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Open String Theory As Dissipative Quantum Mechanics*

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ABSTRACT

We argue that vacuum configurations of open strings correspond to Caldeira-Leggett models of dissipative quantum mechanics (DQM) evaluated at a delocalization critical point. This connection reveals that critical DQM will manifest reparametrization invariance (inherited from the conformal invariance of string theory) rather than just scale invariance. This connection should open up new ways of constructing analytic and approximate solutions of open string theory (in particular, topological solitons such as monopoles and instantons).

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1. Introduction

In this paper we establish a connection between open string theory and the Caldeira-Leggett model of dissipative quantum mechanics, evaluated at a delocalization critical point. This reduces the problem of finding non-trivial open string ground states (magnetic monopoles, for instance) to the problem of finding gauge potentials in which the long time quantum motion of a non-relativistic charged particle subject to dissipation manifests scaling behavior. Since very little seems to be known about the latter problem, this mapping unfortunately does not solve any string theory problems directly. It does however provide new insights into the structure of both the open string ground state problem and dissipative quantum mechanics and suggests new lines of attack which may eventually prove fruitful. At the very least it is amusing and chastening to discover that a significant string theory problem is identical to a problem studied in conventional condensed matter physics.

In order to keep the exposition compact we will assume that the reader is aquainted with both the essentials of the Caldeira-Leggett approach to dissipative quantum mechanics [1,2,3,4] and the boundary state approach to the physics of open strings [5,6,7,8]. We reproduce the essential formulae in Sections 2 and 3 and the reader who is familiar with either of the two subjects should have no difficulty in following our arguments. In Section 4 we discuss reparametrization symmetry which plays an important role in the analysis of these theories. In Section 5 we illustrate our ideas by considering some simple dissipative quantum systems that are relevant to string theory applications. Section 6 contains discussion and suggestions for further work.

2. Dissipative Quantum Mechanics

Let us begin by summarizing the formulation of dissipative quantum mechanics. The quantum mechanics of a non-relativistic particle subject to a quite general force is described by the standard Euclidean path integral

$$Z_{QM} = \int [\mathcal{D}X(t)]\, e^{-S_{QM}[X]}$$

with

$$S_{QM}[X] = \int dt\, \{\frac{1}{2}M\dot{X}^2 + V(X)\} + i\int dt\, A_\mu(X)\,\dot{X}^\mu\,. \tag{2.1}$$

The coordinates X^μ parametrize D-dimensional Euclidean space, $V(X)$ is the scalar potential and $A_\mu(X)$ is the gauge vector potential. The gauge interaction is through a Wilson line factor and, if the gauge group is non-abelian, this factor must be generalized to the trace of a path-ordered exponential.

If the quantum variable, X^μ, is in some sense macroscopic (a good example is the trapped magnetic flux in a Josephson junction) there will typically exist an infinite collection of degrees of freedom to which X^μ is at least weakly coupled and which give rise to dissipative effects on the motion of X^μ. The details of the interaction with the environment are often unknown but in the classical limit, dissipation can be described by simply adding a friction term such as $-\eta\dot{X}^\mu$ (where η is a phenomenological dissipation constant) to the equation of motion for X^μ. It is then natural to ask whether there is a correspondingly simple and universal way to express the *quantum* effects of weak dissipation. A practical motivation for this question is to assess whether coherent quantum effects, such as tunneling, can really be observed in macroscopic quantum systems.

Caldeira and Leggett [1] addressed this problem via the following simple model: In addition to the system (2.1) described by the coordinates X^μ, let there be a bath of harmonic oscillators, q_α, with a distribution of frequencies, ω_α. Let them be coupled linearly to the coordinate X^μ with coupling strengths C_α. These simple assumptions are justified when any single degree of freedom of the environment is only weakly perturbed by the system. The bath of oscillators then describes the excitations around the equilibrium configuration of the environment. This is often the case and these models are applicable to a variety of dissipative problems (in particular weak coupling to *each* mode in the environment does not imply weak dissipation in the system) [1].

If the distribution of oscillator parameters satisfies the functional condition

$$\sum_\alpha \frac{C_\alpha^2}{2\omega_\alpha}\,\delta(\omega-\omega_\alpha) = \frac{\eta\,\omega}{\pi} \tag{2.2}$$

it turns out that, when the oscillator coordinates are "integrated out" of the classical equations of motion for X^μ, they supply the canonical $-\eta\dot{X}^\mu$ friction term. By scaling the oscillator coupling strengths, the friction constant η can be adjusted to any desired value. One can study more general dissipative systems than the linear friction considered here (for example a term proportional to $\dot{X}^2$ could be included) by generalizing the spectral density (2.2) of the oscillator bath. We will see, however, that it is the case of linear dissipation which is relevant in the context of open string theory.

The quantum mechanics of this coupled system is described by a Euclidean path integral over the X^μ and the oscillator coordinates q_α. Since the dependence of the action on the q_α is only quadratic, they can be explicitly integrated out of the quantum mechanical path integral just as they could be eliminated from the classical equations of motion. The result is a new path integral over the X^μ alone, in which the net quantum mechanical effect of friction is contained in a non-local term whose strength is set by the classical friction constant η:

$$Z_{QM}^\eta = \int [\mathcal{D}X(t)]\, e^{-S_Q[X]-S_\eta[X]}$$

where

$$S_\eta[X] = \frac{\eta}{4\pi} \int_{-\infty}^{\infty} dt\, dt' \, \frac{\big(X(t) - X(t')\big)^2}{(t-t')^2} \, . \tag{2.3}$$

The η-term is non-local and consequently it is difficult to extract from the path integral the interesting qualitative physics, such as the long time behavior of Green's functions. We will come back to this in Section 5. For the moment we note that if the original path integral is taken over paths periodic in t with period β, as one does to compute the thermal partition function, the only effect is to change S_η to

$$\begin{aligned} S_\eta^\beta[X] &= \frac{\eta}{4\pi} \int_0^\beta dt \int_{-\infty}^{\infty} dt' \, \frac{\big(X(t) - X(t')\big)^2}{(t-t')^2} \\ &= \frac{\pi\eta}{4\beta^2} \int_0^\beta dt\, dt' \, \frac{\big(X(t) - X(t')\big)^2}{\sin^2\big(\frac{\pi}{\beta}(t-t')\big)} \, . \end{aligned} \tag{2.4}$$

The first equality is a direct transcription of the periodicity condition and the second follows from a well known summation formula.

3. Boundary State Formalism for Open Strings

Somewhat surprisingly, the same path integral allows one to study non-trivial ground states of the open string. The connection comes through the operator approach to calculating the Polyakov path integral on worldsheets with complicated topology. In this approach, states in the closed string Hilbert space are associated with basic worldsheet topological fixtures (boundaries and handles) and, just as a complicated worldsheet is constructed by gluing together such fixtures, the corresponding path integral is obtained by taking inner products of the states associated with the fixtures.

The simplest topological fixture on a worldsheet is a single connected boundary. It can only appear in a theory with open strings. In this Section we review how the associated boundary state is constructed. A detailed treatment can be found in [7] and [8] so we will be brief.

