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TREE LEVEL CONSTRAINTS ON CONFORMAL FIELD THEORIES AND STRING MODELS*

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ABSTRACT

Simple tree level constraints for conformal field theories which follow from the requirement of crossing symmetry of four-point amplitudes are presented, and their utility for probing general properties of string models is briefly illustrated and discussed.

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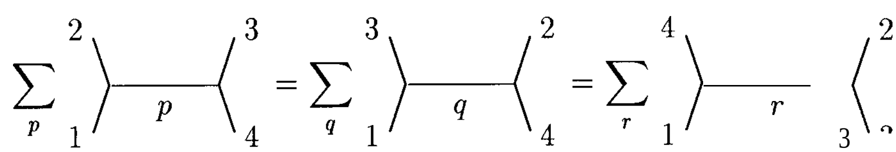
In our current understanding, a string model, or more precisely a classical solution to the string equations of motion which may serve as a vacuum for perturbation theory, consists of familiar spacetime machinery (string coordinates, RNS fermions, reparametrization ghosts, etc.) together with a two-dimensional conformal field theory (CFT) which describes the string's "internal" degrees of freedom. The possible choices of CFT's (and correspondingly string models) are myriad and largely unknown, with no prospects for a complete classification in the near future. Given this situation, it makes sense to examine general properties of CFT's and determine if these imply any general features of the space-time physics described by the string. For example, the existence of certain holomorphic algebras in a CFT (Virasoro, Kac-Moody, super-conformal, etc.) are directly related to the existence of space-time symmetries (gauge, SUSY, etc.). The proof by Dixon, Kaplunovsky and Vafa that the standard model cannot be embedded within a generalized type II string [1] exemplifies the use of such relations to constrain the possible physics obtainable in string models.

There are other highly constraining general properties of CFT's whose impact on space-time physics is much more difficult to determine. In particular a consistent CFT must have modular invariant amplitudes. It is this requirement, for example, which restricts the possible gauge groups of the lo-dimensional heterotic string to only a small handful. The consequences of modular invariance can only be studied systematically, however, within simple classes of string models, for example those constructed out of free world-sheet bosons or fermions, symmetric orbifolds, symmetric collections of minimal models, etc. This is because one loop modular invariance involves all of the string states, not just the massless (or small mass) states of interest for low energy phenomenology. Moreover, it has not proven possible to implement these conditions at an operator level; thus a complete model or class of models must be constructed before connections with space-time physics can really be made.

As an alternative to directly studying modular invariance in the hope of deducing general properties of string models, I would like to advocate examining another necessary and highly constraining requirement of a consistent CFT, that of crossing symmetry (duality) of tree amplitudes. Unlike modular invariance, crossing symmetry of a given amplitude involves only a subset of the fields in the CFT and may be analyzed directly at the operator level. A complete treatment requires more detailed knowledge of a CFT than we would like to assume for finding general properties of string theories. Instead, I'll present simple constraints derived from crossing symmetry which only involve the most immediate quantities of interest in a CFT, namely the conformal dimensions of the primary fields, and their fusion rule algebra [2,3] (i.e., the knowledge of which fields couple to which). The derivation of these relations has been given elsewhere [4] (in pedagogical fashion with examples and related work); Here I'll confine myself to presenting the constraints and briefly discussing their utility and applicability.

Any four-point function of primary fields on the plane, $\langle\phi_1\phi_2\phi_3\phi_4\rangle$, may be factorized into sums of products of three-point functions in three different ways; crossing symmetry is the requirement that these three channels give the same result [2]. Diagrammatically we require,



(1)

This-gives rise in particular to the following simpler constraint [4,5],

$$A4 = N\left(\sum_{j=1}^{4} \Delta_j\right) + N(N-1)/2 - \sum_{i=1}^{N}(d^i_{12} + d^i_{13} + d^i_{14}) \quad ; \quad M \in Z \geq 0 \quad . \tag{2}$$

Here Δ_j are the dimensions of the external states, d^i_{12}, d^i_{13}, d^i_{14} the dimensions of primary fields appearing in the 12, 13 and 14 channels respectively, and N is the number of primaries (of the full holomorphic algebra of the CFT) which appear in each of the three channels (crossing symmetry requires that there be the same number). Knowledge of the fusion rule algebra is implicit in determining which primary states appear as intermediates. Fusion rule coefficients, N_{ijk}, [3] which are > 1, or distinct primaries appearing in a given amplitude which have the same conformal dimensions modulo integers, both lead to stronger versions of eqn.(2) (see [4] for details). The analogous relations for higher point amplitudes are easily derived but give no new information beyond the constraints from all nonvanishing four-point amplitudes in the theory. The less stringent conditions corresponding to eqn.(2) with M allowed to be any (possibly negative) integer, correspond to those found previously by Vafa [6], and can be deduced from the polynomial constraints of Moore and Seiberg[7]; These approaches, which focus on the modular properties of amplitudes, quite generally give no information about the integer part of the conformal dimensions.

