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PAIR PRODUCTION FROM PHOTON-PULSE COLLISIONS*

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ABSTRACT

The high energy expansion for extended targets developed previously is used to describe to the problem of electron pair production by photons, a problem of interest in linear collider design. The treatment is valid throughout the range of possible parameters, from the classical regime to the extreme quantum domain. The differences between this process and pair production in a constant external field are stressed and the physics of the process discussed.

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1. Introduction

Electromagnetic radiation from high energy charged particles as they traverse strong external fields has been well studied, including the transition between classical and quantum regimes. The process of pair creation by photons, related by crossing to the above reaction, has also been well studied. Interest in these processes has been revived in recent years by prospects of building high energy linear colliders for electron and positron beams in the TeV region. In particular, the fractional energy loss to radiation as one beam pulse is accelerated by the electromagnetic field of a pulse through which it crosses at the interaction point is an important parameter in the design of a collider; this fractional loss is known as *beamstrahlung*.[1,2] Himel and Siegrist[3] called attention to the quantum regime for beamstrahlung and treated it by adapting earlier quantum calculations of synchrotron radiation by an electron in a uniform magnetic field. Following the important paper by Himel and Siegrist, there has been a flurry of activity[4–7] seeking simpler and more general means of calculation in order to provide better insights into the corrections due to field inhomogeneities and different charge distributions in the pulses, as well as to develop a better physical understanding of the process and the scaling laws characterizing the transition from the classical to the quantum regime.[4]

A study of the effects of pulse geometry[8] and the development of a quantum theory of multiple photon emission[9] has led to a suggestion of a collider that can operate not only as an e^+e^- machine but also as a photon-electron and a photon-photon collider, by modifying the pulse shape.[10]

Pisin Chen[11] has recently called attention to the potential importance of pair production by the beamstrahlung photons as they traverse the beam pulses in which they are produced. Using the results of Baier and Katkov[12] and Tsai and Erber,[13] Chen showed that this effect could lead to serious background difficulties under certain conditions. This suggestion stimulated us to calculate the probability of pair production by photons using the direct approach of high energy scattering theory developed in Ref. 4. The results of this investigation are the subject of this paper. We confirm previously published results in regions of overlap, extend them to more realistic pulse geometries, and provide a simple physical picture of this process and of its relation via crossing[14] to beamstrahlung. Working independently, Jacob and Wu[15] have also developed a high energy scattering approach to beamstrahlung and to pair production in the extreme quantum limit of beam-beam interactions.[#1]

Our work is valid in all regimes consistent with the assumption of small disruption, i.e. small angular deflection of the electrons as they traverse the other beam pulse; in particular, it contains the extreme quantum and the classical cases as limits. The parameters for most conceptual designs for conceivable TeV colliders[16–18] are such that they operate in the transition region between the quantum region and the classical limit. Since this regime has not been investigated, we will pay particular attention to it through our choice of "typical" collider parameters. The next step in estimating the effects of pair production would be to fold the photon production spectrum of Ref. 9 with the pair production probabilities derived here to include the effects of multi-photons and multi-pairs. We plan to discuss these topics in a later paper.

#1 Our result, which satisfies crossing symmetry, differs by a numerical factor from that of Jacob and Wu who treated only the spin zero case. We all agree for beamstrahlung. However, our result for the case of scalar pair production is larger by $8/2^{1/3} \sim 6.35$ in the extreme quantum limit that they treated.

It is useful to recall the results of beamstrahlung from Ref. 4. Denoting the ratio of the final to initial electron energies by $x \equiv p_f/p_i$, so that the photon energy is $k = (1-x)p_i$, the differential probability of a *spinless* electron emitting a photon and ending up with momentum fraction x is given by

$$\frac{dP(beam)}{dx} = 2\,\alpha\, y\, C_b\, G(u_b)\ , \tag{1.1}$$

where

$$G(u) = \frac{1}{2u} \int_u^\infty dv \left[3v - 2u - \frac{u^4}{v^3}\right] Ai(v)\ . \tag{1.2}$$

$Ai(v)$ is the Airy function,[19] and the variables are given by

$$u_b \equiv \left[C_b \left(\frac{1-x}{x}\right)\right]^{2/3} \qquad\qquad y = \frac{N\alpha}{mB}\ . \tag{1.3}$$

The scaling variable C_b can be expressed in the pulse rest frame and the center-of-mass frame respectively as

$$C_b = \frac{m^2 L}{2yp_i} = \frac{m\ell_0}{4\gamma y}\ . \tag{1.4}$$

In the center-of-mass frame the length of the pulse is ℓ_0 and γm is the incident electron energy. The corresponding quantities in the pulse rest frame are $L = \gamma\ell_0$ and $p_i = 2\gamma^2 m$. Our calculation will be carried out in this latter frame.

The subscript b emphasizes that u_b and C_b are defined for the beamstrahlung process. All of these scaling variables will play an important role in our development. The quantity y is proportional to the square root of the luminosity per pulse;

values of $y \sim 10^2 - 10^3$ are envisaged for future high energy colliders. Following Ref. 4, the average fractional energy loss is written as

$$\delta = \frac{2}{3}\,\alpha\, y\, \frac{F(C)}{C} = \delta_{classical}\, F(C)\ , \tag{1.5}$$

The scaling variable $C_b \to \infty$ in the classical limit, where $F(C_b \gg 1) \to 1$, whereas $C_b \to 0$ in the extreme quantum limit, where $F(C_b \ll 1) \sim 0.83\, C_b^{1/3}$.

These equations were derived for a particular geometry of the collision process, namely a head-on collision of an electron with a uniformly charged cylindrical pulse of radius B containing N positrons.[20] Only collisions with impact parameter $b < B$ are included since these are the ones of interest for studying energy loss to radiation in e^+e^- colliders. End effects are also neglected as small since $B \ll L$. The full contributions for spin 1/2 electrons will be discussed in the next section.

