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Quark Mass Ratios in Semi-Leptonic Quark Decays*

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ABSTRACT

An ambiguity in the calculation of CKM matrix elements from semi-leptonic decay rates is resolved: to every choice of scales for the quark masses in the phase-space factor and in the QCD-correction factor, there corresponds a specific QCD-correction factor. This factor is modified in such a way as to make the final result independent of the scales. Specific expressions are given for the case of on-shell quark masses and for the case where both masses are taken at a single common scale. A calculation of $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ is carried out in view of these clarifications.

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The above-diagonal elements of the Cabibbo - Kobayashi - Maskawa (CKM) matrix are best determined from semi-leptonic meson decays. The two elements $|V_{cb}|$ and $|V_{ub}|$ are determined from B decays. The B-meson, or correspondingly the b-quark, is heavy enough to allow the use of the spectator quark model for the calculation:

$$\Gamma(B \to X_q e \nu_e) = \Gamma(b \to q e \nu_e), \tag{1}$$

with $q = c$ or u. When the electron mass is neglected and all quarks are taken to be much lighter than the W-boson, the spectator quark model gives:

$$\frac{BR(b \to q e \nu_e)}{\tau_b} = \frac{G_F^2 m_b^5}{192\pi^3} F_{ps}(\rho) F_{QCD}(\rho) |V_{qb}|^2, \tag{2}$$

with F_{ps} and F_{QCD} a phase-space factor and a QCD-correction factor respectively. Both factors depend on the ratio

$$\rho = \frac{m_q^2}{m_b^2}. \tag{3}$$

In this work we address the following question: What is the appropriate ratio ρ to be used in the calculation of the semi-leptonic b-decay?

Quark masses are *running* masses which depend on the energy scales at which they are taken, $m_q = m_q(\mu_q)$. The ratio ρ should really be written as

$$\rho = \frac{[m_q(\mu_q)]^2}{[m_b(\mu_b)]^2}. \tag{4}$$

The above question can be rephrased into the following one: What are the scales μ_q and μ_b that should be used? The answer is [1] that we may choose any two scales μ_q and μ_b. *To every choice of scales, there corresponds a specific QCD-correction factor.* The functional dependence of F_{QCD} on ρ is modified in such a way as to make the product $F_{ps}(\rho)F_{QCD}(\rho)$ independent of the choice of scales.

We demonstrate the modification of $F_{QCD}(\rho)$ with two specific examples. We define ρ^s as

$$\rho^s \equiv \frac{[m_q(m_b)]^2}{[m_b(m_b)]^2}, \tag{5}$$

namely the ratio between the quark masses taken at a *single* common scale. We define ρ^o as

$$\rho^o \equiv \frac{[m_q(m_q)]^2}{[m_b(m_b)]^2}, \tag{6}$$

namely the ratio between the *on-shell* masses. The relation between the two ratios, to first order in α_s, is given by

$$\rho^s = \rho^o \left[1 + \frac{2\alpha_s}{\pi} \ln(\rho^o)\right]. \tag{7}$$

As the difference is $O(\alpha_s)$, the phase-space factor (which is by definition zeroth order in α_s) has the same functional dependence on the respective ρ's for both cases:

$$F^o_{ps}(x) = F^s_{ps}(x). \tag{8}$$

The expression for F_{ps} is well known:

$$F_{ps}(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln(x). \tag{9}$$

However, the relation (7) implies that the QCD-correction factor is different for the two cases. The ρ-dependent terms in F_{ps} induce a modification of F_{QCD}. To first order in x we get:

$$F^o_{QCD}(x) = F^s_{QCD}(x) - \frac{16\alpha_s}{\pi} x \ln(x). \tag{10}$$

Such a modification is necessary to give

$$F_{ps}(\rho^o) F^o_{QCD}(\rho^o) = F_{ps}(\rho^s) F^s_{QCD}(\rho^s), \tag{11}$$

which guarantees that the decay width is independent of the choice of scales in ρ.