It is easy to obtain the boundary state, $|B\rangle$, for strings propagating in flat empty spacetime, where the boundary just imposes the usual Neumann condition on the worldsheet fields:

$$\partial_\tau \phi^\mu(\sigma,\tau)\Big|_{\tau=0} = 0\,, \qquad 0 \le \sigma \le 2\pi\,.$$

(We have chosen cylindrical coordinates on the worldsheet near the boundary, which is placed at $\tau = 0$.)

If we make the usual closed string mode expansion of the fields,

$$\phi^\mu(\sigma,\tau) = q^\mu - 2ip^\mu\tau + i\sum_{m\neq 0}\frac{1}{m}\left[\alpha^\mu_m e^{-m\tau-im\sigma} + \tilde{\alpha}^\mu_m e^{-m\tau+im\sigma}\right] \tag{3.1}$$

we can see that the Neumann condition reduces to the following set of conditions on the left- and right-moving oscillators and on the zero modes:

$$\begin{aligned} p^\mu &= 0\,, \\ \alpha^\mu_{-m} + \tilde{\alpha}^\mu_m &= 0\,, \qquad \forall m \neq 0\,. \end{aligned} \tag{3.2}$$

Taking into account the oscillator commutation relations

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] &= m\,\delta_{m+n,0}\,\delta^{\mu\nu}\,, \\ [\alpha^\mu_m, \tilde{\alpha}^\nu_n] &= 0\,. \end{aligned} \tag{3.3}$$

and defining the α_n and $\tilde{\alpha}_n$ with $n > 0$ as annihilation operators and those with $n < 0$ as creation operators, we can see that the state which satisfies the operator form of the boundary conditions is:

$$|B\rangle_{free} = \exp\{-\sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m}\} \, |0\rangle \, , \tag{3.4}$$

where $|0\rangle$ is the $SL(2,C)$ invariant vacuum of the closed string,

$$p^{\mu} \, |0\rangle = 0 \, ; \qquad \alpha^{\mu}_{m} \, |0\rangle = \tilde{\alpha}^{\mu}_{m} \, |0\rangle = 0 \, , \qquad \forall m > 0.$$

By sandwiching a closed string propagator between two such states one can, for example, construct the Polyakov path integral on the cylinder.

Now suppose that, instead of having a string propagating in flat empty spacetime, we want to consider a situation in which some spacetime degrees of freedom of the *open* string are excited. In the Polyakov path integral one accounts for open string spacetime background fields by including worldsheet interactions localized at worldsheet boundaries. The canonical example is the Wilson line factor associated with spacetime background gauge fields. In the presence of boundary interactions, the worldsheet fields no longer satisfy simple linear boundary conditions and simple operator arguments of the kind presented above do not suffice to construct the interacting boundary state. Fortunately, a perturbatively well-defined path integral algorithm for obtaining $|B\rangle$ exists.

In [7] it is shown how the boundary state in an arbitrary spacetime gauge field is formally given in terms of a path integral for a certain one-dimensional field theory. This is obtained by viewing the string as an infinite collection of oscillators and solving the boundary conditions using elementary properties of simple harmonic oscillator quantum mechanics. The result is

$$|B\rangle = \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \alpha_{-m} \cdot \tilde{\alpha}_{-m}\Big) \int [\mathcal{D}X^{\mu}(s)] \, \mathrm{tr} \, P\exp\big[-S_{reg} - S_0 - S_A - S_{ls}\big] \, |0\rangle \, , \tag{3.5}$$

where

$$
\begin{aligned}
S_0[X] &= \frac{1}{8\alpha'} \int_0^{2\pi} \tfrac{ds}{2\pi} \int_0^{2\pi} \tfrac{ds'}{2\pi} \; \frac{\left(X(s)-X(s')\right)^2}{\sin^2(\frac{s-s'}{2})} \,, \\
S_A[X] &= \frac{i}{2} \int_0^{2\pi} \tfrac{ds}{2\pi} \; A_\mu(X(s)) \, \frac{dX^\mu(s)}{ds} \,, \\
S_{ls}[X] &= \int_0^{2\pi} \tfrac{ds}{2\pi} \; \alpha(s) \cdot X(s) \,, \\
& \qquad \alpha^\mu(s) = \sum_{m=1}^{\infty} i \left(\tilde{\alpha}^\mu_{-m} \, e^{-ims} + \alpha^\mu_{-m} \, e^{ims} \right) , \\
S_{reg}[X] &= \int_0^{2\pi} \tfrac{ds}{2\pi} \, \frac{1}{2} M \, \dot{X}(s)^2 \,.
\end{aligned}
\tag{3.6}
$$

The path integral is over scalar fields $X^\mu(s)$ defined on S_1, parametrized by $s \in [0, 2\pi]$, corresponding to the boundary. There should ultimately be nothing special about choosing the parameter length of the boundary equal to 2π. This path integral describes an interacting one-dimensional field theory with a non-local but scale-invariant kinetic term, S_0, and an interaction term, S_A, of power counting dimension one. Perturbation theory in S_A is therefore renormalizable, not finite. Strictly speaking we need a regulator term, of dimension greater than one, to render the path integral finite and that is the role of S_{reg} (which is also just the standard non-relativistic quantum mechanical kinetic term). Eventually we must remove the regulator by taking the limit $M \to 0$ and removing divergences by renormalization counterterms.

The non-local kinetic term is quite peculiar. It has the above simple form only if the worldsheet fields have *no* interactions on the body of the worldsheet. This means that as far as closed string fields are concerned we are considering the trivial configuration of flat spacetime. Only the open string fields are allowed to have spacetime condensates, leading to interactions only on worldsheet boundaries. S_A weights the path integral with a Wilson line factor based on the chosen spacetime gauge potential, A_μ. We could, of course, accommodate a non-abelian gauge field by including the usual path-ordering and gauge trace instructions. More general local interaction terms are possible, and in particular, it eventually turns out to be essential to include the dimension zero (super-renormalizable) interaction

$$S_V = \int_0^{2\pi} \frac{ds}{2\pi}\, V\big(X(s)\big) \tag{3.7}$$

corresponding to a spacetime condensate of the open string tachyon.

The role of the linear source term is to turn the path integral into a functional of the left- and right-moving closed string creation operators, α^μ_{-m} and $\tilde{\alpha}^\mu_{-m}$. Since they all commute with each other they can be treated as c-numbers. The whole functional acts on the oscillator ground state $|0\rangle$ and, when expanded in powers of the creation operators, yields a state in the closed string Hilbert space. The oscillator Gaussian standing in front of the path integral arises naturally enough in the careful derivation [7] of the above expressions. In the free case ($A_\mu = 0$) the path integral is Gaussian and can be done explicitly and one can readily verify that the extra Gaussian factor is needed to reproduce the known free boundary state (3.4).