Physical Interpretation:

There is a simple way to understand the general form of the main results for both beamstrahlung and pair production. The transverse coherence length of the radiation, $\ell_\perp$, is defined as the path length of the electron corresponding to its acquiring a transverse momentum $\sim m$ from the electric field. Since the widths of the photon radiation pattern and the pair production patterns are also $\sim m$, the radiation can be coherent only from a finite length of the curving path, namely

$$\ell_\perp \sim \left|\frac{m}{eE_\perp}\right| \sim \frac{L}{2y} = \frac{\ell_0}{2y}\gamma\,. \tag{1.6}$$

The *longitudinal* coherence (or radiation) length ℓ_z, is related by the uncertainty principle to the reciprocal of the longitudinal momentum transfer:

$$\ell_z \sim \frac{1}{|q_z|} \sim \frac{p}{m^2} \tag{1.7}$$

This is the length of the target that the electron scatters from coherently during the radiation process. The ratio of Eqs. (1.6) and (1.7) leads to (1.4) and to the physical interpretation of C_b as discussed in prior references.

The occurrence of the combination $C_b(1-x)/x)$ in (1.3) can be understood by considering the minimum value of the longitudinal momentum transfer for a given momentum fraction x. Invoking energy conservation, $E_i = k + E_f$, we find

$$\begin{aligned} q_z^{min} &= p_i - k - p_f \approx \frac{m^2}{2p_i} + \frac{m^2}{2p_f} \\ &= \frac{m^2 k}{2p_i p_f} = \frac{m^2}{2p_i}\left(\frac{1-x}{x}\right)\,. \end{aligned} \tag{1.8}$$

This suggests a corresponding scaling parameter

$$C_x \equiv \ell_\perp q_z^{min} = C_b\left(\frac{1-x}{x}\right) \equiv u^{3/2} . \tag{1.9}$$

We can now consider the process of pair production by an incident photon which is related to beamstrahlung by crossing symmetry. *Crossing* involves the continuation in the matrix element from an outgoing to an incoming photon, and the reverse for the incident electron:

$$\begin{gathered} p_i \rightarrow -p_+ \qquad\qquad p_f \rightarrow +p_- \\ k \rightarrow -k \quad . \end{gathered} \tag{1.10}$$

Under this transformation, it is easy to see that the scaling variable becomes

$$\begin{aligned} C_x &= \ell_\perp * (m^2/p_z^f - m^2/p_z^i) = C_b(1-x)/x \\ &\rightarrow \ell_\perp * (m^2/p_z^- + m^2/p_z^+) = C_p/x(1-x), \end{aligned} \tag{1.11}$$

where in the latter formula,

$$C_p = m^2 L/(2\ y\ k\ ) . \tag{1.12}$$

The subscript p refers to pair production. Crossing symmetry relates matrix elements. However in this case, no phase space factors are affected by the crossing transformation since we are calculating in the small disruption approximation, i.e., the limit of high energy and small scattering angles. Therefore, crossing applies also to the differential probability Eq. (1.1)—a claim we explicitly verify in the following calculation. For spinless electrons we need only take into account that we average over the two polarization states for the incident photon, but sum over the emitted photon states in the beamstrahlung process; this multiplies the result (1.1) by a factor of one-half. Therefore, we expect the probability of pair production by a photon with momentum k to be given by

$$\frac{dP(pair)}{dx} = \alpha \; y \; C_p \; G(u) \; , \tag{1.13}$$

where the function $G(u)$ is defined as in Eq. (1.2) but with u now given in terms of the C_p as given by the last form of Eq. (1.9), $u_p = [C_p/x(1-x)]^{2/3}$.

The variables C_b and C_p are central in our work. They are the reciprocals of the corresponding variables χ as used in Refs. 11–13. It is seen from Eqs. (1.3), (1.4) and (1.13), that $1/C$ is proportional to the energy of the incident particle times the strength of the transverse electric field at the edge of the pulse; i.e. $|eE_{\perp}| = 2N\alpha/LB$, and

$$\frac{1}{C} = \frac{p|eE_{\perp}(B)|}{m^3} \; . \tag{1.14}$$

We shall show by explicit calculation that (1.13) is the correct pair production result. The careful reader may find it remarkable that even though the pair production calculation is in detail very different from the beamstrahlung calculation at every stage, the final results are related in the simple manner described above—as they indeed must be. The detailed calculations will be carried through both for a cylindrical uniformly charged pulse and for a constant electric field. Throughout most of our discussion, we shall assume cylindrical symmetry. However, in the next section, a general formula, valid for slowly varying fields, will be constructed and sensitivities to field variations displayed. Spin 1/2 electrons will be treated as well as the scalar case.

2. Review of New Results and Crossing Symmetry

In this section we summarize our results, discuss some of their implications and connect and contrast pair production with the earlier result for beamstrahlung. Henceforth we will work exclusively in the *pulse rest frame.* The differential probability[21] for both processes for *spinor* electrons can be expressed as

$$\frac{dP}{dx} = \alpha\, y\, C\, G(u,x) \tag{2.1}$$

with

$$G(u,x) = \int \frac{d^2b}{\pi B^2} \int\limits_{u\left(\frac{E^2(B)}{E^2(b)}\right)^{1/3}}^{\infty} dv\, Ai(v) \left\{ \left[\frac{2v}{u} \left(\frac{E^2(b)}{E^2(B)} \right)^{1/3} - 1 \right] S_{NF}(x) + S_F(x) \right\} . \tag{2.2}$$

For pair production we have $C = C_p$ and

$$u = u_p \equiv \left[\frac{C}{x(1-x)} \right]^{2/3} , \tag{2.3}$$

while for beamstrahlung, crossing leads to $C = C_b$ and

$$u = u_b \equiv \left[\frac{C(1-x)}{x} \right]^{2/3} . \tag{2.4}$$

The spinor factors are also related in a simple way:

$$\text{non-flip: } S_{NF}(x) = \begin{cases} \frac{p_+}{p_-} + \frac{p_-}{p_+} = \frac{x^2+(1-x)^2}{x(1-x)} & \text{pair production} \\ \frac{p_f}{p_i} + \frac{p_i}{p_f} = \frac{1+x^2}{x} & \text{beamstrahlung} \end{cases} \tag{2.5}$$

$$\text{flip: } S_F(x) = \begin{cases} \frac{k^2}{p_+p_-} = \frac{1}{(x(1-x)} & \text{pair production} \\ \frac{k^2}{p_ip_f} = \frac{(1-x)^2}{x} & \text{beamstrahlung .} \end{cases} \tag{2.6}$$

The square of the transverse electric field at impact parameter $\vec{b}$ in the rest frame of the target pulse is $E^2(b)$; (2.1) and (2.2) are of course Lorentz invariant. Equation (2.2) assumes that the fields are slowly varying in the transverse direction (on a scale of $\delta b \sim B\,[Cy^3]^{-1/2} \ll B$) in which case the calculation is found to be simply the sum of suitably weighted contributions from small elements of area $\delta^2 b$ with $0 < b < B$. The extension to ribbon pulses proceeds exactly as in Ref. 8.

