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BOUNDS ON NONLINEAR MOTION FOR A FINITE TIME*

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ABSTRACT

Recent improvements in numerical methods to compute canonical transformations make it feasible to set interesting bounds on the motion of nonlinear Hamiltonian systems over a finite interval of time.

INTRODUCTION

The existence of invariant surfaces in phase space has a direct bearing on the stability of motion in cases of low dimension, but only an indirect implication in cases of higher dimension. In systems with at most four phase-space dimensions, invariant tori permanently confine the motion of a phase-space point. In higher dimensions, an invariant torus does not divide phase space into disjoint regions and therefore does not confine orbits. Since a system with p degrees of freedom having a Hamiltonian with periodic time dependence has a phase space of dimension $2p+1$, we are confronted with the latter situation in most problems of accelerator theory-for instance, in betatron oscillations in two degrees of freedom.

In the absence of confining surfaces, there can be a slow drift from one region of phase space to a remote region, the so-called Arnol'd Diffusion. Nekhoroshev' derived a lower bound on the time for the orbit to stay in a prescribed region. The bound increases exponentially as the nonlinear perturbation strength tends to zero. Although this may suggest that Arnol'd Diffusion can be very slow in appropriate circumstances, the Nekhoroshev theorem does not give useful quantitative information in practical problems. The perturbation strength has to be ridiculously small to yield stability times long enough to be interesting.

Nekhoroshev's argument depends on making a canonical transformation to new action-angle variables such that the actions are nearly (but not exactly) constant. The transformation is constructed by high-order perturbation theory. To analyze its properties rigorously, one is led to a string of involved and pessimistic analytic estimates. This feature is responsible for the poor quantitative result.

Our proposal is to avoid the pessimistic estimates by computing the canonical transformation numerically, using an accurate nonperturbative method. We have studied various nonperturbative methods, including iterative solution of the Hamilton-Jacobi equation,[2] smooth interpolation of tracking data,[3] and a new method based on iterative solution of a functional equation.[4] The latter is particularly efficient, and appears to be adequate to the task in the case of realistic accelerator models.

The determination of a transformation which makes actions invariant is equivalent to determination of an invariant torus. Thus, our goal is to compute approximate invariant tori of high accuracy. Unlike exact invariant tori, these approximate tori foliate open regions of phase space; i.e., they form continuous families, with a member of the family passing through every point of the region considered. This property has an important role in both the calculation and the application of approximate invariant surfaces.

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BOUNDING THE MOTION THROUGH A CHANGE OF VARIABLE

We first show how to bound the original phase-space coordinates in terms of bounds on new coordinates. We work in canonical polar coordinates $(\mathbf{I}, \Phi)$, which is to say the action-angle coordinates of the unperturbed system. The unperturbed system is integrable, with Hamiltonian $H_o(\mathbf{I}, s)$. The Hamiltonian of the fully perturbed system is

$$H(\mathbf{I}, \Phi, s) = H_o(\mathbf{I}, s) + V(\mathbf{I}, \Phi, s) \quad , \tag{1}$$

where s is the time, or, equivalently, arc-length along a closed reference orbit. Both H_o and V are periodic in s with period C. Bold-faced letters represent vectors of dimension p equal to the number of degrees of freedom. For betatron motion in a lattice with beta functions $\beta_i(s)$, the Cartesian phase-space coordinates x_i and $p_i = x_i'$ are related to action-angle coordinates by the equations

$$\begin{aligned} x_i &= [2I_i\beta_i]^{1/2}\cos\Phi_i \quad , \\ p_i &= -[2I_i/\beta_i]^{1/2}\left[\sin\Phi_i - \frac{\beta_i'}{2}\cos\Phi_i\right] \quad , \quad i = 1, 2 \quad , \end{aligned} \tag{2}$$

where primes denote derivatives with respect to s. The unperturbed Hamiltonian for betatron motion is

$$H_o(\mathbf{I}, s) = \sum_{i=1}^{2} I_i/\beta_i(s) \quad . \tag{3}$$

We make use of a norm for p-dimensional vectors v defined as

$$\| \mathbf{v} \| = \sum_{i=1}^{p} |v_i| \quad . \tag{4}$$

A compatible norm for p x p matrices M is

$$\| M \| = \sup_{\mathbf{v}} \left[\frac{\| M\mathbf{v} \|}{\| \mathbf{v} \|} \right] \quad . \tag{5}$$