The net result of all of this is that the path integral for dissipative quantum mechanics and the path integral for the open string boundary state are essentially identical after appropriate reinterpretations of quantities. The scalar and vector potentials in the quantum mechanics problem are interpreted as spacetime tachyon and gauge fields in the string problem. The dissipation constant is reinterpreted as the inverse of the string Regge trajectory slope (*i.e.* string tension) by the relation $\eta = \frac{1}{2\pi\alpha'}$. We know from study of the string problem that the perturbative expansion parameter for this path integral is α'. This means that we cannot expand around the zero dissipation, pure quantum mechanics limit. The mass in the standard quantum mechanical kinetic term is interpreted as a cutoff scale in the string path integral (M has worldsheet dimensions of length, so the cutoff is removed as $M \to 0$). The temperature in the dissipative quantum mechanics corresponds to the parameter length of the boundary in the string case.

4. Reparametrization Invariance

To discuss string physics, we must scale the cutoff parameter M to zero and carry out a renormalization program to deal with the divergences which arise. Equivalently, we can say that the string physics is contained in the *infrared* behavior of the theory with a fixed cutoff scale. If we take the renormalization group approach, we will find that the effective interactions, *i.e.* the vector potential A_μ (marginal) and scalar potential V (relevant), will in general "flow" as we integrate out larger and larger distance scales relative to the cutoff. To have a sensible interpretation as spacetime fields corresponding to a definite solution of the open string field equation, they should in fact *not* flow, which is to say that they should sit at a renormalization group fixed point. This is the stringy origin of equations of motion for spacetime fields.

A more precise characterization of the fixed point theory can be extracted from some technical string theory considerations: Revert to the particle physics point of view and carry out renormalization by scaling the short distance cutoff to zero, adding counterterm interactions as needed to keep finite scale physics fixed. Let $|B, A\rangle$ be the resulting boundary state. Its construction depends implicitly on a choice of parametrization for the boundary, but in order for $|B, A\rangle$ to be an acceptable state to insert into the closed string worldsheet, it must be independent of parametrization choice. Technically, this is expressed by requiring that $|B, A\rangle$ be annihilated by the combinations of left- and right-moving Virasoro generators that generate reparametrizations of the boundary:

$$(L_n - \tilde{L}_{-n})\,|B, A\rangle = 0\ , \qquad -\infty \leq n \leq \infty\ . \tag{4.1}$$

Since the boundary state is a functional of the oscillator creation operators α^μ_{-n} ($\tilde{\alpha}^\mu_{-n}$) and since the L_n ($\tilde{L}_n$) can be represented as differential operators acting on these variables, (4.1) amounts to a set of reparametrization Ward identities on the boundary state path integral and its renormalization scheme. Detailed examination shows that they include the condition that the vector potential sit at its renormalization group fixed point [8]. Normally, a renormalization group fixed point is associated with global scale invariance (critical point scaling behavior). The point we want to make is that in this one-dimensional problem, scaling is promoted to a higher symmetry, essentially local reparametrization invariance. This is analogous to the appearance of conformal invariance at critical points of two-dimensional theories.

There is a subtlety, however. Despite the fact that the boundary state is annihilated by *all* the worldsheet generators of boundary reparametrizations, it turns out that the one-particle-irreducible Green's functions of the critical one-dimensional theory are strictly invariant only under the finite dimensional subgroup $SL(2,R)$. In fact this is a good thing: strict reparametrization invariance would require the Green's functions to be trivial [10]. This is analogous to the situation in two-dimensional conformal field theory where, because of the conformal anomaly, Green's functions are strictly invariant only under $SL(2,C)$, rather than the full conformal group. The reason here is not the anomaly (the usual conformal anomaly cancels between the left- and right-moving parts of the reparametrization generators $L_n - \tilde{L}_{-n}$) but the peculiar properties of the non-local kinetic term. This is explained in detail in [8] and we will confine ourselves here to an account of the essential physical ideas.

Let us consider the invariance properties of the various terms in the action of the one-dimensional path integral. Under a reparametrization $s \rightarrow t(s)$ a local operator of dimension d transforms as $\mathcal{O}_d(s) = (\frac{dt}{ds})^d\, \widetilde{\mathcal{O}}_d(t)$. The integral of an operator of dimension one is therefore invariant. The coordinate $X^\mu(s)$ behaves like an operator of dimension zero, and its derivative like an operator of dimension one. The vector potential interaction term, S_A, is the integral of a dimension one operator and therefore strictly reparametrization invariant. The non-relativistic kinetic term S_{reg} and the scalar potential term S_V are integrals of a dimension two and a dimension zero operator respectively and therefore *not* invariant. The regulator term is irrelevant (dimension greater than one) and should have no influence on the physics when the regulator scale is taken to zero. The scalar potential term is relevant (perturbative dimension less than one) and must be chosen just right so as not to disturb the invariance of S_A. The mechanics of doing this are detailed in [8] and we will not repeat the arguments here. We simply assert that as far as the local terms in the action are concerned, full reparametrization invariance is a possible symmetry of the one-dimensional theory.

The non-local kinetic, or dissipation, term is another matter. We can readily show that it is invariant only under a finite dimensional subgroup of the reparametrization group. Consider first the non-local term evaluated on the infinite line (the zero temperature case for the dissipative quantum mechanics application):

$$S_{nl}^{\infty} = \frac{1}{8\alpha'\pi^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \, \frac{\left(X(t)-X(t')\right)^2}{(t-t')^2} \,. \tag{4.2}$$

Under a reparametrization $\tilde{t}=f(t)$ the fields transform as $\widetilde{X}(\tilde{t})=X(t)$. The action in terms of the transformed fields is easy to obtain

$$\begin{aligned} S_{nl}^{\infty} &= \frac{1}{8\alpha'\pi^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \, \frac{\left((\widetilde{X}(f(t))-\widetilde{X}(f(t'))\right)^2}{(t-t')^2} \\ &= \frac{1}{8\alpha'\pi^2}\int_{-\infty}^{\infty} d\tilde{t} \int_{-\infty}^{\infty} d\tilde{t}' \, \frac{\left(\widetilde{X}(\tilde{t})-\widetilde{X}(\tilde{t}')\right)^2}{(\tilde{t}-\tilde{t}')^2}\left\{\frac{\left(f(t)-f(t')\right)^2}{(t-t')^2\, f'(t)\, f'(t')}\right\} . \end{aligned} \tag{4.3}$$

The condition for form invariance under reparametrization is thus

$$(t-t')^2\, f'(t)\, f'(t') = \left(f(t)-f(t')\right)^2 . \tag{4.4}$$

With a bit of arithmetic, one shows that the solution of this equation is

$$f(t) = \frac{at+b}{ct+d}\,, \qquad\qquad ad-bc=1\,. \tag{4.5}$$

These are nothing but the $SL(2,R)$ mappings of the real line into itself. The infinitesimal generators of the full reparametrization group are

$$\mathcal{D}_n = t^{n+1}\,\frac{d}{dt}\,, \qquad\qquad -\infty \le n \le \infty \tag{4.6}$$

and the infinitesimal generators of $SL(2,R)$ form the subalgebra

$$\big\{\mathcal{D}_{-1}, \mathcal{D}_0, \mathcal{D}_1\big\} = \big\{\frac{d}{dt}, t\frac{d}{dt}, t^2\frac{d}{dt}\big\}\,. \tag{4.7}$$

The invariance of S^{∞}_{nl} under $\mathcal{D}_{-1}$ and $\mathcal{D}_0$ (translations and global rescaling) is fairly obvious, while the invariance under $\mathcal{D}_1$ is non-trivial. The effect of the non-local kinetic term is therefore to reduce the explicit invariance of the n-point functions of the critical one-dimensional theory from full reparametrization invariance to $SL(2,R)$. As explained in [8], this is perfectly consistent with the boundary state being invariant under the *full* set of boundary reparametrization generators.