The QCD-correction factor is conventionally written as:

$$F_{QCD}(x) = 1 - \frac{2\alpha_s}{3\pi} f(x). \tag{12}$$

The functions $f^o(x)$ and $f^s(x)$ that should be used for $F^o_{QCD}(\rho^o)$ and $F^s_{QCD}(\rho^s)$ respectively are given, to first order in x, by:

$$\begin{aligned} f^o(x) =& \pi^2 - \frac{25}{4} + (18 + 8\pi^2)x + 24x\ln(x), \\ f^s(x) =& \pi^2 - \frac{25}{4} + (18 + 8\pi^2)x. \end{aligned} \tag{13}$$

The zeroth order term $\pi^2 - 25/4$ is known from previous calculations. The absence of an $x\ln(x)$ term from $f^s(x)$ is an important check on our first-order calculation [2]: the masses m_q and m_b, when divided by a common scale, can be regarded as coupling constants. One does not expect a logarithmic dependence on the coupling constants when they are all taken at a single common scale. With no logarithmic term in $f^s(x)$, the coefficient 24 for the $x\ln(x)$ term in $f^o(x)$ could be predicted: it is the product of a factor of 2 from the relation between ρ^o and ρ^s (eq. (7)), a factor of 8 from the linear term in the phase-space factor (eq. (9)) and a factor of 3/2 from the definition of $f(x)$ (eq. (12)).

For the case of on-shell masses, we were able to find the analytical expression for the QCD correction to all orders in the mass ratio. We now give the expression for the function $h^o(\rho) \equiv F_{ps}(\rho)f^o(\rho)$:

$$\begin{aligned} h^o(\rho) = & -(1-\rho^2)\left(\frac{25}{4} - \frac{239}{3}\rho + \frac{25}{4}\rho^2\right) + \rho\ln\rho\left(20 + 90\rho - \frac{4}{3}\rho^2 + \frac{17}{3}\rho^3\right) \\ & + \rho^2\ln^2\rho\,(36+\rho^2) + (1-\rho^2)\left(\frac{17}{3} - \frac{64}{3}\rho + \frac{17}{3}\rho^2\right)\ln(1-\rho) \\ & -4(1+30\rho^2+\rho^4)\ln\rho\ln(1-\rho) - (1+16\rho^2+\rho^4)\left[6Li_2(\rho) - \pi^2\right] \\ & -32\rho^{3/2}(1+\rho)\left[\pi^2 - 4Li_2(\sqrt{\rho}) + 4Li_2(-\sqrt{\rho}) - 2\ln\rho\ln\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right]. \end{aligned} \tag{14}$$

The dilogarithm function $Li_2(x)$ is defined as in ref. [3]:

$$Li_2(x) = -\int_0^x \frac{\ln(1-z)}{z} dz = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots \quad for\ |x| \le 1. \tag{15}$$

All previous calculations [4] of the QCD correction factor (beyond zeroth order in the quark mass ratio) were numerical: their starting point was the QED corrections to the differential cross section for muon decay, which is analytically known [5]. It was modified to describe QCD corrections to quark decays by the replacement $\alpha_{EM} \to \frac{4}{3}\alpha_s$. The correction to the decay rate was then calculated by a *numerical* integration over the differential cross section. In the original QED calculation, the masses are by definition on-shell masses: lepton masses are experimentally measurable, and these physical masses identify with the on-shell masses. Consequently, the existing calculations of QCD-corrected quarks decays correspond to the mass ratio between on-shell masses. Moreover, our result for the on-shell case (eq. (14)) indeed agrees with the numerical integration results given for specific cases in ref. [4].

We now apply the above results to the actual calculation of the matrix elements. For the $b \to c$ case we use

$$\rho_c^o = \frac{[m_c(m_c)]^2}{[m_b(m_b)]^2} = (0.30 \pm 0.02)^2. \tag{16}$$

This gives

$$\begin{aligned} F_{ps}(\rho_c^o) =& 0.52 \pm 0.04, \\ f^o(\rho_c^o) =& 2.51 \pm 0.06. \end{aligned} \tag{17}$$

The QCD correction factor depends on the value of α_s as well. We take $\alpha_s = 0.20 \pm 0.02$:

$$F^o_{QCD}(\rho_c^o) = 0.89 \pm 0.01. \tag{18}$$

We note that the QCD correction factor given in the literature, η_0', indeed corresponds to the on-shell case:

$$\eta_0' \equiv F^o_{QCD}(\rho^o_c). \tag{19}$$

We take:

$$\begin{aligned} BR(b \to c e \nu_e) &= 0.115 \pm 0.004, \\ \tau_b &= (1.18 \pm 0.14) \times 10^{-12} sec, \\ m_b &= 4.9 \pm 0.3\ GeV, \end{aligned} \tag{20}$$

and get:

$$|V_{cb}|^2 = (2.1 \pm 0.7) \times 10^{-3} \implies |V_{cb}| = 0.046 \pm 0.008. \tag{21}$$

