moody characters and minimal models. In these latter cases, however, we lack an explicit operator representation (the analog of free fermions) and so have not succeeded to date in systematically constructing consistent theories. Some new methods must be developed before systematic construction of such models becomes practical. To cite one possibility currently under investigation: To some extent it is possible to

* Note: In the second paper of ref. [10] the authors claim to have constructed a nonsupersymmetric model with chiral fermions and massless adjoint scalar fields. As the gauge group appears to be at level 1 and, moreover, the adjoint scalars are built from the Kac-Moody currents themselves, this contradicts the general arguments given in sect. 2. We suspect that there is some problem with this model but have not familiarized ourselves sufficiently with the formalism to locate it.

address the questions of crossing symmetry and modular invariance (at least at the one loop level) just in terms of the most readily accessible quantities in Kac-Moody theories or minimal models, that is the conformal dimensions and fusion rules. Appropriately matching up these quantities for the holomorphic and antiholomorphic theories being sewn together it is possible, with only modest additional information, to construct non-trivial asymmetric modular invariants,[27] though not as yet in any systematic fashion.

5. Summary and Conclusions

The most important problem in superstring theory, if we wish to treat it seriously as a theory of everything, is the identification of the underlying principles of string dynamics which perhaps pick one vacuum state- that in which we live— from out of the multitude of classical string vacua which we know to exist. Given this situation, constructing particular new string models is of use only if they incorporate considerably different low energy phenomenology than the classes of models previously studied, or if they teach us something about the general properties of string vacua and the space in which string dynamics operates.

The constructions we have given here satisfy both of these criteria. Heterotic string models in which the gauge group is realized as a higher level ($k \geq 2$) Kac-Moody algebra-on the string world-sheet can incorporate particles in group representations not present if $k = 1$, thereby opening new possibilities for string phenomenology. Such models have not appeared in the past for largely technical reasons- namely the difficulty in insuring modular invariance in left-right asymmetric constructions not built from free bosons. In the numerous examples given here, this hurdle was surmounted within a construction employing free real fermions. The most realistic model presented includes gauge group SO(10) realized at level 2, N=l space-time supersymmetry, four generations of chiral fermions, and massless scalar fields in the adjoint of SO(10) which could serve to break the symmetry spontaneously to the standard model gauge group. The appearance of massless adjoint scalars together with N=l supersymmetry or chiral fermions is not possible if the gauge group is realized at level 1. Thus heterotic string models incorporating higher level Kac-Moody algebras open the possibilities for embedding standard GUT's within string models, and remove much of the string motivation behind models such as flipped SU(5)⊗U(1) whichcan break to the standard model without adjoint Higg's fields.

It should be emphasized that our main purpose in the present work is not to promote the virtues of models whose gauge symmetry at the Planck scale is a grand unified group rather than the standard model one. Indeed the latter case can also be realized via a higher level Kac-Moody algebra with new phenomenological features which should be explored. Rather we wish to emphasize the broader point that the possibilities for string vacua remain largely unexplored and those considered to date likely do not even constitute a representative sample. In particular, we should not focus solely on compactifications (i.e., basically left-right symmetric modifications) of the ten-dimensional heterotic string in trying to deduce general phenomenological features of string models or in specifying the parameter space for string dynamics. It may turn out, once the necessary technical hurdles are overcome, that it is left-right asymmetric conformal field theories which are in fact the basis for the bulk of string vacua.

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APPENDIX

In this appendix we summarize the necessary formulae and results needed for constructing consistent string models out of free world-sheet fermions. The focus will be squarely on the case of four-dimensional, heterotic string models in light-cone gauge, built from free real fermions with an $[SU(2)]^6$ symmetric form for the world-sheet supercurrent. The formalism for more general cases- type II or bosonic strings, mixed real and complex fermions (with arbitrary boundary conditions), other forms for the supercurrent, etc.- are all similar and given in the literature.