The virtue of the relations given in eqn.(2) are their simplicity and dependence only on the most immediately accessible objects in a CFT, the dimensions and fusion rules. As such, they prove to be of great practical use for constraining these quantities in any conformal field theory with a finite number of primary fields appearing in some four-point amplitudes (the theory need not be restricted to a finite total number of primary fields in order for eqn.(2) to apply). For example: we know the dimensions of the primary fields in many CFT's for which we have only partial knowledge of the fusion rules. The complete fusion rules can in principle be found using the differential equations which follow as a consequence of whatever holomorphic algebra is present, or by explicitly evaluating the modular transformation properties of the characters and using Verlinde's relation [3]. In practice, however, it proves much simpler to apply Vafa's conditions or their generalizations given in eqn.(2) for some convenient four-point amplitudes and thereby determine the allowed fusion rules for the given conformal dimensions. For example, the SU(3) level 2 WZW model has six primary fields corresponding to the representations I, $3, \bar{3}, 6, \bar{6}, 8$ with dimensions 0, $4/15$, $4/15$, $2/3$, $2/3$, $3/5$, respectively. Finite group theory limits the possible fusion rules somewhat, e.g., a 6 and a $\bar{6}$ might fuse to a I and/or 8 as might two 8's. Demanding crossing symmetry (e.g., in the form given in eqn.(2)) for the amplitudes (6666) and (8888) one easily finds that the only consistent possibility is $6 \cdot \bar{6} \sim I$ and $8 \cdot 8 \sim I + 8$.

To what extent can we employ crossing symmetry to probe the general structure of string vacua? One approach is to solve a class of string models, identify some interesting general features of this class, and then use constraints such as eqn.(2) to see to what degree these results can be extended to models constructed from general CFT's. To illustrate, consider the case of four-dimensional heterotic string models in light cone gauge constructed solely from free world-sheet bosons. The one loop partition function for the bosonized RNS fermions and internal degrees of freedom is [8],

$$Z_{int}(\tau,\bar{\tau}) = \eta^{-22}\bar{\eta}^{-10} \sum_{\mathbf{Q}\in\Gamma-\mathbf{S}} exp(i\pi\tau\mathbf{Q}_L^2 \ - i\pi\bar{\tau}\mathbf{Q}_R^2 \ + 2\pi i\mathbf{Q}\cdot\mathbf{S})\ .$$

Here, Q = $(\mathbf{Q}_L \,|\, \mathbf{Q}_R)$ and S = (1, 0, . . . 0); modular invariance requires that $\mathbf{Q}^2$ be an odd integer and that Γ be a Lorentzian, self-dual lattice of dimension (10,22). The first component of each lattice vector is restricted to take integer or half integer values (since it represents the two bosonized RNS fermions in the light-cone gauge in 4-dimensions); by self-duality this is equivalent to $2\mathbf{S} \in \Gamma$.

There are additional constraints on Γ from world-sheet supersymmetry , but for simplicity I'll consider here only two simple features of these models which don't depend on these details and which will serve to illustrate possible uses of eqn.(2). First note that the existence of a vector in Γ of the form (0, k | k, 0.. .O) with $\mathbf{k}^2 << 1$ gives rise not only to a small mass space-time tensor state but to an entire tower of such states because all multiples of this vector appear in Γ. In other words, within this class of models the existence of a small mass tensor state necessarily implies decompactification at that energy scale. Second, consider the effects of including a vector $\mathbf{g}$ = (1, a | 0,. . . 0) $\in \Gamma$ with $\mathbf{a}^2$ = 1. This leads directly to gauge bosons coming from the world-sheet supersymmetric half of the heterotic string (the space-time vector index is contributed by exciting one of the string coordinates in the bosonic sector). It also necessarily means that the resulting model is non-chiral. A massless fermion of positive helicity arises from a lattice vector in Γ of the form $f_+ = (3/2, b \mid d)$ with $b^2 = 3/4, d^2 = 2$ and $a \cdot b = \pm 1/2$ (Γ is integral so $g \cdot f_+ \in Z$). Given g, f_+ and $2S$ in Γ all linear combinations, in particular the vector $f_- = (1/2, b \mp a \mid d)$ are also in Γ. But this vector represents a massless fermion of opposite helicity to f_+ with the same gauge quantum numbers (encoded in d). Thus within this class of models, gauge bosons from the supersymmetric half of the string imply the absence of chiral fermions.