Both norms satisfy the triangle inequality, and

$$\| M\mathbf{v} \| \leq \| M \| \, . \, \| \mathbf{v} \| \quad . \tag{6}$$

In Hamiltonian-Jacobi theory, one considers canonical transformations $(\mathbf{I}, \Phi) \rightarrow (\mathbf{J}, \Psi)$ induced by a generating function $\mathbf{J} \cdot \Phi + \mathbf{G}(\mathbf{J}, \Phi, s)$, which is periodic in s with period C. The equations relating old and new variables are

$$\mathrm{I} = \mathbf{J} + G_{\Phi}(\mathbf{J}, \Phi, s) \quad , \tag{7}$$

$$\Psi = \Phi + G_{\mathbf{J}}(\mathbf{J}, \Phi, s) \quad , \tag{8}$$

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where subscripts indicate partial derivatives. Here we emphasize the transformation of the action variables at a sequence of homologous values of s, say $s = 0, C, 2C, \ldots$. Since G is periodic in s, we may then suppress reference to s-dependence and write (7) as

$$\mathbf{I} = \mathbf{J} + \mathbf{u}(\mathbf{J}, \Phi) \quad , \tag{9}$$

where $\mathbf{u}(\mathbf{J}, \Phi) = G_{\Phi}(\mathbf{J}, \Phi, \mathbf{0})$.

We are interested in deviations from initial values under time evolution, namely,

$$\mathbf{I}_s - \mathbf{I}_o = \mathbf{J}_s - \mathbf{J}_o + \mathbf{u}(\mathbf{J}_s, \Phi_s) - \mathbf{u}(\mathbf{J}_o, \Phi_o) \quad , \quad s = 0, c, 2C, \ldots \tag{10}$$

'For an upper bound we add and subtract $\mathbf{u}(\mathbf{J}_o, \Phi_s)$, then apply the triangle inequality for the norm (4) to obtain

$$\| \mathbf{I}_s - \mathbf{I}_o \| \leq \| \mathbf{J}_s - \mathbf{J}_o \| + \| \mathbf{u}(\mathbf{J}_s, \Phi_s) - \mathbf{u}(\mathbf{J}_o, \Phi_s) \| + \| \mathbf{u}(\mathbf{J}_o, \Phi_s) - \mathbf{u}(\mathbf{J}_o, \Phi_o) \| \quad . \tag{11}$$

To bound the second term on the right-hand side we apply the mean value theorem and Eq. (6); to bound the third term we use again the triangle inequality. The result is

$$\| \mathbf{I}_s - \mathbf{I}_o \| \leq (1 + \sup_{\mathbf{J}, \Phi} \| \mathbf{u}_{\mathbf{J}}(\mathbf{J}, \Phi) \|) \cdot \| \mathbf{J}_s - \mathbf{J}_o \| + 2 \sup_{\Phi} \| \mathbf{u}(\mathbf{J}_o, \Phi) \| \quad , \tag{12}$$

where $\mathbf{u}_{\mathbf{J}}$ is the p x p Jacobian matrix of u. Here we have assumed that $\mathbf{u}$ has continuous derivatives with respect to $\mathbf{J}$ in the annular region

$$|J_i - J_{oi}| \leq |J_{si} - J_{oi}|, \quad i = 1, \ldots, p \quad . \tag{13}$$

The supremum over $\mathbf{J}$ in (12) refers to this region, while the supremum over Φ refers to the interval $[0, 2\pi]$ for each component. The region (13) is contained in the generally larger region

$$\| \mathbf{J} - \mathbf{J}_o \| \leq \| \mathbf{J}_s - \mathbf{J}_o \| \quad , \tag{14}$$

so that (12) also holds if the supremum is taken over region (14). For the following it is convenient to use (14).