The particle content of a string model is most succinctly given by its one-loop vacuum amplitude. In a mathematically consistent and physically sensible theory this object must be a modular invariant function of the complex modular parameter τ and behave like a physical partition function. In the present case the general solution for the internal degrees of freedom can be written,

$$Z(\tau,\bar{\tau}) = \sum_{\alpha_j,\beta_i=0}^{1} exp[2\pi i \sum(\alpha_j V_j^1 + \beta_i(k_{ij}\alpha_j + V_i^1 - \tfrac{1}{2}\mathbf{V}_i \cdot \overline{\alpha\mathbf{V}}))] \cdot \\ Tr[exp(2\pi i(\tau H_{\alpha V} - \bar{\tau}\tilde{H}_{\alpha V}))exp(-2\pi i\beta\mathbf{V}\cdot\mathbf{N}_{\alpha V})] \quad , \tag{A.1}$$

subject to,

$$k_{ij}, V_i^l \in \{0, \frac{1}{2}\}$$

$$k_{ij} + k_{ji} = \frac{1}{2}\mathbf{V}_i \cdot \mathbf{V}_j \qquad (mod\ 1) \tag{A.2}$$

$$k_{ii} + k_{i0} + V_i^1 - \frac{1}{4}\mathbf{V}_i \cdot \mathbf{V}_i = 0 \qquad (mod\ 1) \tag{A.3}$$

$$4\sum_{l=1}^{64} V_i^l V_j^l V_k^l = 0 \qquad (mod\ 1) \qquad \forall \quad i,j,k \tag{A.4}$$

The chief inputs defining a given model are Lorentzian vectors, Vi, of dimensions (20,44) which define the boundary conditions on the 20 real fermions in the supersymmetric half of the string and 44 real fermions in the bosonic half. The components of the $\mathbf{V}_i$ take values 0 or $\frac{1}{2}$ corresponding to anti-periodic or periodic boundary conditions, respectively. All dot products involving these vectors are defined with Lorentzian signature. The set of vectors $\{\mathbf{V}_i\}$ appearing in (A.l) are chosen to be linearly independent and must include the vector with all components $\frac{1}{2}$, denoted $\mathbf{V}_0$. In addition, the boundary conditions must be chosen consistent with the world-sheet supersymmetry of the supersymmetric half of the heterotic string. For the purposes of this work, the $[SU(2)]^6$ symmetric realization of the world-sheet supercurrent in terms of fermions proves by far the most useful in that it permits the most freedom in choosing different periodic and anti-periodic boundary conditions. To be consistent with this form of the supercurrent all of the Vi's must satisfy the "triplet" constraint[4,5], i.e., be of the form,

$$\begin{gathered}(aa(A_1B_1C_1)(A_2B_2C_2)\dots(A_6B_6C_6)|\dots\dots) \\ \text{with } A_i + B_i + C_i = a \qquad (mod\ \ 1) \ .\end{gathered} \tag{A.5}$$

The sum over different values for the α_j in (A.l) gives all sets of boundary conditions around the closed string generated by the linear combinations of {Vi}. This splits the partition function into different sectors labeled by the vectors $\alpha\mathbf{V}$, with Hamiltonians $H_{\alpha V}$ for the bosonic half of the string, $\tilde{H}_{\alpha V}$ for the supersymmetric half,. and a vector of number operators, $\mathbf{N}_{\alpha V}$, for the 64 fermions. The overbar indicates that only the fractional part of the linear combinations, $\alpha\mathbf{V} \equiv \sum_i \alpha_i\mathbf{V}_i$, of components is kept in defining $\alpha\mathbf{V}$. The sum over different values of the β_i generates all sets of boundary conditions $\beta\mathbf{V}$ around the other non-contractible loop of the world-sheet torus. This serves to project some of the world-sheet states out of the physical spectrum. The precise projections depend on the values chosen for the set of parameters k_{ij}, and the definition of the exponential of fermion number operators appearing in (A.l). The latter is straight forward except for the presence of fermion zero modes for which the exponential of the number operator is replaced by a product of zero mode operators, γ^l, which, properly normalized, satisfy the commutation relations of gamma matrices. A consistent and technically convenient definition is[15] ($\mathbf{N}'_{\alpha V}$ denotes the vector of number operators for the non-zero modes),