The consequences of $SL(2,R)$ invariance are easy enough to state for two-, three- and four-point functions. An $SL(2,R)$ transformation is a reparametrization so the transformation properties of operators under $SL(2,R)$ are determined by their dimension, d:

$$\begin{aligned} \mathcal{O}_d(t) &= \Big(\frac{d\tilde{t}}{dt}\Big)^d\,\tilde{\mathcal{O}}_d(\tilde{t})\,, \\ \tilde{t} &= \frac{at+b}{ct+d}\,, \qquad\qquad ad-bc=1\,. \end{aligned} \tag{4.8}$$

The question of which dimensions, d, are allowed is a dynamical one and the answer depends on the particular theory. It is then rather easy to show that $SL(2,R)$ invariance has the following consequences for the first few n-point functions:

$$\begin{aligned} \langle \mathcal{O}_d(t_1)\mathcal{O}_d(t_2)\rangle &= \frac{c}{(t_1-t_2)^{2d}}\,, \\ \langle \mathcal{O}_{d_1}(t_1)\mathcal{O}_{d_2}(t_2)\mathcal{O}_{d_3}(t_3)\rangle &= \frac{C_{123}}{(t_1-t_2)^{d_1+d_2-d_3}(t_2-t_3)^{d_2+d_3-d_1}(t_3-t_1)^{d_3+d_1-d_2}}\,, \\ \langle \mathcal{O}_{d_1}(t_1)\ldots\mathcal{O}_{d_4}(t_4)\rangle &= \Big\{\prod_{i<j}\frac{1}{(t_i-t_j)^{d_i+d_j-\frac{d}{3}}}\Big\}F\Big(\frac{(t_1-t_2)(t_3-t_4)}{(t_1-t_3)(t_2-t_4)}\Big)\,, \end{aligned} \tag{4.9}$$

where $d=\sum_i d_i$ and $F(x)$ is an arbitrary function. Note that if one were to impose invariance under the full reparametrization group the only possible solution would be that all these Green's functions vanish identically.

For reference, we show how this works for the case where the action is defined on a circle (appropriate for finite temperature dissipative quantum mechanics or string theory applications). On a periodic interval of length 2π, the non-linear term in the action is

$$S_{nl}^{2\pi} = \frac{1}{8\alpha'} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\theta'}{2\pi} \, \frac{\left(X(\theta) - X(\theta')\right)^2}{\left[\sin\left(\frac{\theta-\theta'}{2}\right)\right]^2} \,. \tag{4.10}$$

Now consider a reparametrization of the interval, $\tilde{\theta} = f(\theta)$. The boundary fields transform as scalars, $\widetilde{X}^\mu(\tilde{\theta}) = X^\mu(\theta)$ and it easily follows that $S_{nl}^{2\pi}$ is form invariant under (4.8) provided that $f(\theta)$ is restricted by

$$\sin^2\left(\frac{f(\theta) - f(\theta')}{2}\right) = \sin^2\left(\frac{\theta-\theta'}{2}\right) f'(\theta)\, f'(\theta') \,. \tag{4.11}$$

This condition is satisfied by mappings in the $SU(1,1)$ subgroup of the Möbius transformation group:

$$\tilde{z} = \frac{az+b}{\bar{b}z+\bar{a}} \,, \qquad\qquad |a|^2 - |b|^2 = 1 \,. \tag{4.12}$$

They map the unit circle into itself and if we set $z{=}e^{i\theta}$ and $\tilde{z}{=}e^{i\tilde{\theta}}$, (4.12) defines a reparametrization of the circle, $\tilde{\theta}{=}f(\theta)$. So, for the one-dimensional theory defined on the circle, the manifest symmetry of the Green's functions of the critical theory will be $SU(1,1)$ restricted to the unit circle, which is a three-parameter subgroup of the full reparametrization group.

The points we are trying to make can be summarized as follows: A cutoff one-dimensional field theory underlies both the string theory boundary state and dissipative quantum mechanics. In the string theory application it is necessary to remove the cutoff, or to take the infrared limit, and adjust the couplings so that the theory sits at a renormalization group fixed point. In dissipative quantum mechanics such fixed points should correspond to points of transition between qualitatively different long time behavior (localized *vs.* delocalized, as we shall see) and are of course of considerable interest. One knows that, when viewed from string theory, the infrared, or renormalized, one-dimensional theory manifests an enhanced symmetry related to reparametrization invariance. Roughly speaking we expect the fixed point to possess the symmetry left unbroken by the marginal terms in the action. Marginal *local* operators leave full reparametrization invariance unbroken, but the non-local kinetic term breaks it down to a finite dimensional subgroup ($SL(2,R)$ or $SU(1,1)$) of reparametrizations. Thus the Green's functions of the critical theory, whatever its interpretation, will be explicitly invariant under a finite-dimensional subgroup of the reparametrization group (but larger than just translations and global rescaling). In the string theory application we furthermore must insist that the boundary *state* be fully reparametrization invariant which results in a set of Ward identities for the one-dimensional field theory [8] (with the requirement of $SL(2,R)$ invariance contained as a subset of these symmetry conditions). The infinite dimensional symmetry associated with the critical theory can perhaps be utilized to directly construct non-trivial fixed point theories in one dimension, although since the symmetry is broken at the classical level it is not clear to us at present how to proceed.

The two points we hope eventually to exploit are, first, that string theory is identical to critical dissipative quantum mechanics and, second, that such one-dimensional critical theories possess an extended symmetry related to reparametrizations. The latter point seems not to have been noticed in the condensed matter literature on critical dissipative quantum mechanics. The former connection is apparently completely new.

5. Mobility, Delocalization and All That

Our goal is to find interesting examples of the critical points discussed in the previous sections. Some small steps in the right direction can be made by consulting the condensed matter and string theory literature. In the former, attention has focussed on the periodic scalar potential which can describe either an electron (or muon) moving in a lattice or the trapped flux in a SQUID. The central issue there has been localization: In standard quantum mechanics, no matter how strong the periodic potential, because of quantum tunneling, the particle diffuses as if it were free on long time scales. The question is whether dissipation can undo this coherent quantum effect and produce localized long-time behavior. In the string literature, the focus has been on the effects of the vector potential (spacetime gauge field) with the scalar potential (open string tachyon) regarded as an ancillary nuisance. All that has been achieved is an exact treatment of the gauge field strength case plus the evaluation of a few orders in a perturbative expansion in α'. Larger questions of, say, the existence and properties of soliton sectors remain untouched. In this section we will collect a few results from the condensed matter literature, put them into the more general context we have been discussing and try to draw some conclusions for string applications.