The assumption that $\mathbf{u}$ has derivatives with respect to $\mathbf{J}$ in region (13) is not innocuous. If $\mathbf{u}$ were the ideal transformation, such as to make $\mathbf{J}$ a constant of the motion, then $\mathbf{u}$ would not have $\mathbf{J}$ derivatives at all. Phase space is shot through with resonances of arbitrarily high order, which prevent differentiability. The closer that $\mathbf{u}$ approximates the ideal transformation, the smaller the region in which $\mathbf{u}$ may be differentiable with respect to $\mathbf{J}$.

Given any AJ, let n be the largest integer such that

$$\| \mathbf{J}_s - \mathbf{J}_o \| \leq \mathrm{A\,J}, \quad s = 0, C, \ldots, nC \quad . \tag{15}$$

If n is infinite, then for all $s = mC$, m = 0, 1, . . . we have the inequality

$$\| \mathbf{I}_s - \mathbf{I}_o \| \leq (1 + \sup_{\|\mathbf{J} - \mathbf{J}_o\| \leq \delta J, \Phi} \| \mathbf{u}_{\mathbf{J}}(\mathbf{J}, \Phi) \|) \Delta J + 2 \sup_{\Phi} \| \mathbf{u}(\mathbf{J}_o, \Phi) \| \quad . \tag{16}$$

If n is finite, then

$$\| \mathbf{J}_{(n+1)C} - \mathbf{J}_o \| > \Delta J \quad , \tag{17}$$

and therefore

$$\| \mathbf{J}_{(n+1)C} - \mathbf{J}_{nC} \| + \| \mathbf{J}_{nC} - \mathbf{J}_{(n-1)C} \| + \cdots + \| \mathbf{J}_C - \mathbf{J}_o \| > \Delta J \quad , \tag{18}$$

which implies that

$$(n+1) \max_{m=0,\ldots,n} \| \mathbf{J}_{(m+1)C} - \mathbf{J}_{mC} \| > \Delta J \tag{19}$$

Thus, the bound (16) will hold for all s = mC with $m \leq n$, where n may be infinite, but in any case

$$n + 1 > \frac{\mathrm{A\,J}}{\max_{m=0,\ldots,n} \| \mathbf{J}_{(m+1)C} - \mathbf{J}_{mC} \|} \quad . \tag{20}$$

To complete the argument, we. have to invoke the specific dynamics of the system to find an upper bound δJ for the denominator of (20):

$$\max_{m=0,\ldots,n} \| \mathbf{J}_{(m+1)C} - \mathbf{J}_{mC} \| \leq \delta J \quad . \tag{21}$$

When such a bound has been found, we shall know that the system is stable in the sense of Eq. (16) for a number of turns n, where

$$n + 1 > \frac{\Delta J}{\delta J} \quad . \tag{22}$$

We next describe a technique for finding δJ.

APPLICATION OF A FUNCTIONAL EQUATION FOR INVARIANT TORI

The required bound (21) may be calculated from the Hamilton-Jacobi equation through the methods of Ref. 2. Here we describe an alternative method, which is more direct and efficient. It is stated in terms of the time evolution map for a full turn, $s \rightarrow s + C$. We write the map in terms of action-angle coordinates as follows:

$$\mathbf{I}' = \mathbf{I} + \mathbf{B}(\mathbf{I}, \Phi) \quad , \tag{23}$$

$$\Phi' = \Phi + \mathbf{A}(\mathbf{I}, \Phi) \tag{24}$$

For an integrable system, there is a choice of coordinates such that $\mathbf{B} = 0$ and $\mathrm{A} = \mathbf{A(1)}$ is independent of Φ. For a general system, we try to find a change of action variable $\mathbf{I} \rightarrow \mathbf{J} = \mathbf{J}(\mathbf{I}, \Phi)$ such that the new action is nearly (if not exactly) constant in time. In terms of the new variable, the map is written as

$$\mathbf{J}' = \mathbf{J} + \mathbf{B}^{(1)}(\mathbf{J}, \Phi) \quad , \tag{25}$$

$$\Phi' = \Phi + \mathbf{A}^{(1)}(\mathbf{J}, \Phi) \quad . \tag{26}$$

We seek to make $\mathbf{B}^{(1)}$ as small as possible; in any event, much less than **B.** For our purposes, it is not necessary to transform the angle variable. There is no harm in the circumstance that **J** and Φ do not form a canonical pair.