A number of simple features can be deduced from (2.1) to (2.6):

1. Crossing symmetry applies directly to matrix elements. It is expressed here for the differential probabilities because the final transverse momentum integrals cover the same regions of phase space in the two processes; for high energies the electron and photon are both essentially massless and each has two spin polarization states.

2. In the extreme quantum limit $C \rightarrow 0$; using the definition (1.14), we see that this is the high energy, strong field limit. In this limit, $u \rightarrow 0$ in (2.2), which then simplifies to

$$G(u \rightarrow 0, x) \approx \frac{2}{u}\, S_{NF}(x) \left(\int_0^\infty v\, Ai(v) dv \right) \int_0^{B^2} \frac{db^2}{B^2} \left\{ \frac{E^2(b)}{E^2(B)} \right\}^{1/3} , \qquad (2.7)$$

indicating that the spectrum in x is broad and that all impact parameters give significant contributions for both constant fields and uniform charge densities for which $E(b) \propto b$. The total probability is

$$P_{tot} = 2\,\alpha\, y\, C^{1/3} \cdot g_1 \cdot (5 \cdot g_2) \cdot a \ , \qquad (2.8)$$

where

$$g_1 \equiv \int_0^\infty v\, Ai(v)\, dv \;=\; \left[3^{1/3}\Gamma(1/3)\right]^{-1} = 0.259 \tag{2.9}$$

$$5 \cdot g_2 \equiv \int_0^1 \frac{x^2 + (1-x)^2}{x(1-x)} \, [x(1-x)]^{2/3} dx = \frac{5}{7} \, \frac{\Gamma^2(2/3)}{\Gamma(4/3)} \tag{2.10}$$

$$= 5 \int_0^1 [x(1-x)]^{2/3} dx = 5 \; (0.293)$$

and

$$a = \begin{cases} 1 & \text{for a constant field} \\ 3/4 & \text{for a constant charge density} \;. \end{cases} \tag{2.11}$$

The factor 5 was explicitly introduced in (2.10) because that is the ratio of the production probabilities for spin 1/2 to spin 0 electrons resulting from the non-flip factor $S_{NF}(x)$ of (2.5) in the $x-$integrand.

Note that the beamstrahlung probability in this limit differs from pair production only by the spectral integral

$$g_b \equiv \int_0^1 dx \left(\frac{x}{1-x}\right)^{2/3} \frac{1+x^2}{x} \approx 6 \;,$$

which is larger than (2.10) by a factor of ~ 4 because of the soft photon tail present in the beamstrahlung spectrum.

3. In the extreme classical limit one has $C \to \infty$ which by Eq. (1.14) is the case when the field strength and energy are not too large. In this limit we see a sharp contrast between pair production and beamstrahlung. In the former case, by (2.3) $u \to \infty$ and the integral over the Airy function in (2.2) falls

exponentially. For beamstrahlung on the other hand, (2.4) shows that the dominant contribution, as discussed in Ref. 4, comes from very soft photons, $(1-x) \sim 1/C$, leading to the familiar classical result.

We also see in (2.2) that when $u \to \infty$, most of the contribution to $G(u)$ comes from a thin ring near the outer edge of the beam. This is the case if the charge distribution is localized and if $E^2(b)$ increases as we move out from the center $(b = 0)$ to the edge $(b \approx B)$. It is in this region that the lower limit of the dv integral falls to its smallest value. Physically this is the demand that in order to transfer sufficient momentum to make a pair in the weak field situation, the incident photon must be near the edge of the pulse in the region of the maximum field.

In contrast to this result, for a constant field all impact parameters contribute to (2.2) and one obtains a very different result. Setting $E^2(b) = E^2(B)$, and introducing the asymptotic limit of the Airy function,

$$Ai(v) \xrightarrow[v \to \infty]{} \frac{1}{2\sqrt{\pi v^{1/2}}} \, e^{-(2/3)v^{3/2}} \;, \tag{2.12}$$

one can integrate (2.2) directly

$$G(u) \sim \frac{1}{2\sqrt{\pi u^{3/2}}} \left[1 \,,\, \frac{3}{u^{3/2}} \right] \, e^{-(2/3)u^{3/2}} \;, \tag{2.13}$$

where the two factor choices in the bracket correspond to a constant field and to a uniform charge density, respectively. For large C, the exponential restricts $x \sim 1/2$ according to (2.3), that is the pair share the energy equally, and we obtain

$$
\begin{aligned}
P_{tot} &= \frac{\alpha\, y}{16}\, \sqrt{3/2}\; e^{-8C/3} && \text{constant field} \\
P_{tot} &= \frac{3}{4C}\, (P_{tot})_{\mathrm{const.field}} && \text{uniform charge} \quad .
\end{aligned}
\tag{2.14}
$$

In this limit the pair production probability for spin 1/2 electrons is 6 times larger than that for the spin 0 case. This factor of 6 can be understood simply from (2.2), (2.5), and (2.6) . The dominant contribution in the curly bracket in (2.2) reduces to $[S_{NF}(x) + S_F(x)]$ when $v \to u$ and $b \to B$. Also since $x \simeq 1/2$ in this classical limit, we see that $\{S_F(1/2)/S_{NF}(1/2)\} = 2$, indicating that 2/3 of the contribution comes from spin flip. The non-flip term itself is twice as large as it is in the spin zero case because two spin states contribute.

4. Having gained a cursory overview of the physics of this process, we now see how to derive equations (2.1) and (2.2) for a general but slowly varying field. Simply average the result of the constant field calculation over all impact parameters but take into account the slow variation of the field strength; for a constant field (and suppressing inessential spin factors) we have

$$
G_c(u) = \int_0^{B^2} \frac{db^2}{B^2}\; G_c(u,b)
$$

$$
G_c(u,b) \equiv \int_u^{\infty} dv\; Ai(v) \left(\frac{2v}{u} - 1 \right)
$$

Recognizing that $u \propto C^{2/3} \propto (1/E(b)^2)^{1/3}$, we simply introduce the appropriate u for each impact parameter

$$u \to u(b) = u(B) \left(\frac{E^2(B)}{E^2(b)} \right)^{1/3}$$

which reproduces (2.2).