The condensed matter literature defines a useful quantitative measure of localization called mobility. Consider our one-dimensional theory defined on the open line and define

$$\begin{aligned} K(t) &= \langle (X(t)-X(0))^2 \rangle \\ &= \int \frac{d\omega}{\pi}\, \mathrm{tr}\, S(\omega)\,(1-\cos\omega t) \end{aligned} \tag{5.1}$$

where $S(\omega)$ is the frequency-space propagator and the trace is over coordinate-space indices. If the long-time behavior of this quantity is bounded by a constant, the particle is localized; if it grows without limit, the particle is delocalized. In the condensed matter literature, logarithmic growth, obviously a transition between the two extremes, is regarded as the sign of critical behavior and the numerical value of the critical mobility is defined as the coefficient of the logarithm. (Since the conditions for a fixed point include Ward identities for all n-point functions, we should perhaps bear in mind that things could be more complicated.) It is instructive to compute the mobility in a constant homogeneous abelian electromagnetic field, an exactly soluble example of considerable importance in string theory.

In a convenient gauge, the interaction term for this theory has the simple form

$$S_A = \int dt \, \frac{1}{2} F_{\mu\nu} X^\mu \dot{X}^\nu \, . \tag{5.2}$$

The other terms in the action are also simple quadratics and can easily be Fourier transformed to yield the frequency-space inverse propagator:

$$S^{-1}_{\mu\nu}(\omega) = (M\omega^2 + \eta|\omega|)\delta_{\mu\nu} - \omega F_{\mu\nu} \, . \tag{5.3}$$

The first term comes from the standard kinetic term (regulator), the second from the non-local kinetic term and the third from the gauge potential. Both the non-local kinetic and the gauge terms scale as the first power of ω, as is appropriate for dimension-one operators. The non-local kinetic term is distinguished from the local operators by its non-analytic behavior (via the absolute value instruction) at $\omega = 0$. This non-analyticity, as we shall see, has a critical effect on long-time behavior. The explicit expression for $K(t)$ is

$$\begin{aligned} K(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{\pi} (1 - \cos\omega t) \, \mathrm{tr} \left\{ \frac{1}{M\omega^2 + \eta\,|\omega| + \omega F} \right\} \\ &= \int_0^{\infty} \frac{d\omega}{\pi\omega} (1 - \cos\omega t) \, \mathrm{tr} \left\{ \frac{1}{M\omega + \eta + F} + \frac{1}{M\omega + \eta - F} \right\} \\ &= \int_0^{\infty} \frac{d\omega}{\pi\omega} (1 - \cos\omega t) \sum_{a=1}^{\frac{D}{2}} \frac{4(M\omega + \eta)}{(M\omega + \eta)^2 + f_a^2} \end{aligned} \tag{5.4}$$

where if_a are the eigenvalues of the field strength matrix and D is the spacetime dimension. For generic values of the parameters, it is easy to estimate that as

$t \to \infty$

$$K(t) \to \frac{4}{\pi} \log t \sum_{a=1}^{\frac{D}{2}} \frac{\eta}{\eta^2 + f_a^2} \, . \tag{5.5}$$

In the limit $\eta \to 0$, we can evaluate $K(t)$ exactly, with the result

$$K(t)\big|_{\eta=0} = \sum_{a=1}^{\frac{D}{2}} \frac{1}{f_a} \left(1 - e^{-f_a t/M}\right) \, . \tag{5.6}$$

In the further limit that $F_{\mu\nu}$ is turned off, we have

$$K(t)\big|_{F,\eta=0} = \frac{D}{2M} t \, . \tag{5.7}$$

The interpretation of these results is simple. For $\eta{=}F{=}0$, we recover the standard quantum mechanics of a free massive particle whose coordinate undergoes Brownian motion ($K(t) \to t$). This is delocalized behavior. For $\eta{=}0$ but $F_{\mu\nu}{\neq}0$ we are dealing with the standard quantum mechanics of a particle in a constant magnetic field. We of course expect the particle to be stuck in a particular Landau orbit and to manifest localized behavior. In fact, we find $K(t) \to const.$, with the value of the constant essentially given by the size of a Landau orbit. In the generic case , $K(t)$ grows logarithmically with a coefficient (the mobility) which depends on the non-zero values of η and $F_{\mu\nu}$. Roughly speaking, the coupling to the bath of oscillators (dissipation) causes the particle to wander slowly among the Landau orbits. In addition, we know from other string theory work [7], that the boundary state built on a constant Abelian gauge field strength satisfies all the reparametrization invariance Ward identities. This guarantees that the underlying one-dimensional field theory is at a critical point and verifies that logarithmic growth of $K(t)$ is characteristic of the transition between localized and delocalized behavior. It is easy to see that operator n-point functions of the theory satisfy the requirements of $SL(2,R)$ invariance, albeit in a rather trivial way. Since the action is quadratic, only the connected two-point function is non-zero. As usual,

X^μ does not have a well-defined dimension, but $\dot{X}^\mu$ behaves like a dimension-one operator. By taking derivatives of $K(t)$, we see that

$$\left\langle \dot{X}(t)\dot{X}(0)\right\rangle \sim \frac{1}{t^2} \tag{5.8}$$

which is just what is required by $SL(2,R)$ invariance for dimension-one operators.

Another example which has been studied rather thoroughly, this time in the condensed matter context, is that of dissipative quantum mechanics in the presence of a periodic scalar potential. In string language, this is the problem of open strings in the presence of a periodic open string tachyon background, but no gauge field. In the absence of dissipation, quantum tunneling makes all states delocalized and the question is whether sufficiently strong dissipation creates localized states. (Note that this is more or less the inverse of the story for the background gauge field where the standard quantum-mechanical problem had localized states and arbitrarily small dissipation caused delocalization.). Since the potential is not quadratic, the problem is not exactly soluble and approximations must be made. Several authors [2,3,4] have studied this problem, all concluding that there is indeed a localization transition. We will briefly summarize the renormalization group argument of Fisher and Zwerger [4] and try to show how it fits into the more general context we have been developing.

To be precise, we wish to study the problem defined by the Euclidean action $S_0 + S_1$ with

$$\begin{aligned} S_0 &= \frac{\eta}{4\pi}\int_{-\infty}^{\infty} dt\, dt' \, \frac{\left(X(t)-X(t')\right)^2}{(t-t')^2} + \frac{M}{2}\int_{-\infty}^{\infty} dt\ \dot{X}(t)^2 \ , \\ S_1 &= V\int_{-\infty}^{\infty} dt\ \cos\!\left(2\pi\frac{X(t)}{X_0}\right) . \end{aligned} \tag{5.9}$$

For simplicity, we imagine here that X has only one component. It is convenient to write S_0 in momentum-space form:

$$S_0 = \frac{1}{2}\int \frac{d\omega}{2\pi}\, |\tilde{X}(\omega)|^2\, S_\Lambda(\omega)\ ,$$
$$S_\Lambda = \frac{\gamma|\omega|}{2\pi} + \frac{\omega^2}{\Lambda}\ , \tag{5.10}$$
$$\gamma = \frac{\eta X_0^2}{2\pi}\ , \qquad \Lambda = \frac{4\pi^2}{M X_0^2}\ .$$