When $B^{(1)}$ is known, a δJ as in Eq. (21) can be found immediately. Recalling the definition of n in Eq. (15), we have

$$\| \mathbf{J}_{(m+1)C} - \mathbf{J}_{mC} \mathbf{II} = \| \mathbf{B}^{(1)}(\mathbf{J}_{mC}, \Phi) \| \leq \sup_{|\mathbf{J}-\mathbf{J}_o| < \Delta J} \| \mathbf{B}^{(1)}(\mathbf{J}, \Phi) \| = \delta J \, , \quad 0 < \Phi < 2\pi \tag{27}$$

for $m = 0, 1, \ldots,$ n.

As in the previous section, the change of variable is represented as in Eq. (9). If the transformation were ideal, making **J** an invariant, then Eq. (9) would actually represent an invariant surface in the space of the original variables. At constant **J,** it expresses **I** as a function of Φ; i.e., it constitutes an explicit representation of the invariant torus, with different tori corresponding to different values of **J.** Consequently, it is easy to find an equation that $\mathbf{u}(\mathbf{J}, \Phi)$ must satisfy in the ideal case. We merely demand that when $(\mathbf{I}, \Phi)$ lies on the surface, so does its image under the map, (I', Φ'). This requirement yields the functional equation to determine u:

$$\mathbf{W}(\mathbf{u};\ \mathbf{J}) = \mathbf{0} \, , \tag{28}$$

where

$$\mathbf{W}(\mathbf{u};\ \mathbf{J}) = \mathbf{u}(\Phi) - \mathbf{u}(\Phi + \mathbf{A}(\mathbf{J} + \mathbf{u}(\Phi), \Phi)) + \mathbf{B}(\mathbf{J} + \mathbf{u}(\Phi),\ \Phi) \quad . \tag{29}$$

Here we have suppressed reference to the J-dependence of u; the J-dependence is induced through the presence of **J as an** explicit parameter in (28).

An approximate solution of (28) yields an approximate invariant'surface and a change of variable (9). The function $\mathbf{B}^{(1)}$, expressing through (25) the time evolution of the new variable, is obtained as the solution of the following equation:

$$\mathbf{B}^{(1)} + \mathbf{u}(\mathbf{B}^{(1)} + \mathbf{J}, \Phi + \mathbf{A}(\mathbf{J} + \mathbf{u}(\mathbf{J}, \Phi), \Phi)) = \mathbf{u}(\mathbf{J}, \Phi) + \mathbf{B}(\mathbf{J} + \mathbf{u}(\mathbf{J}, \Phi), \Phi) \, . \tag{30}$$

This equation is easily solved for $\mathbf{B}^{(1)}$ by Newton's method, at each (**J**, 9). In the lowest approximation,

$$\mathbf{B}^{(1)} \approx \mathbf{W} \, . \tag{31}$$

That is, the order of magnitude of $\mathbf{B}^{(1)}$ is given by the residual of the approximate solution of (28). As we shall see, it is possible to construct rather accurate approximate solutions of (28), which yield very small values for $\mathbf{B}^{(1)}$. This $\mathbf{B}^{(1)}$ may give a δJ in (27) so small as to yield an interesting lower limit for n through (22). If $\mathbf{B}^{(1)}$ is not sufficiently small, one can repeat the whole process, applying a functional equation such as (28) for the map (25), (26) to obtain a second change of variable. The composition of the two changes of variable would then play the role of (9) in the discussion of the previous section.

A method for numerical solution of (28) is treated in a forthcoming paper.[5] The idea is to write $\mathbf{u}(\Phi)$ as a truncated Fourier series in Φ. The Fourier transform of W in (28) then gives a set of nonlinear equations for the Fourier coefficients of **u**, which can be solved by iteration. Convergence of the iteration suffers if too many Fourier modes are included. This is an intrinsic feature of the equation, associated with small divisors. Thus, there is a technical limit to achievable accuracy in solving (28).