Similarly one can slice a pulse into disks of varying thickness to allow for a field varying along the direction of the collision axis $0 < z < L$ and write $G(u)$ as an integral

$$\int_0^L \frac{dz}{L} \, G(u,z)$$

where in $G(u,z)$ one introduces the (slowly varying) z dependence of $E^2(b,z)$. This is the case of varying field that has been treated previously in Ref. 8.

5. *Cross-Over:* Coherent production of pairs can be very much larger than incoherent production for energies and currents envisioned for some of the future collider designs. By equating the relevant second result in Eq. (2.14) with N times the Bethe-Heitler rate, we find that they are comparable for $C \sim 3.5$ for typical collider parameters.[22] The decrease in coherent production for larger C is evident from Eq. (2.14) . Indeed, as C increases from 3 to 6, the coherent production drops by a factor of $\sim 5,000$.

Several important consequences of the above features are summarized in Figs. 1, 2, and 3. The rapid decrease in the pair production cross section with increasing $C > 1$ is shown in Fig. 1. In Fig. 2, one sees that as C increases, the x-distribution is more and more peaked at $x = 1/2$. This narrowing effect suppresses the very soft pairs that could be an important source of background. Figure

3 illustrates the importance of including electron spin in the calculation. The general topic of experimental consequences of our results will be discussed in more detail at a later note.

3. Scalar Electrons

In this section we derive an expression for the matrix element for pair production by a photon incident upon a pulse of N electrons; only pairs originating inside the pulse ($b < B$) are considered. This case is treated first to illustrate our approach with minimal algebraic complexity. In a later section the case of Dirac electrons will be discussed. The general form of the matrix element of interest is the integral

$$M = \left\langle \phi_f^{(-)} \middle| \vec{A} \cdot \vec{J} \middle| vac \right\rangle \ , \tag{3.1}$$

where $\vec{A}$ is the photon field, $\vec{J}$ is the electron current and $\phi_f^{(-)}$ is the final (incoming) scattering eigenstate of the electron-positron pair in the static external field of the pulse. For simplicity we will assume that the pulse is a cylinder of length L and radius B. The calculation will be carried out in the rest frame of the pulse following the analagous treatment of beamstrahlung given in Refs. 4 and 8. The main technical difference between these two problems lies in the nature of the boundary conditions: incoming waves for both members of the pair in the present case, and in beamstrahlung the electron line has both an incoming and an outgoing wave in the initial and final state wave functions. Let us now introduce several kinematic variables that will prove useful. Then we will go into a detailed calculation of the relevant wave functions, matrix elements, and cross sections for our problem. As in the study of beamstrahlung, it will prove to be necessary to retain corrections of order $(1/k)^2$ to the leading terms, or one order beyond the standard eikonal approximation, due to the extended nature of the target.

Kinematics and Variables:

The incident photon will have an energy of k and will be assumed to have no transverse momentum, $k_\perp = 0$. The final electron will have an energy denoted by $p^- = xk$ with components $p_\perp^-$ non-zero and $p_z^- = p^- - [(p_\perp^-)^2 + m^2]/(2p^-)$.

The final positron has an energy denoted by $p^+ = (1-x)k$ with $p_\perp^+$ and $p_z^+ = p^+ - [(p_\perp^+)^2 + m^2]/2p^+$.

It is also convenient to define transverse center of mass and relative coordinates for the pair:

$$\Sigma_\perp \equiv p_\perp^+ + p_\perp^- \qquad\qquad \Delta_\perp \equiv p_\perp^+ - p_\perp^-$$
$$\text{and} \qquad LZ \equiv (L - z) , \tag{3.2}$$

where Z is a dimensionless variable that will be a useful quantity since it measures the fractional distance that the pair travels in the electric field of the pulse after being created at the point z. A measure of disruption that occurs naturally in our calculation is provided by the ratio

$$a = \frac{N\alpha L}{2B^2 kx(1-x)} , \tag{3.3}$$

which is assumed small in our calculation.

The momentum transfer to the pulse is defined to be $q = k - p^- - p^+$. Its longitudinal component can be expressed as

$$
\begin{aligned}
q_z &= \frac{m^2+(p_\perp^-)^2}{2p^-}+\frac{m^2+(p_\perp^+)^2}{2p^+} \\
&= \frac{1}{2kx(1-x)}[m^2+(1-x)(p_\perp^+)^2+x(p_\perp^-)^2] \; . \qquad (3.4) \\
&= \frac{1}{8kx(1-x)}[4m^2+\Sigma_\perp^2+\Delta_\perp^2+2(1-2x)\,\Sigma_\perp\cdot\Delta_\perp] \; .
\end{aligned}
$$

Approximate Wave Functions:

The Klein-Gordon equation for a scalar particle of mass m in an external vector potential A_μ is

$$[m^2-(i\partial-eA)^2]\,\phi \;=\; 0 \; . \qquad (3.5)$$

In the rest frame of the pulse there is only a static field and the spatial K-G equation for each member of the pair can be written as

$$\left[(E-V)^2+\vec{\nabla}^2-m^2\right]\phi(x)=0 \; . \qquad (3.6)$$

The solution will be written in the form

$$\phi(x) \;=\; \exp(i\Phi(x)) \; , \qquad (3.7)$$

where Φ satisfies the equation

$$(E-V)^2-m^2 \;=\; \left(\vec{\nabla}\Phi(x)\right)^2-i\vec{\nabla}^2\Phi \; . \qquad (3.8)$$

For the problem of interest, we must solve this equation in the limit of large energies for the requisite boundary conditions, and must exhibit the solutions to the requisite accuracy.