$$
\begin{aligned}
e^{-2\pi i\beta \mathbf{V}\cdot\mathbf{N}_{\alpha V}} &\equiv \Gamma^{\alpha\mathbf{V}}_{\overline{\beta\mathbf{V}}} e^{-2\pi i\beta\mathbf{V}\cdot\mathbf{N}'_{\alpha V}} \\
\Gamma^{\alpha\mathbf{V}}_{\overline{\beta\mathbf{V}}} &\equiv \prod_j (\Gamma^{\alpha V}_j)^{\beta_j} \\
\Gamma^{\alpha V}_j &\equiv \prod_i (g^i_j)^{\alpha_i}
\end{aligned}
\tag{A.6}
$$

$$
g^i_j \equiv (i)^{\frac{n}{2}}\gamma^{l_1}\gamma^{l_2}\dots\gamma^{l_n} \quad \text{with } l_1 < l_2 < \dots l_n \text{ and } V^{l_k}_i = V^{l_k}_j = \frac{1}{2} .
$$

The states surviving the projections in (A.l) are then the ones satisfying,

$$
[\mathbf{V}_i\cdot\mathbf{N}'_{\alpha V} + \frac{1}{4}(1-\Gamma^{\alpha V}_i)]\mid phys\rangle = (\sum_j k_{ij}\alpha_j + V^1_i - \frac{1}{2}\mathbf{V}_i\cdot\overline{\alpha\mathbf{V}})\mid phys\rangle \quad (mod\ 1).
\tag{A.7}
$$

The massless physical states — which are the only relevant ones for low energy phenomenology- are those whose vertex operators have conformal dimensions $(\frac{1}{2},1)$ in addition to satisfying (A.7). The operators creating the ground state of the $\alpha\mathbf{V}$ sector from the 0 sector (i.e., the Neveu-Schwarz sector, which has dimensions $(0,0)$) have conformal dimensions $(\frac{1}{2}\sum_{i=1}^{20}(\overline{\alpha V^i})^2, \frac{1}{2}\sum_{i=21}^{64}(\overline{\alpha V^i})^2)$. Each Neveu-Schwarz fermion contributes conformal dimension $\frac{1}{2}$.

Before discussing practical considerations of string model building a few words about multi-loop modular invariance are in order. Any modular transformation on a higher genus Riemann surface can be generated by a sequence of Dehn twists about the non-contractible loops on the surface. The most general type of Dehn twist appears already at genus 2 so it is sufficient to consider two-loop modular invariance. The determinants for complex fermions with arbitrary boundary conditions on higher genus Riemann surfaces are known, along with thier behavior under modular transformations.[28] Four independent sets of boundary conditions appear in a two-loop amplitude. If the boundary condition vectors satisfy an additional "quartic" constraint,

$$\sum 8V_i^l V_j^l V_k^l V_n^l = 0 \qquad (mod\ 1) \qquad \forall \ \ i,j,\ k,n \ , \qquad (A.8)$$

then any term with fixed boundary conditions contributing to the two-loop amplitude can be expressed in terms of complex fermions and hence its behavior under modular transformations determined explicitly. Using this fact it has been shown directly in these cases that the string models constructed according to the equations above are multi-loop modular invariant.[7,15] Note also that if (A.8) is satisfied then all of the operators in eqn.(A.6) commute and so no subtleties arise in the analysis due to operator ordering.