To attain arbitrary accuracy, we take advantage of the fact that the number of Fourier modes allowed in a convergent iterative solution of (28) increases if **B** is decreased in magnitude. A second functional equation, based on the map (25), (26) can be solved on a larger mode set. In principle, the process can be repeated indefinitely, so as to produce an infinite sequence of coordinate transformations and functional equations, the latter being solved on ever larger mode sets. This sequence is similar to the sequence employed in Moser's proof[6] of the Twist Theorem (K. A. M. Theorem for maps of an annulus), but convergence will be much faster in our case, since we solve the functional equations nonperturbatively rather than in lowest-order perturbation theory.

Each functional equation has an approximate solution $\mathbf{u}(\mathbf{J}, \Phi)$, which is defined and differentiable with respect to action in a domain $\| \mathbf{J} - \mathbf{J}_o \| <$ AJ, but AJ shrinks as the number of Fourier modes in the approximate solution is increased. To set a large lower bound for n as in (22), we depend on δJ decreasing more rapidly than A J as successive coordinate changes are carried out.

NUMERICAL REALIZATION OF THE METHOD

To assure ourselves that the above proposal is within reason, we tried it first for betatron motion in one degree of freedom, in a lattice consisting of four sextupoles. The lattice is one cell of the Berkeley Advanced Light Source (ALS), an example in. which the computation of invariant surfaces has proved to be relatively difficult because of its strong sextupoles. In a report? for the previous ICFA Beam Dynamics Workshop, we treated this example through solution of the Hamilton-Jacobi equation by the shooting method. Extending the computer program used for that work, we have obtained long-term bounds on the motion in a region of substantial nonlinearity. We refer the reader to Ref. 2 for information about the lattice and the method for solving the Hamilton-Jacobi equation.

As an example, we consider betatron motion in'the horizontal plane, and solve the Hamilton-Jacobi equation for $u(J_o, \Phi)$ at $J_o = 10^{-6}$ meters, which corresponds to x-displacements about 20% of those at the dynamic aperture as estimated from short-term tracking (1000 turns or so). At this value of J_o, the variation of $u(J_o, \Phi)$ with Φ (sometimes called the "smear") is about $\pm$ 7% of J_o, the average of u being zero. We ask for a lower bound on the number of turns n such that $|I_s - I_o|$ does not vary by more than an additional 7% of J; i.e., we choose AJ in (16) so that

$$|I_s - I_o| < 3 \sup_{\Phi} |u(J_o, \Phi)| \quad . \tag{32}$$

This choice of allowed range is reasonable, since we conjecture that u is actually close to a true invariant surface; in fact, an orbit from tracking lies close to the surface for 1000 turns, the maximum discrepancy at s = 0, C, . . . , 1000 C being $\Delta I/J_o \approx$ 1.6 x 10^{-6}. If u were an exact invariant surface, we would have $|I_s - I_o| \leq 2$ sup $|u|$. If I_o is any point on the surface

$Z = J_o + u(J_o, \Phi)$, we find that (32) will be satisfied for at least n turns where

$$n = 2.4 \times 10' \quad . \tag{33}$$

The length of the lattice is 16.4 meters, so that this corresponds to motion for 1.3 seconds.

It might be argued that this is an example in a J-dimensional phase space (1-1/2 degrees of freedom), so that the motion should actually be stable, being confined by a K.A.M.. surface. One should remember, however, that it would be impossible to prove the existence of the surface, since the K.A.M. proof fails for the strong nonlinearities of this example. Thus, the result (33) can be considered the best information currently available for the example studied.

We regard the value (33) as quite satisfactory, in view of the fact that we have carried out only one canonical transformation. It is certain that a second transformation will provide a larger lower bound for n, by several orders of magnitude. Moreover, with a second transformation we can work at larger amplitudes, close to the estimated dynamic aperture.