4. Uniformly Charged Cylindrical Pulse

Neglecting end effects, the potential for a cylindrically symmetric charge distribution is

$$V(x) = V(z,b) \qquad b^2 \equiv x^2 + y^2 \tag{4.1}$$

for $0 < z < L$, and zero otherwise. We will consider the case of a uniformly charged cylinder with

$$V(z,b) = V_0 b^2 \qquad V_0 = \frac{N\alpha}{LB^2} \,. \tag{4.2}$$

For the electron, the leading term in $\Phi^-(x)$ must be $\overrightarrow{p^-} \cdot \overrightarrow{x}$; the electron phase function to order $(1/p^-)$ is

$$\Phi^-(x) = \overrightarrow{p^-} \cdot \overrightarrow{x} + V_0 L b^2 Z + \frac{2}{3p^-} V_0^2 \, L^3 \, b^2 \, Z^3 + \frac{1}{p^-} V_0 \, L^2 \, p_\perp^- \cdot b_\perp \, Z^2 \,, \tag{4.3}$$

as can be seen by direct substitution into Eq. (3.8). The final state boundary condition is explicit in the dependence on Z.

The positron phase function to the same order is

$$\Phi^+(x) = \overrightarrow{p^+} \cdot \overrightarrow{x} - V_0 L b^2 Z + \frac{2}{3p^+} V_0^2 \, L^3 \, b^2 \, Z^3 - \frac{1}{p^+} V_0 \, L^2 \, p_\perp^+ \cdot b_\perp \, Z^2 \,, \tag{4.4}$$

in which the opposite sign of the charge is also explicit. Certain higher order real as well as imaginary terms have been dropped since they will not affect our results to the order that we will work.

It is interesting to calculate the "local" value of the momentum defined as $\overrightarrow{p}^{\pm}_{loc} \equiv \overrightarrow{\nabla}\Phi^{\pm}$. The results are:

$$
\begin{aligned}
(p^-_{loc})_z &= p^-_z - V_0 b^2 - \frac{2}{p^-} V_0^2 L^2 b^2 Z^2 - \frac{1}{p^-} V_0 \, L \, p^-_\perp \cdot b_\perp \, Z \\
(p^+_{loc})_z &= p^+_z + V_0 b^2 - \frac{2}{p^+} V_0^2 L^2 b^2 Z^2 + \frac{1}{p^+} V_0 \, L \, p^+_\perp \cdot b_\perp \, Z \\
(p^-_{loc})_\perp &= p^-_\perp + 2V_0 LZ \; b_\perp + \frac{4}{3p^-} V_0^2 L^3 Z^3 \; b_\perp + \frac{1}{p^-} V_0 \, L^2 \, Z^2 \; p^-_\perp \\
(p^+_{loc})_\perp &= p^+_\perp - 2V_0 LZ \; b_\perp + \frac{4}{3p^+} V_0^2 L^3 Z^3 \; b_\perp - \frac{1}{p^+} V_0 \, L^2 \, Z^2 \; p^+_\perp
\end{aligned}
\tag{4.5}
$$

In the evaluation of the matrix element, an essential element is the total phase of the product of the wave functions. Including the phase of the photon wave function $A(\overrightarrow{r})$, and using $\overrightarrow{q} = \overrightarrow{k} - \overrightarrow{p}^- - \overrightarrow{p}^+$, it can be written in the form

$$
\begin{aligned}
\Phi_{tot} &= \overrightarrow{k} \cdot \overrightarrow{r} - \Phi^-(x) - \Phi^+(x) \\
&= + \overrightarrow{q} \cdot \overrightarrow{r} - \frac{2}{3kx(1-x)} V_0^2 \, L^3 \, b^2 \, Z^3 + \frac{1}{kx(1-x)} V_0 \, L^2 \, Z^2 \, \overrightarrow{b_\perp} \cdot \overrightarrow{p}_\perp(x) \, ,
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
p_\perp(x) &= x p^+_\perp - (1-x) p^-_\perp \\
&= \tfrac{1}{2} \Delta_\perp - \tfrac{1}{2} (1-2x) \Sigma_\perp \, .
\end{aligned}
\tag{4.7}
$$

Notice that the zeroth order term linear in V_0 has cancelled in the total phase function. This is very different from the structure achieved at this point in the beamstrahlung calculation; this is due to the different boundary conditions.

The total phase can be rewritten in more convenient form as $(q_\perp = -\Sigma_\perp)$

$$\Phi_{tot} = +q_z z + \left[\overrightarrow{b_\perp} \cdot \overrightarrow{A}_\perp + b^2 A_s \right] , \tag{4.8}$$

where

$$\begin{aligned} \overrightarrow{A}_\perp &= \overrightarrow{\Sigma_\perp} - \frac{V_0 L^2 Z^2}{kx(1-x)} \overrightarrow{p}_\perp(x) , \\ &= \overrightarrow{\Sigma_\perp}[1 + aZ^2(1-2x)] - aZ^2 \overrightarrow{\Delta_\perp} \\ &\simeq \overrightarrow{\Sigma_\perp} - aZ^2 \overrightarrow{\Delta_\perp} \\ A_s &= +\frac{2V_0^2 L^3 Z^3}{3kx(1-x)} = +\frac{4}{3} a V_0 L Z^3 . \end{aligned} \tag{4.9}$$

Matrix Element- Stationary Phase:

The matrix element now achieves the form

$$M = eL \int_0^1 dZ \int^B d^2b \; \overrightarrow{\epsilon} \cdot \overrightarrow{P}(Z,b) \;\; \exp[i\Phi_{tot}(Z,b)] \;\; , \tag{4.10}$$

where the factor $\overrightarrow{P}(Z,b)$ is the gauge invariant local current

$$\overrightarrow{P}(Z,b) = \left[\overrightarrow{p^+_{loc}}(Z,b) - \overrightarrow{p^-_{loc}}(Z,b) \right] . \tag{4.11}$$

In component form these are given by Eq. (4.5).