Are these results true for general string models? The space-time degrees of freedom can again be represented by free world-sheet bosons; the internal degrees of freedom are now replaced by some unspecified CFT. Consider first a model with a light mass tensor state. This corresponds to a primary field in the CFT, call it ϕ, with small conformal dimension, $\Delta_\phi << 1$. In the free boson case this implied decompactification because the operator with small conformal dimension, $e^{i\mathbf{k}\cdot\mathbf{\Phi}}$, generates through its OPE's an entire tower of small dimension states, $e^{in\mathbf{k}\cdot\mathbf{\Phi}}$ ($\mathbf{\Phi}$ representing the free boson fields). In a general CFT we need to explore the possible conformal dimensions appearing in the operator algebra generated by ϕ. To do this we can apply eqn.(2) to the four-point function $\langle\phi\phi\hat{\phi}\hat{\phi}\rangle$, where $\hat{\phi}$ is the conjugate operator to ϕ. If only a single intermediate state primary appears in each channel of this amplitude (i.e.,N = 1, $\phi\hat{\phi} \sim I, \phi\phi \sim \phi_2$) then eqn.(2) implies $\Delta_{\phi_2} = 4\Delta_\phi$. Considering then eqn.(2) for the amplitude $\langle\phi\hat{\phi}\phi_i\hat{\phi}_i\rangle$ for $i = 2, 3, \ldots$ we find again an entire tower of states with small dimensions, $\Delta_{\phi_i} = i^2\Delta_\phi$, indicating decompactification. On the other hand, if more than one primary appears in each channel of $\langle\phi\phi\hat{\phi}\hat{\phi}\rangle$ then eqn.(2) may be satisfied with the dimensions of the intermediate state primaries all of order 1, so it is possible for ϕ to be the only small dimension field in the theory. Thus in a general string model a small mass tensor state necessarily implies decompactification if the $\phi\phi$ OPE contains a single primary field , but otherwise not.

Now consider the second case; Does a gauge boson from the supersymmetric half of a heterotic string always preclude chiral fermions? This has already been proved to be the case in ref.[1] using knowledge of the super Kac-Moody algebra generated by the RNS fermions and the dimension (1/2,0) field which gives rise to the gauge boson. We can prove the same result exactly as we did for the free boson case above, without using any knowledge of the holomorphic algebra generated by the fields, by using eqn.(2) to provide all of the information we need about the operator algebra. The states of momenta a and b are replaced by operators in the general CFT, a and β, with dimensions 1/2 and 3/8 respectively. The operator algebra generated by the analogs of g and f+ again contains the analog of $\mathbf{f}_-$, a fermion of opposite helicity to $\mathbf{f}_+$ with the same gauge quantum numbers . In this case level matching (i.e., modular invariance under $\tau \rightarrow \tau + 1$) guarantees that the internal left moving piece of $\mathbf{f}_-$ (i.e., the analog of b $\mp$ a) has conformal dimension 3/8 (mod 1). Applying eqn.(2) to $\langle\alpha\hat{\alpha}\beta\hat{\beta}\rangle$ we can fix the integer piece and find the dimension to be 3/8, i.e., the fermion is massless, completing the argument as before. Only a single primary contributes in each channel of the amplitude because α is a holomorphic operator.

The two examples given above, while of only limited interest in their own right, accurately illustrate the utility and limitations of using the constraints in eqn.(2) to study the properties of classical string vacua. Since eqn.(2) relates states of different conformal dimensions it is especially useful for examining small mass scales in string models, e.g., studying conditions implying decompactification; finding relations between purely massless states is less direct. As we have seen, in order to apply eqn.(2) to a given amplitude we need (or must assume) some information about the number of primary fields appearing in the amplitude. Knowledge of the structure of an amplitude for either the left or right movers is generally sufficient for this purpose, since the number of primaries appearing is necessarily the same for the holomorphic and antiholomorphic parts of any amplitude. Eqn.(2), and crossing symmetry in general, is particularly powerful for studying holomorphic fields, even those of large conformal dimension which may generate complicated or unknown algebras, It is worth recalling that the requirement of crossing symmetry was used in the original proof that dimension (1,0) fields must generate a Kac-Moody algebra[9].

Finally, let me conclude by stressing a basic property of the crossing symmetry constraints, including eqn.(2). There is apriori a different constraint for each nonvanishing four-point amplitude. Even in a complicated theory, some of these will likely be simple and of practical use. Taken together with other necessary structures such as world-sheet supersymmetry, some very interesting and uniquely stringy constraints on space-time physics may result.

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