The constant γ is a dimensionless measure of dissipation and Λ is a large cutoff frequency. It is simplest to replace S_Λ^{-1} by $S^{-1} = \frac{\gamma}{2\pi}|\omega|$ and instead cut off all frequency integrals at Λ. The renormalization group transformation amounts to replacing Λ by a smaller cutoff μ by integrating out modes with frequency $\mu \leq \omega \leq \Lambda$ at the price of changing the parameters V and γ. The approximation consists in only including diagrams that are first-order in V in the integrating-out process. The calculation is summarized in Figure 1. The propagators are coincident-point two-point functions including only the integrated-out frequencies and have the value $G = \frac{2}{\gamma}\log(\Lambda/\mu)$. The resulting flow equation for V is

$$V_\mu = V_\Lambda(\frac{\Lambda}{\mu})e^{-\frac{1}{2}G} = V_\Lambda\,(\frac{\mu}{\Lambda})^{\frac{1}{\gamma}-1}\ . \tag{5.11}$$

In this approximation, and probably in an exact treatment [3], γ does not flow [4]. If $\gamma<1$, V_μ scales to zero in the infrared limit, the neglect of higher powers of V is justified and the physics at long time scales is surely delocalized. On the other hand if $\gamma>1$, V_μ grows in the infrared limit, the neglect of higher powers of V is not justified and some totally different infrared fixed point, which we are not in a position to describe, but which probably corresponds to localized behavior, will be reached. The value $\gamma=1$ would appear to be a critical point (indeed, in this approximation, a critical line, since V can take any value) for delocalization. This is in some respects a familiar story: potentials with a short enough spacetime period ($\gamma<1$) are irrelevant (scaling dimension <1) while potentials with long enough period ($\gamma>1$) are relevant and potentials with a critical period ($\gamma=1$) are marginal.

According to what we have said earlier, the Green's functions of the critical theory should possess explicit $SL(2,R)$ invariance. More precisely, they should satisfy an infinite set of Ward identities which guarantee that the string boundary state built on this theory is strictly reparametrization invariant. To show this is actually quite a challenging problem as, unlike in the critical theory based on constant background gauge field strength, there are nontrivial interactions and nontrivial higher-point Green's functions. A possible strategy for analysing this question relies on the fact that, at the critical value of γ, the interaction term is of dimension one and should be equivalent to a fermion bilinear of the $\tilde{\psi}\psi$ type by the usual bosonization argument. In fermionic language the action would be quadratic and therefore exactly soluble. This is essentially the tack taken by Guinea *et.al.* [3]. They evaluate the mobility two-point function $K(t)$ and show that it behaves logarithmically, just as we found for the constant gauge field strength case, with a mobility coefficient which depends on the potential strength V in a simple way. For the reasons outlined above, this is consistent with $SL(2,R)$ invariance at the two-point function level. We are trying to extend this argument to a demonstration that the boundary state for this critical theory satisfies the full set of reparametrization invariance Ward identities. We will report our results elsewhere. In the Appendix we will show, using the techniques of [8], that these Ward identities are indeed satisfied at the level of approximation of the renormalization group calculation of [4].

6. Discussion

Although we have hardly given a proof, we think we have offered significant evidence that open string theory and dissipative quantum mechanics at an infrared fixed point are essentially the same subject. The search for such infrared fixed points is a condensed matter problem with a fairly lengthy history and a modest record of success. The string theory connection implies that the dissipative quantum mechanics fixed point theory possesses a high degree of symmetry (beyond the expected scale and translation invariance) inherited eventually from the conformal invariance of string theory. One might hope that symmetry considerations would permit the direct construction of nontrivial fixed point theories, much as conformal invariance permits the construction in two dimensions of the $c<1$ discrete series and its relatives. At the moment, we have no concrete proposals along these lines, but a clearer understanding of the symmetry properties of a theory must make it easier to study its solutions.

The string theory connection also points out the importance of studying the dissipative quantum mechanics problem in the presence of gauge vector potentials. The reason, of course, is that in the string interpretation of this system, the vector potential corresponds to spacetime gauge fields while the scalar potential corresponds to spacetime (open string) tachyon fields. A fixed point theory with nontrivial values of the potentials corresponds to a classical ground state of the open string field theory with the corresponding spacetime distribution of gauge and tachyon fields. While one could look for states with only the tachyon field excited, our main interest is in theories with a nontrivial spacetime distribution of gauge fields and a monopole or instanton interpretation. In particular, one would like to know whether string field theory possesses topological charge sectors, and, if so, whether there exists something like a Bogomolny bound for the action in a given sector. (The latter question is particularly interesting since, contrary to ordinary field theory, there is no universal analytic form for the action of a given configuration.) Little or nothing is currently known about these issues and we hope that this paper will help to stimulate research in the right directions.

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APPENDIX

Ward Identity for Periodic Scalar Potential

As an exercise, we will evaluate the full set of reparametrization invariance Ward identities, as defined in [8], for a theory whose interactions come from a scalar potential only. The Ward identities are calculated to first order in the scalar potential but diagrams with any number of loops are included. This is the order of approximation used by Fisher and Zwerger in their approximate construction [4] of the renormalization group fixed point for this theory. We find that the scalar potential must be periodic for the Ward identities to hold, with the same period as required by the renormalization group argument. Thus, at the fixed point, an infinite number of generators are conserved, a fact that would not have been apparent from the renormalization group argument alone. We do the calculation in a pedestrian way with a naive cutoff to give an instructive example of how regulation and renormalization are dealt with. This should be a useful amplification of [8], where we used special methods which don't work, for instance, for fermions or for higher loop orders. The diagrammatic approach described here should be applicable in all cases (though it may be unpleasant to work out explicitly).

The reparametrization Ward identities derived in [8] are

$$0 = \delta_n\Gamma - \sum_{m=1}^{n-1} \tfrac{1}{2}m(n-m)\,\phi_m\cdot\phi_{n-m} - \sum_{m=1}^{n-1} \tfrac{1}{2}m(n-m)\,\frac{\partial^2 W}{\partial\alpha_{-m}\cdot\partial\alpha_{m-n}} + in\frac{\partial^2 W}{\partial X_0\cdot\partial\alpha_{-n}}, \tag{A.1}$$

where Γ is the effective action, W is the connected generating functional and δ_n denotes a variation under an infinitesimal reparametrization by $f_n(s)=ie^{ins}$. According to (A.1) the effective action is not required to be reparametrization invariant (the naive result). The situation is complicated by the fact that the symmetry is broken at the classical level and the Ward identity expresses this broken invariance. The second and third terms in (A.1) give the instruction to subtract off the explicit variation of the classical kinetic term and all diagrams with that variation inserted. Finally the last term in (A.1) comes from the response of the zero mode projection to the reparametrization. (These matters are discussed in more detail in [8].)