If eqn.(A.8) is not satisfied then we can deduce the modular behavior of the two-loop amplitude from the corresponding complex fermion calculation only up to a sign, and the number operators defined in eqn.(A.6) for different sectors of -the theory need not commute. Failure to appreciate these subtleties led at one time to the conclusion that the quartic constraint (A.8) is required for a consistent theory,[5,7] but this proves not to be the case. A careful account of the operator structure showed that eqns.(A.1)-(A.4) are sufficient for one-loop consistency.[15] In more recent work, Sonoda[21] has shown that in any conformal field theory multi-loop consistency is guaranteed if all four-point tree amplitudes in the theory are crossing symmetric, and all one-point one-loop amplitudes behave correctly under modular transformations. The sufficiency of the constraints (A.2)-(A.4) for multi-loop consistency can be infered from this result. Any four-point tree amplitude within a string model given by eqn. (A.l) involves only three independent boundary condition vectors $\alpha\mathbf{V}$ (the fourth is the sum of the other three) and so it is not difficult to see that the "cubic" constraint (A.4) is sufficient to guarantee the crossing symmetry of any four-point amplitude. That one-point one-loop amplitudes in these models behave correctly under modular transformations was implicitly shown in the analysis of ref. [15]. The modular invariance of the partition function (A.1) was examined at the operator level, and holds fermion by fermion except for over all phases which cancel once all of the fermions are considered together. Treated simply as a function of τ all the contributions to the partition function from any sectors with Ramond fermions identically vanish. Thus to derive nontrivial results one must in fact consider (A.1) at an operator level, allowing for operator insertions which can give a non-vanishing result. In ref. [15] this was achieved (in slightly roundabout fashion) by considering particular two-loop amplitudes in the limit where they factor into products of one-point one-loop amplitudes.

-The practical construction and analysis of string models using the formalism sketched here is much simpler than the equations given above might imply. Having already guaranteed modular invariance we need not consider the partition function any further. In fact, for low energy phenomenology we need only consider the physical state projection conditions (A.7) for a small subset of the sectors of the theory- those which can contribute massless states. The most efficient approach to constructing a model with certain desired features is the following. First, specify a self consistent set of vectors $\mathbf{V}_i$ ignoring the k_{ij}'s. Multiplying the constraints (A.2) and (A.3) by 2 to eliminate the k_{ij} dependence we find three simple conditions from (A.2)-(A.4) which must be satisfied in addition to the triplet constraint (A.5): 1) the number of non-zero components (counting left minus right) of each $\mathbf{V}_i$ is a multiple of 8; 2) the number of non-zero components (counting left minus right) common to any two vectors $\mathbf{V}_i$ and $\mathbf{V}_j$ is a multiple of 4; and 3) the number of non-zero components common to any three vectors is a multiple of 2. The first condition is necessary for modular invariance under $\tau \rightarrow \tau + 1$ (level matching) and the other two guarantee that the first condition is satisfied for any linear combination of the vectors {Vi}.

The vectors $\mathbf{V}_i$ are best chosen in two stages. First, include vectors which generate all sectors of the forms required to produce the specific particles desired. For example, massless states can arise only from vectors $\alpha\mathbf{V}$ with 0 or 8 $\frac{1}{2}$'s in the first 20 components and 0, 8 or 16 $\frac{1}{2}$'s in the last 44. The first two components (V_i^1 and V_i^2) correspond to the space-time degrees of freedom; the values 0 and $\frac{1}{2}$ indicating space-time bosons and fermions, respectively. The unique sector, up to reordering of components, which can contribute space-time gravitinos is,

$$\mathbf{V}_1 = \left(\tfrac{1}{2}\tfrac{1}{2}(\tfrac{1}{2}00)^6 \mid (0)^{44}\right) .$$

Massless vector bosons require vectors whose first 20 components vanish. The form of these vectors give a good idea of what sort of gauge group can be realized and, in turn, what sort of vectors should be included to give massless fermions or scalars inspecific representations. Having chosen vectors which plausibly provide the particles desired, one should then add vectors (consistent with those already chosen) which can project out states which we do not want and, at the same time, not contribute unwanted particles in the new sectors added. For example all the gravitinos but one can be projected out to guarantee N=l space-time supersymmetry, or a vector can be added which correlates the space-time and gauge degrees of freedom, thereby permitting chiral representations. The number of sectors grows exponentially with the number of-vectors. Accordingly, a ten or twenty line computer program to check for self consistency and identify sectors which can contribute massless states is highly recommended.