The phase Φ_{tot} is quadratic in the impact parameter for a long uniform cylindrical pulse. Since the coefficient of b^2 is very large in units of the radius of the pulse, we will carry out the d^2b integral via the method of stationary phase. To do this it is necessary to solve for the stationary impact parameter $\vec{b}_0$, where

$$\vec{\nabla}_\perp \Phi_{tot}(Z, b_0) = 0 \ . \tag{4.12}$$

This gives

$$\begin{aligned} \vec{b}_0 &= -\frac{\vec{A}_\perp}{2\,A_s} \\ &= \frac{3}{8V_0 LZ}\left[\vec{\Delta}_\perp - \frac{1}{aZ^2}\vec{\Sigma}_\perp\right] \ , \end{aligned} \tag{4.13}$$

which fixes the "classical" impact parameter in terms of the final pair momenta, the point of production Z, and the coordinate and energy of the incident photon. The Z-dependence is a reflection of the curved classical trajectories of the pair. Note that since $b_0 \leq B$, the momentum transfers are restricted (otherwise the stationary point does not exist). Expanding the impact parameter around this value allows the transverse integration to be done, and the result is

$$M = -i\,\frac{3\pi e}{4aV_0}\int_0^1 \frac{dZ}{Z^3}\ \vec{\epsilon}\cdot\vec{P}(Z)\ \ \exp[i\Phi_{tot}(Z)] \ . \tag{4.14}$$

To leading order in $1/p$, the phase $\Phi_{tot}(Z)$ is

$$\begin{aligned} \Phi_{tot}(Z) &\equiv \Phi_{tot}(Z, b_0(Z)) \\ &= +q_z L - \left[q_z LZ + \frac{1}{4A_s}\vec{A}_\perp\cdot\vec{A}_\perp\right] \end{aligned} \tag{4.15}$$

and after inserting the values from Eq. (4.9) and (3.4)

$$
\begin{aligned}
\Phi_{tot}(Z) - q_z L &= -\frac{LZ}{32kx(1-x)}(4T_1 - 3T_2) \\
T_1 &= \Delta_\perp^2 - 2(1-2x)\Delta_\perp \cdot \Sigma_\perp + \Sigma_\perp^2 + 4m^2 \\
T_2 &= \Delta_\perp^2 - \frac{2}{aZ^2}\Delta_\perp \cdot \Sigma_\perp + \frac{1}{a^2 Z^4}\Sigma_\perp^2 \ .
\end{aligned}
\tag{4.16}
$$

The current, as given by (4.5) and (4.11) and evaluated at the stationary point, becomes

$$
\begin{aligned}
\overrightarrow{\epsilon_\perp} \cdot \overrightarrow{P}(Z) &= \overrightarrow{\epsilon_\perp} \cdot \left[\overrightarrow{\Delta_\perp} - 4V_0 L Z \overrightarrow{b_0}(Z)\right] \\
&= -\frac{1}{2}\,\overrightarrow{\epsilon_\perp} \cdot \left[\overrightarrow{\Delta_\perp} - \frac{3}{aZ^2}\overrightarrow{\Sigma_\perp}\right] \ .
\end{aligned}
\tag{4.17}
$$

The square of the matrix element, averaged over the incident photon polarization, is of the form ($e^2 = 4\pi\alpha$)

$$
\tfrac{1}{2}\sum_{pol} M^* M = 2\pi\alpha \left(\frac{3\pi}{4aV_0}\right)^2 \int_0^1 \frac{dZ_1\, dZ_2}{(Z_1\, Z_2)^3}\, S\ \exp[i\Delta\Phi_{tot}(Z_1, Z_2)] \ , \tag{4.18}
$$

where the polarization sum that we require is

$$
S(Boson) = \sum_{pol} \overrightarrow{\epsilon} \cdot \overrightarrow{P}(Z_1) \times \overrightarrow{\epsilon} \cdot \overrightarrow{P}(Z_2) \ , \tag{4.19}
$$

and the phase difference is

$$
\Delta\Phi_{tot}(Z_1, Z_2) = \Phi_{tot}(Z_1) - \Phi_{tot}(Z_2) \ . \tag{4.20}
$$

Polarization, Phase Differences and Limits:

Since the polarization vectors are transverse, their sum becomes

$$S(Boson) = \vec{P}_\perp(Z_1)\cdot\vec{P}_\perp(Z_2)\ , \tag{4.21}$$

which after some manipulation can be written in the form

$$\begin{aligned} 4\,S(Boson) &= \left(\overrightarrow{\Delta_\perp} - \frac{3}{aZ_1Z_2}\overrightarrow{\Sigma_\perp}\right)^2 - \frac{3(Z_1-Z_2)^2}{a(Z_1Z_2)^2}\overrightarrow{\Sigma_\perp}\cdot\overrightarrow{\Delta_\perp} \\ &= \left(\overrightarrow{\Delta'_\perp}\right)^2 - 9w^2(\overrightarrow{\sigma_\perp})^2 - \frac{3w^2}{(Z_1Z_2)^{1/2}}\overrightarrow{\sigma_\perp}\cdot\overrightarrow{\Delta'_\perp}\ , \end{aligned} \tag{4.22}$$

where for convenience, we have introduced the quantities $w = Z_1 - Z_2$ and

$$\overrightarrow{\Delta'_\perp} = \overrightarrow{\Delta_\perp} - \frac{3}{aZ_1Z_2}\overrightarrow{\Sigma_\perp} \qquad \overrightarrow{\sigma_\perp} = \frac{1}{a(Z_1Z_2)^{3/2}}\overrightarrow{\Sigma_\perp}\ . \tag{4.23}$$

The phase difference also achieves a simple form in terms of these variables. Straightforward algebra leads to

$$\begin{aligned} -\Delta\Phi_{tot}(Z_1,Z_2))] &= \frac{Lw}{32kx(1-x)}\left[16\,m^2 + (\overrightarrow{\Delta_\perp} - \frac{3}{aZ_1Z_2}\overrightarrow{\Sigma_\perp})^2 + \frac{3w^2}{a^2(Z_1Z_2)^3}\overrightarrow{\Sigma_\perp}\cdot\overrightarrow{\Sigma_\perp}\right] \\ &= \frac{Lw}{32kx(1-x)}\left[16\,m^2 + (\overrightarrow{\Delta'_\perp})^2 + 3w^2(\overrightarrow{\sigma_\perp})^2\right] \\ &\equiv sw + \frac{1}{3}r^3w^3\ , \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} s &= \frac{L}{32kx(1-x)}\left[16\,m^2 + (\overrightarrow{\Delta'_\perp})^2\right] = a\left[1 + \frac{1}{16\,m^2}(\overrightarrow{\Delta'_\perp})^2\right] \\ r^3 &= \frac{L}{32kx(1-x)}3(\overrightarrow{\sigma_\perp})^2 = \frac{3}{16}a\,(\frac{\overrightarrow{\sigma_\perp}}{m})^2\ . \end{aligned} \tag{4.25}$$

Introducing the scaling variables y and C_x, these become

$$
\begin{aligned}
s &= y C_x \left[1 + \frac{1}{16m^2}(\Delta'_\perp)^2\right] \\
r^3 &= y^3 C_x \left(\frac{\sigma_\perp}{\sigma_{max}}\right)^2 ,
\end{aligned}
\tag{4.26}
$$

where we introduce $\sigma_{max} \equiv 4my/3$.