For betatron motion in two degrees of freedom, we have applied the method of Sec. 3. We have solved (28) by the Fourier method, using Newton's iteration to solve the nonlinear equation for the Fourier coefficients of $\mathbf{u}$. For the map functions, A and B, we call a symplectic tracking code, tracking for one turn at each required value of the arguments of the functions. This is more efficient and more accurate than using an explicit formula for the map (for instance a polynomial-Fourier representation, as in Ref. 7), unless the lattice is fairly long.

To the present time, we have obtained the new map function $\mathbf{B}^{(1)}$ only in the lowest approximation (31), so that the results quoted are correct only to a few percent. Full results, including the construction of a second coordinate change and a second map function $\mathbf{B}^{(2)}$, should be available in the near future. We report results for the same ALS lattice that was discussed above, taking the same initial amplitude for s-motion and one-half that much for y-motion:

$$J_{o1} = 10^{-6} m, \quad J_{o2} = 5 \cdot 10^{-7} m \, . \tag{34}$$

Calculation of $\mathbf{u}(\mathbf{J}_o, \Phi)$ shows that the y-motion induces a little more variation in I_1 than we had before, almost 11% rather than 7% of J_{o1}. We find

$$\begin{aligned} \sup_{\Phi} |u_1(\mathbf{J}_o, \Phi)| &= 0.107 \cdot 10^{-6} m \, , \\ \sup_{\Phi} |u_2(\mathbf{J}_o, \Phi)| &= 0.0567. \, 10^{-6} m \quad , \end{aligned} \tag{35}$$

in other words, about 11% smear in each coordinate. As in Eq. (32), we look for a lower bound on n such that

$$\| \mathbf{I}_s - \mathbf{b} \| < 3 \sup_{\Phi} \| \mathbf{u}(\mathbf{J}_o, \Phi) \| \quad , \tag{36}$$

'again with I, being any point on the surface $\mathbf{I} = \mathbf{J}_o + \mathbf{u}(\mathbf{J}_o, \Phi)$. Applying (22) and (27) in the approximation (31), we find that (36) will be satisfied for at least n turns, where

$$n \approx 1.8 \cdot 10^5 \, . \tag{37}$$

As expected, the results for two degrees of freedom are not yet as impressive as for one. The equations are more difficult to solve, perhaps because the small divisors have a stronger effect. Again, we are confident of obtaining much better results after a second change of coordinates. Fortunately, the coordinate changes are much easier to implement in the method of Sec. 3 than in the canonical Hamilton-Jacobi formalism.

The approximate invariant surface $\mathbf{I} = \mathbf{J}_o + \mathbf{u}(\mathbf{J}_o, \Phi)$ that yielded the result (37) agrees rather well with tracking. In tracking for 1000 turns, and checking discrepancy with the surface at every 100 turns, we found

$$\epsilon_1 < 1.1 \cdot 10^{-7}, \; \epsilon_2 < 2.2 \cdot 10^{-7} \tag{38}$$

with

$$\epsilon_i = \frac{|I_i^{tr} - I_i|}{[J_{o1}^2 + J_{o2}^2]^{1/2}} \quad , \tag{39}$$

where (I?', Φ^{tr}) are coordinates from tracking, and $\mathbf{I} = \mathbf{J}_o + \mathbf{u}(\mathbf{J}_o, \Phi^{tr})$ is the corresponding point on the surface. Of course, much more accurate invariant surfaces will be produced after a second transformation.

CONCLUSION

Preliminary results baaed on one change of coordinates show that interesting bounds on betatron motion in two degrees of freedom can be anticipated. Refinement of the method through successive coordinate changes should allow one to establish stability for much longer times than can be established by tracking, and with better control of numerical error. The program is much facilitated by a new scheme for computation of approximate invariant surfaces and associated coordinate changes. The scheme, based on iterative solution of a functional equation, is so compact and efficient as to be feasible (in two degrees of freedom) for realistic models of large accelerators. The extension to three degrees of freedom is probably possible, but has not yet been explored.

Finally, we should mention that the required computer codes are rather short. The codes are not written for a specific dynamics; they call a tracking code for the specific dynamics, so that the user can apply any tracking code suited to the problem at hand.

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