After writing down a promising set of vectors {V;} the important sectors must be considered in some detail to determine if values for the k_{ij}'s exist which give the desired projections (c.f.,(A.7)). The parameters k_{ij} for i > j are independent; the others are fixed by the constraints (A.2) and (A.3). In a given model only a few of these parameters significantly affect the physical spectrum. The redundancy of the other k_{ij} (many need not be specified at all) is due to the high degree of symmetry in the fermionic construction.[29] In most cases we can ignore the precise γ orderings in eqn. (A.7) because the resulting minus signs can be absorbed into different choices for the redundant k_{ij}'s. With some practice the important features of even very complicated models may be determined with a minimal amount of tedious analysis.

REFERENCES

1. P. Candelas, G. Horowitz, A. Strominger, and E. Witten, *Nucl.Phys.* **B258** (1985), 46

2. L. Dixon, J. Harvey, C. Vafa, and E. Witten, *Nucl.Phys.* **B261** (1985), 651 and, *Nucl.Phys.* B274 (1986), 285 ; L. Ibáñez, H. Nilles, F. Quevedo, *Phys.Lett.* **187B** (1987), 25.

3. K. Narain, *Phys.Lett.* **169B** (1986), 41.

4. H. Kawai, D.C. Lewellen, S.-H.H. Tye, *Phys.Rev.Lett.* 57 (1986), 1832.

5. H. Kawai, D.C. Lewellen, S.-H.H. Tye, *Nucl.Phys.* **B288** (1987), 1.

6. W. Lerche, D. Lüst, A.N. Schellekens, *Nucl.Phys.* B287 (1987), 477; L. Lerche, A.N. Schellekens, and N.P. Warner, *Phys.Rep.* **177** (1989), 1.

7. I. Antoniadis, C. Bachas, C. Kounnas, *Nucl.Phys.* **B289** (1987), 87; I. Antoniadis, C. Bachas, *Nucl.Phys.* *I3298* (1987), 586.

8. K. Narain, M. Sarmadi, C. Vafa, *Nucl.Phys.* **B288** (1987), 551.

9. D. Gepner, *Nucl.Phys.* **B296** (1988), 757, *Phys.Lett.* **199B** (1987), 380.

10. A.H. Chamseddine, J.P. Derendinger, *Nucl.Phys.* **B301** (1988), 381; A.H. Chamseddine, J.P. Derendinger, M. Quiros, *Nucl.Phys.* **B311** (1988), 140.

11. D. Sénéchal, *Phys.Rev.* **D39** (1989), 3717.

12. see e.g., P. Langacker, *Phys.Rep.* 72 (1981), 185.

13. S.M. Barr, *Phys.Lett.* **112B** (1982), 219; J.P. Derendinger, J.H. Kim, D.V. Nanopoulos, *Phys.Lett.* **139B** (1984), 170; I. Antoniadis, J. Ellis , J. Hagelin, D.V. Nanopoulos, *Phys.Lett.* **194B** (1987), 231.