This upper limit on $\sigma_\perp$ derives from the demand that the stationary point exists, i.e. that $b_0(Z) \leq B$. This requirement is simplified by noting that the phase difference $\Delta\Phi$ forces the two creation points Z_1 and Z_2 to be very close to each other. Thus in the expression for $b_0(Z)$, Z can be replaced by the geometric mean position, that is $Z = (Z_1 Z_2)^{1/2}$. This leads to

$$
\begin{aligned}
\vec{b}_0 &= -\frac{3}{8V_0 L (Z_1 Z_2)^{1/2}} \left[\overrightarrow{\Delta_\perp} - \frac{1}{a Z_1 Z_2}\overrightarrow{\Sigma_\perp}\right] \\
&= -\frac{3}{8V_0 L (Z_1 Z_2)^{1/2}} \left[\overrightarrow{\Delta'_\perp} + \frac{2}{a Z_1 Z_2}\overrightarrow{\Sigma_\perp}\right] .
\end{aligned}
\tag{4.27}
$$

Since the disruption parameter a is small, the second term is much larger than the first and the limit can be expressed as

$$
\sigma_\perp = \frac{4V_0 L}{3}\, b_0 \leq \frac{4V_0 L}{3}\, B = \frac{4}{3} m\, y \equiv \sigma_{max} . \tag{4.28}
$$

Final State Sum:

The square of the matrix element, averaged over the polarization and integrated over the transverse momentum of the pair, is denoted as

$$
\int\!\!\int M * M \equiv \int \frac{d^2 p_\perp^-}{(2\pi)^2} \frac{d^2 p_\perp^+}{(2\pi)^2} \, \tfrac{1}{2} \sum_{pol} M^* M . \tag{4.29}
$$

The differential cross section is achieved by dividing by the normalization factors for the three wave functions and multiplying by the energy constraint integral:

$$
\begin{aligned}
d\sigma &= K\,dx \int\!\!\int M * M \\
K\,dx &\equiv [8kp^+p^-]^{-1} \int \frac{dp_z^+ dp_z^-}{(2\pi)^2} 2\pi\delta(\Delta E) \\
K &= [16\pi k^2 x(1-x)]^{-1} \ .
\end{aligned}
\tag{4.30}
$$

Our next task is to evaluate

$$
\int\!\!\int M * M = 4A \int \frac{d^2p_\perp^-}{(2\pi)^2}\frac{d^2p_\perp^+}{(2\pi)^2} \int_0^1 \frac{dZ_1\, dZ_2}{a^2(Z_1\, Z_2)^3} S \ \exp[i\Delta\Phi_{tot}(Z_1, Z_2)] \ , \tag{4.31}
$$

where

$$
A = \frac{\pi\alpha}{2}\left(\frac{3\pi}{4V_0}\right)^2 \tag{4.32}
$$

Interchanging orders and changing the momentum integrations to the local coordinates (and picking up a factor of $1/4$ from the change of variable)

$$
\int\!\!\int M * M = A \int_0^1 \frac{dZ_1\, dZ_2}{a^2(Z_1\, Z_2)^3} \int \frac{d^2\Delta'_\perp\, d^2\Sigma_\perp}{(2\pi)^2\,(2\pi)^2} S \ \exp[i\Delta\Phi_{tot}(Z_1, Z_2)] \ . \tag{4.33}
$$

The polarization trace S and the phase difference $\Delta\Phi_{tot}(Z_1, Z_2)$ become functions of w only. All other dependence on the coordinates Z_1 and Z_2 has been absorbed into $\Delta'_\perp$. Making a change of integration variable from $\Sigma_\perp$ to $\sigma_\perp$ now allows the limitation due to the radius B to also become independent of $Z_{1,2}$ so that

$$\int\!\!\int M * M \;=\; A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\frac{d^2\sigma_\perp}{(2\pi)^2} \int\limits_0^1 dZ_1\, dZ_2\, S\left[w^2\right] \exp\left[-i\left(sw + \tfrac{1}{3}\, r^3 w^3\right)\right] \tag{4.34}$$

with the restriction $\sigma_\perp^2 \le \sigma_{max}^2$, and where

$$4\,S\left[w^2\right] \;=\; (\Delta'_\perp)^2 - 9\,(\sigma_\perp)^2\, w^2\ . \tag{4.35}$$

The last term in S, given in Eq. (4.22), involves a cross term between $\Delta'_\perp$ and $\sigma_\perp$ and is zero after performing the angular integrations. Thus

$$\int\!\!\int M * M \;=\; A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\frac{d^2\sigma_\perp}{(2\pi)^2}\, S\left[-\frac{d^2}{ds^2}\right] \int\limits_0^1 dZ_1\, dZ_2\, \exp\left[-i\left(sw + \tfrac{1}{3}\, r^3 w^3\right)\right]\ , \tag{4.36}$$

and changing one of the Z's to w and integrating the other, we find

$$\int\!\!\int M * M \;=\; 2A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\frac{d^2\sigma_\perp}{(2\pi)^2}\, S\left[-\frac{d^2}{ds^2}\right] \int\limits_0^1 dw\,(1-w)\, \cos\left(sw + \tfrac{1}{3}\, r^3 w^3\right)\ . \tag{4.37}$$

Since the parameters r and s are large, both of order y from their definition, the w integral can be well approximated by the Airy integral, so that

$$\int\!\!\int M * M \;=\; 2\pi A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\frac{d^2\sigma_\perp}{(2\pi)^2}\, S\left[-\frac{d^2}{ds^2}\right] \frac{1}{r} Ai\left[\frac{s}{r}\right]\ . \tag{4.38}$$