We will explicitly calculate the terms in (A.1) to first order in the scalar potential strength, V, using a perturbative background field expansion. The interaction vertices are obtained by Taylor expanding $V(X)$ around the background value of the field X:

$$\begin{aligned}\int \tfrac{ds}{2\pi}\, V(X) &= \int \tfrac{ds}{2\pi} V(\phi+\pi) \\ &= \int \tfrac{ds}{2\pi}\, V(\phi) + \int \tfrac{ds}{2\pi}\, \nabla_\mu V(\phi)\, \pi^\mu \\ &\quad + \tfrac{1}{2} \int \tfrac{ds}{2\pi}\, \nabla_\mu \nabla_\nu V(\phi)\, \pi^\mu \pi^\nu + \dots \ .\end{aligned} \tag{A.2}$$

The non-local kinetic term defines our propagator. A little manipulation of the expression in (3.6) allows us to read off the inverse propagator, H_0, in momentum space,

$$S_0[X] = \tfrac{1}{2} \int \tfrac{ds}{2\pi} \tfrac{ds'}{2\pi}\, X^\mu(s) \Big(\sum_{-\infty}^{+\infty} \frac{|m|}{\alpha'} e^{im(s-s')} \Big) X^\mu(s') \ . \tag{A.3}$$

The variation of S_0 under a reparametrization by $f_n(s)$ can be expressed in terms of H_0 in a simple way

$$\begin{aligned}\delta_n S_0[X] &= \frac{1}{\alpha'} \sum_{m=1}^{n-1} m(n-m)\, X_m \cdot X_{n-m} \\ &= -\tfrac{1}{2} \int \tfrac{ds}{2\pi} \tfrac{ds'}{2\pi}\, X^\mu(s) \big[\tfrac{d}{ds} \big(f_n(s) H_0(s-s') \big) - H_0(s-s') f_n(s') \tfrac{d}{ds'} \big] X^\mu(s') \ .\end{aligned} \tag{A.4}$$

A key observation is that the diagrams which contribute to the third term in the Ward identity (A.1) are obtained by inserting the operator in the square brackets in (A.4) into the 1PI diagrams in Γ. (An insertion of $\delta_f H_0 = \frac{d}{ds} f_n H_0 - H_0 f_n \frac{d}{ds}$ on a leg of an N loop 1PI diagram effectively cuts that leg open to give an (N-1) loop contribution to the propagator $\frac{\partial^2 W}{\partial \alpha^2}$, precisely of the form found in (A.1).)

Before proceeding to an explicit evaluation of diagrams we must specify a regulator. Rather than use the nonrelativistic kinetic term suggested in the body of the paper, we find it more convenient for calculations to define a regulated propagator G_0^ϵ by cutting off the high frequency mode contributions in the following way [11]:

$$\begin{aligned} G_0^\epsilon(s) &= \sum_{m\neq 0} \frac{\alpha'}{|m|} e^{ims-|m|\epsilon} \\ &= -\alpha' \log\left(1 - 2e^{-\epsilon}\cos s + e^{-2\epsilon}\right) . \end{aligned} \tag{A.5}$$

Note that the propagator does not involve the zero frequency mode and that for finite cutoff it is not the exact inverse of H_0:

$$\begin{aligned} \int \tfrac{ds''}{2\pi}\, G_0^\epsilon(s-s'')H_0(s''-s') &= \sum_{m\neq 0} e^{im(s-s')-|m|\epsilon} \\ &= \frac{\sinh\epsilon}{\cosh\epsilon - \cos(s-s')} - 1 \\ &= \Delta_\epsilon(s-s') - 1 \\ &\rightarrow 2\pi\delta(s-s') - 1 \quad \text{as } \epsilon \rightarrow 0 \ . \end{aligned} \tag{A.6}$$

In the limit, modulo the effect of the zero mode, we get a delta function as we should.

Now we are ready to analyze the Ward identity diagrams. The first term in (A.1) is the response of $\Gamma[\phi]$ to an infinitesimal reparametrization. We are working only to first order in V so all propagators have to be contracted on a single vertex and the N loop contribution to Γ is given by a single "daisy diagram" with N petals, as shown in Figure 2. The vertex comes from the background field expansion of $V(X)$ at order $2N$ and the Feynman rules give for this diagram

$$\Gamma_N = -\tfrac{1}{2^N N!} \int \tfrac{ds}{2\pi}\, \Box^N V(\phi(s))\ \left[G_0(0)\right]^N \ . \tag{A.7}$$

From the one-dimensional point of view $\Box^N V(\phi)$ is a scalar field so the reparametrization variation of Γ_N is simply

$$\delta_f \Gamma_N = \frac{1}{2^N N!} \int \frac{ds}{2\pi} f'(s) \Box^N V(\phi(s)) \left[G_0(0)\right]^N \quad . \tag{A.8}$$

The second term in the Ward identity (A.1) precisely cancels the explicit variation of the kinetic term in the tree-level effective action (but not the tree-level variation of the scalar interaction term).

The N loop contribution to the third term in (A.1) is represented by a daisy diagram with an insertion of the operator $\delta_f H_0$ on one petal, see Figure 3. There are N petals to choose from so the diagram takes the value

$$\begin{aligned} D_N &= \frac{-1}{2^N (N-1)!} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \frac{ds''}{2\pi} \Box^N V(\phi(s)) \left[G_0(0)\right]^{N-1} G_0(s-s') \delta_f H_0(s'-s'') G_0(s''-s) \\ &= \frac{1}{2^N (N-1)!} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} \frac{ds''}{2\pi} \Box^N V(\phi(s)) \left[G_0(0)\right]^{N-1} \\ &\qquad \times 2 \frac{d}{ds} G(s-s') f(s') H_0(s'-s'') G_0(s''-s) \, . \end{aligned} \tag{A.9}$$

We have to remember that H_0 and G_0 are not exact inverses. There is a residual constant term in (A.6) due to the zero mode and Δ_ϵ is only an approximation to the delta function for finite cutoff. Introducing some shorthand notation

$$g(s) = \left[G_0(0)\right]^{N-1} \Box^N V(\phi(s))$$

we can write

$$D_N = \frac{1}{2^{N-1} (N-1)!} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} g(s) f(s') \left(\Delta(s-s') - 1\right) \frac{d}{ds} G_0(s-s') \, . \tag{A.10}$$

Consider first the piece of this containing the constant part of $G_0 H_0$

$$\begin{aligned} &\frac{-1}{2^{N-1} (N-1)!} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} g(s) f(s') \frac{d}{ds} G_0(s-s') \\ &= \frac{1}{2^{N-1} (N-1)!} \int \frac{ds}{2\pi} \frac{ds'}{2\pi} g(s) G_0(s-s') f'(s') \, . \end{aligned} \tag{A.11}$$

This cancels against the (N-1) loop contribution to the last term in the Ward identity (A.1). That can be seen as follows. First note that

$$in \frac{\partial^2 W}{\partial X_0 \cdot \partial \alpha_{-n}} = - \int \tfrac{ds}{2\pi} f_n'(s) \frac{\partial}{\partial \phi_0} \cdot \frac{\delta W}{\delta \alpha(s)} \tag{A.12}$$

where $\alpha(s)$ is the linear source in (3.6). By definition $\frac{\delta W}{\delta \alpha(s)}$ is the sum of all connected graphs with one external source removed. In terms of the 1PI functional and to first order in V that means daisy diagrams with one open ended external leg. The $\frac{\partial}{\partial \phi_0}$ operation acts on the vertex and simply increases the number of target space derivatives on $V(\phi)$ by one. The (N-1) loop contribution to (A.12) is therefore given by the diagram shown in Figure 4. It takes the value

$$\tfrac{1}{2^{N-1}(N-1)!} \int \tfrac{ds}{2\pi} \tfrac{ds'}{2\pi} f'(s)\, G_0(s-s') \Box^N V(\phi(s)) \left[G_0(0)\right]^{N-1} \tag{A.13}$$

which precisely cancels against (A.11).