14. D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm, *Phys.Rev.Lett.* **151B** (1985), 21.

15. H. Kawai, D.C. Lewellen, J.A. Schwartz, S.-H.H. Tye, *Nucl.Phys.* **B299** (1988), 431.

16. H. Dreiner, J.L. Lopez, D.V. Nanopolous, D.B. Reiss, *Phys.Lett.* **B216** (1989), 283; preprint MAD/TH/89-2 (1989).

17. P. Goddard and D. Olive, *Int.J.Mod.Phys.* **Al** (1986), 303.

18. W. McKay and J. Patera, *Tables of Dimensions, Indices and Branching Rules for Representations of Simple Algebras* (Dekker, New York, 1981).

19. D. Gepner, E. Witten, *Nucl.Phys.* B278 (1986), 493.

20. L. Dixon, V. Kaplunovsky, C. Vafa, *Nucl.Phys.* **B294** (1987), 43.

21. H. Sonoda, *Nucl.Phys.* **B311** (1988), 417.

22. P. Ginsparg, *Phys.Lett.* **197B** (1987), 139.

23. L. Dixon, J.A. Harvey, *Nucl.Phys.* **B274** (1986), 93.

24. H. Kawai, D.C. Lewellen, S.H.H. Tye, *Phys.Rev.* **D34** (1986), 3794.

25. P. Forgacs, Z. Horvath, L. Palla, P. Vecsernyes, *Nucl.Phys.* **B308** (1988), 477.

26. P. Goddard, W. Nahm, D. Olive, *Phys.Lett.* **160B** (1985), 111.

27. D.C. Lewellen, work in progress.

28. L. Alvarez-Gaume,G. Moore, and C. Vafa, *Comm.Math.Phys.* **106** (1986), 1

29. D.C. Lewellen, PhD. thesis.

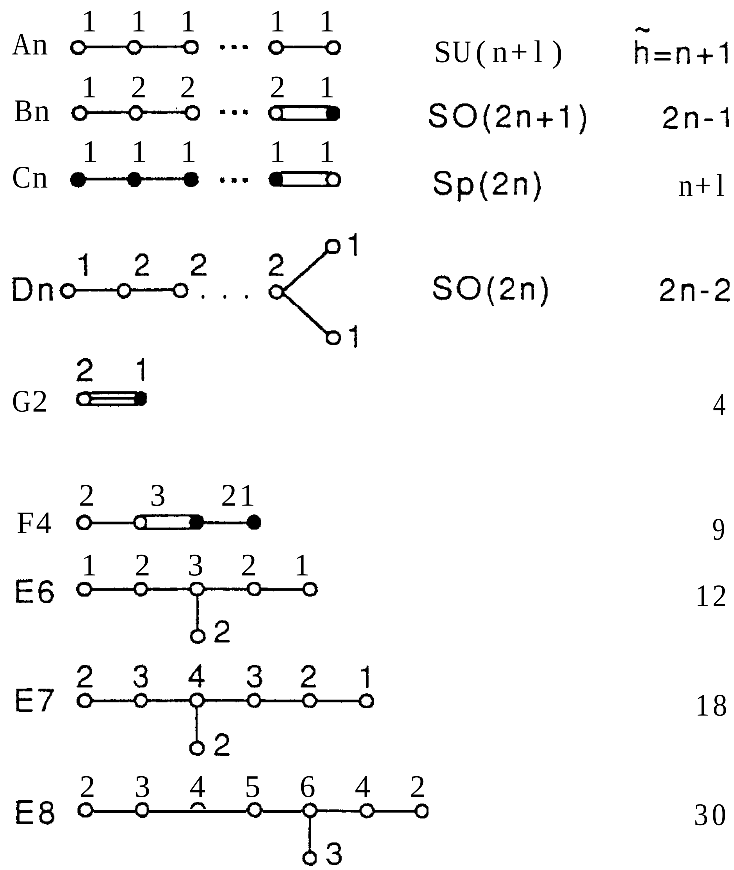

Table 1. Dynkin diagrams for the simple Lie algebras. The simple roots are labeled by the co-marks, m_i, used in eqn.(2.3) and $\tilde{h}$ is the dual Coxeter number of the algebra.