The ratio $|V_{ub}/V_{cb}|$ is free of the uncertainty in m_b:

$$\frac{|V_{ub}|^2}{|V_{cb}|^2} = \left[\frac{BR(b \to u e \nu_e)}{BR(b \to c e \nu_e)}\right] \left[\frac{F_{ps}(\rho^o_c)}{F_{ps}(\rho^o_u)}\right] \left[\frac{F^o_{QCD}(\rho^o_c)}{F^o_{QCD}(\rho^o_u)}\right]. \tag{22}$$

We note that:

a. As the u-quark is lighter than Λ_{QCD}, its mass is not well-defined on-shell. What we would really like to use is the ratio m_u/m_b at a single common scale, as m_u at scales above Λ_{QCD} is well-defined and known:

$$\rho^s_u \approx (8^{+6}_{-4}) \times 10^{-7}. \tag{23}$$

Eqs. (9) and (13) then tell us that indeed ρ^s_u can be safely put to 0. However, to zeroth order in the mass ratio, the different calculations identify:

$$\begin{aligned} F_{ps}(\rho = 0) =& 1, \\ f(\rho = 0) =& \pi^2 - 25/4. \end{aligned} \tag{24}$$

b. The QCD correction factor given in existing literature, η_0'', corresponds to $\rho = 0$. With $\alpha_s = 0.20 \pm 0.02$ we get:

$$F_{QCD}(\rho = 0) \equiv \eta_0'' = 0.85 \pm 0.01. \tag{25}$$

c. The ratio $F_{QCD}(\rho_c)/F_{QCD}(\rho_u)$ does not depend on our choice of α_s.

We get

$$\frac{|V_{ub}|}{|V_{cb}|} = (0.74 \pm 0.03) \left[\frac{BR(B \to X_u e \nu_e)}{BR(B \to X_c e \nu_e)} \right]^{1/2} . \tag{26}$$

At present, there is no measurement of the inclusive semi-leptonic charmless B decay.

To conclude: The need for accuracy in the determination of the quark mixing angles necessitates a refinement of the ingredients involved in the calculation. We concern ourselves with one such aspect: the quark mass ratio that should be used in the calculation of semi-leptonic decay widths. We are interested in the difference between calculations using the ratio between the on-shell masses and those using the ratio between the masses taken at a common energy scale.

There are three possible cases:

a. The ratio is close to 1. In this region the question is unimportant both in principle, as the two possible mass ratios are very close to each other, and in practice, as nature has not provided us yet with such a case.

b. The ratio is close to 0 (relevant to m_u/m_b). Here the question is interesting in principle, as for light quarks there is no well-defined on-shell mass. However, in practice the question is, again, unimportant because the mass ratio can be approximated to zero, and the calculations identify to zeroth order.

c. The ratio is non-negligible, but not too close to 1 (relevant to m_c/m_b). Here the question is important both in principle and in practice. We find that previous calculations, which were all numerical, correspond to the ratio between the on-shell masses.

We give an analytic expression for the QCD-correction factor to all orders in the ratio between on-shell masses. We also give useful approximations to the QCD-correction when either the ratio between on-shell masses or the ratio between the masses at a single scale is used.

REFERENCES

1. Y. Nir, Phys. Lett. 221B (1989) 184.
2. M. Peskin, private communication.
3. L. Lewin, Dilogarithms and associated functions, (Macdonald, London, 1958).
4. N. Cabibbo and L. Maiani, Phys. Lett. 79B (1978) 109;
   A. Ali and E. Pietarinen, Nucl. Phys. B154 (1979) 519;
   N. Cabibbo, G. Corbó and L. Maiani, Nucl. Phys. B155 (1979) 83;
   G. Corbó, Phys. Lett. 116B (1982) 298; Nucl. Phys. B212 (1983) 99.
5. R.E. Behrends, R.J. Finkelstein and A. Sirlin, Phys. Rev. 101 (1956) 866;
   S.M. Berman,Phys. Rev. 112 (1958) 267;
   T. Kinoshita and A. Sirlin, Phys. Rev. 113 (1959) 1652.