The remaining contribution from (A.10) is

$$\tfrac{1}{2^{N-1}(N-1)!} \int \tfrac{ds}{2\pi} \tfrac{ds'}{2\pi}\, g(s) f(s')\, \Delta(s-s')\, \tfrac{d}{ds} G_0(s-s') \; . \tag{A.14}$$

Since Δ_ϵ approximates the delta function we can evaluate $\frac{d}{ds}G_0$ for small values of its argument. To leading order in ϵ, we may therefore make the replacements

$$\begin{aligned} \Delta(s) \to & \quad \frac{2\epsilon}{\epsilon^2+s^2} \; , \\ \tfrac{d}{ds} G_0(s) \to & -\frac{2\alpha' s}{\epsilon^2+s^2} \; . \end{aligned} \tag{A.15}$$

Writing $s_+ = \frac{1}{2}(s+s')$ and $s_- = s-s'$ we obtain after some simple manipulations

$$\begin{aligned} & \tfrac{4\alpha'}{2^{N-1}(N-1)!} \int \frac{ds_+}{2\pi} f'(s_+) g(s_+) \int \frac{ds_-}{2\pi} \frac{\epsilon s_-^2}{(\epsilon^2+s_-^2)^2} + O(\epsilon) \\ & \to \tfrac{\alpha'}{2^{N-1}(N-1)!} \int \tfrac{ds}{2\pi} f'(s)\, g(s) \; . \end{aligned} \tag{A.16}$$

The N loop contribution to the Ward identity is given by the sum of (A.8) and (A.16). The perturbation series (to first order in V) can be neatly summed up if we combine the N loop contribution to (A.16) with the (N-1) loop piece of (A.8). (The one-loop diagram with a $\delta_f H_0$ insertion is paired with the tree-level variation of the effective action, and so on.) Then the Ward identity for a reparametrization by $f(s)$ is written

$$\begin{aligned} 0 &= \sum_{N=0}^{\infty} \tfrac{1}{2^N N!} \left[G_0(0)\right]^N \int \tfrac{ds}{2\pi} f'(s) \Box^N \left(1+\alpha'\Box\right) V\left(\phi(s)\right) \\ &= \int \tfrac{ds}{2\pi} f'(s)\, e^{-\log(1-e^{-\epsilon})\alpha'\Box} \left(1+\alpha'\Box\right) V\left(\phi(s)\right) \quad . \end{aligned} \tag{A.17}$$

This must hold for any choice of $f(s)$ which imposes the condition that $V(\phi)$ be periodic, with the same period as required by the simple renormalization group calculation of Fisher and Zwerger [4], which we reproduced in Section 5. The point we wish to make here is that the fixed point theory satisfies the Ward identities of an infinite-parameter symmetry group, not just global scale invariance (at least to the approximation we have been able to calculate).

FIGURE CAPTIONS

1) These diagrams summarize a renormalization group calculation for DQM in a periodic potential when we neglect contributions beyond first order in V.

2) Working only to first order in V the N-loop contribution to the effective action is given by a single daisy diagram.

3) The reparametrization Ward identity receives a contribution from diagrams with the operator $\delta_f H_0 = \frac{d}{ds} f_n H_0 - H_0 f_n \frac{d}{ds}$ inserted on a leg.

4) This diagram represents the last term in the Ward identity (A.1) which comes from the explicit breaking of reparametrization invariance due to the zero mode projection.

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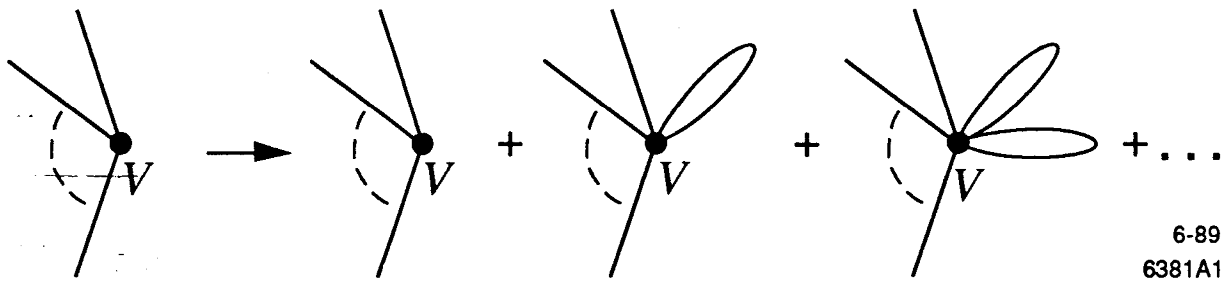

Fig. 1

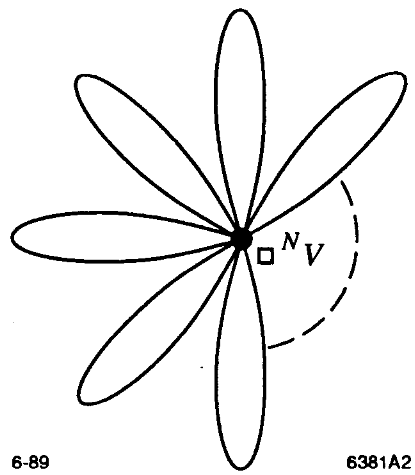

Fig. 2

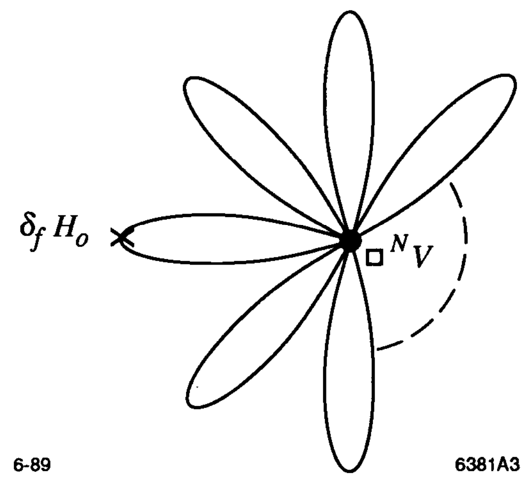

Fig. 3

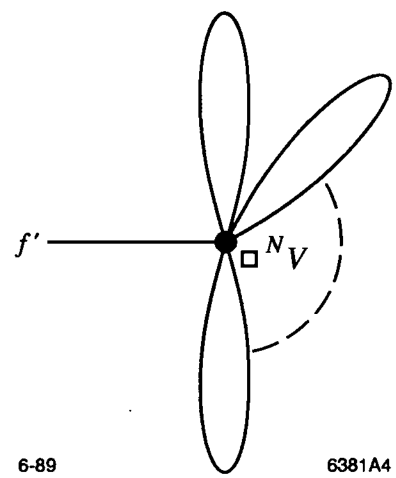

Fig. 4