Finally, using the differential equation satisfied by $Ai(w)$, namely $Ai(w)'' = wAi(w)$, one achieves

$$\int\!\!\int M * M \;=\; 2\pi A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\frac{d^2\sigma_\perp}{(2\pi)^2}\, S\left[-\frac{s}{r^3}\right] \frac{1}{r} Ai\left[\frac{s}{r}\right]\ . \tag{4.39}$$

where, using the definition of r and s, the polarization sum S simplifies to

$$4\,S\left[-\frac{s}{r^3}\right] \;=\; 16m^2 \;+\; 2(\Delta'_\perp)^2 \;. \tag{4.40}$$

Now we proceed to a series of variable changes to make this integral tractable. First define

$$v \;=\; \frac{s}{r} \;=\; \frac{yC_x}{r}\left[1+\frac{1}{16m^2}\,(\Delta'_\perp)^2\right] \;, \tag{4.41}$$

and the integration over $d^2\Delta'_\perp$ can be replaced by an integral over v. Paying attention to the limits of integration, and introducing the value of v at $(\Delta'_\perp)^2 = 0$, namely

$$v_0 \;=\; yC_x/r \;=\; \left(C_x\frac{\sigma_{max}}{\sigma}\right)^{2/3} \quad , \tag{4.42}$$

we find

$$\int\!\!\int M*M \;=\; A_1 \int\limits_0^{\sigma^2_{max}} \frac{d\sigma^2_\perp}{\sigma^2_{max}}\left(\frac{\sigma^2_\perp}{\sigma^2_{max}}\right)^{1/3} \int\limits_{v_0}^{\infty} dv\;\left[2v-v_0\right]Ai(v)\;, \tag{4.43}$$

with

$$A_1 \;=\; \frac{\alpha}{4}\left[\frac{4\pi\, m^2 L\,B}{yC_x}\right]^2 \left(yC_x^{1/3}\right)\;. \tag{4.44}$$

If we write $(\sigma^2_\perp/\sigma^2_{max}) \;=\; t^3$, where $0<t<1$, then $v_0 = u_p/t$, where u_p is the scaling variable defined earlier. The integral now becomes

$$\int\!\!\int M*M \;=\; A_1 \int\limits_0^{1} 3t^3dt \int\limits_{u_p/t}^{\infty} dv\;\left[2v-\frac{u_p}{t}\right]Ai(v)\;, \tag{4.45}$$

The integrations can be interchanged, the dt integration performed, and the result is

$$\int\int M * M \;=\; A_1 \; G(u_p) \; , \tag{4.46}$$

where

$$G(u) \;\equiv\; \frac{1}{2u} \int_u^{\infty} dv \; \left[3\,v - 2\,u - \frac{u^4}{v^3} \right] Ai(v) \; . \tag{4.47}$$

Recall that the scaling variable u_p is defined by

$$u_p \;=\; \left[\frac{C_p}{x(1-x)} \right]^{2/3} \; . \tag{4.48}$$

Collecting the above results, and using the definitions of C, C_x and u, the production probability and cross section become

$$\frac{dP(pair)}{dx} \;=\; \frac{1}{\pi B^2} \frac{d\sigma}{dx} \;=\; \alpha \; y \; C_p \; G(u_p) \tag{4.49}$$

which is the result that was gotten directly from crossing!

---

5. Uniform Electric Field

For this case, we choose the potential to be

$$V(x) = -\vec{E} \cdot \vec{b_\perp} \qquad \vec{E} = E_0 \hat{x} \; . \tag{5.1}$$

In order to characterize the electric field in a dimensionless manner, we will introduce the parameter ϵ by writing

$$E_0 = \frac{2N\alpha}{LB} \epsilon \; . \tag{5.2}$$

If ϵ equals one, then the constant electric field equals the field of the uniformly charged cylinder at the boundary of the pulse, $b = B$ as described elsewhere.

The electron phase function to order $(1/p^-)$ is

$$\Phi^-(x) = \vec{p^-} \cdot \vec{x} - LZ\vec{E}_\perp \cdot \vec{b_\perp} - \frac{L^2Z^2}{2p^-}\vec{E}_\perp \cdot \vec{p^-} + \frac{L^3Z^3}{6p^-}E_0^2 \tag{5.3}$$

as can be seen by direct substitution into the differential equation. Again, the final state boundary condition is explicit in the dependence on Z.

The positron phase function to the same order is

$$\Phi^+(x) = \vec{p^+} \cdot \vec{x} + LZ\vec{E}_\perp \cdot \vec{b_\perp} + \frac{L^2Z^2}{2p^+}\vec{E}_\perp \cdot \vec{p^+} + \frac{L^3Z^3}{6p^+}E_0^2 \; . \tag{5.4}$$

It is again interesting to calculate the local value of the momentum defined as $\vec{p}^{\,\pm}_{loc} \equiv \vec{\nabla}\Phi^\pm$. The result here for the local transverse momentum is

$$(p^\pm_{loc})_\perp = p^\pm_\perp \pm \vec{E}_\perp LZ \; . \tag{5.5}$$

We see that the electron and positron pick up equal and opposite momenta from the constant field as they propagate in the pulse.

The total phase function is

$$\Phi_{tot} = q_z L\,(1-Z) - \frac{L^2 Z^2}{4kx(1-x)} \vec{E} \cdot \left[\overrightarrow{\Delta_\perp} - (1-2x)\overrightarrow{\Sigma_\perp} \right] - \frac{L^3 Z^3}{6kx(1-x)} E_0^2 - \overrightarrow{\Sigma_\perp} \cdot \overrightarrow{b_\perp} \,, \tag{5.6}$$

The b integral over the face of the pulse in the matrix element will force the momentum vector $\overrightarrow{\Sigma_\perp}$ to be small ($\Sigma_\perp \leq 1/B$). This is much smaller than m so that $\Sigma_\perp$ can be neglected in the rest of the phase. Thus the total phase can be separated into the terms

$$\begin{aligned} \Phi_{tot} &= \Phi_{tot}(Z) \; - \overrightarrow{\Sigma_\perp} \cdot \overrightarrow{b_\perp} \\ \Phi_{tot}(Z) &= +q_z L\,(1-Z) - \frac{L^2 Z^2}{4kx(1-x)} \vec{E} \cdot \overrightarrow{\Delta_\perp} - \frac{L^3 Z^3}{6kx(1-x)} E_0^2 \,. \end{aligned} \tag{5.7}$$

The matrix element now achieves the form

$$M = eL \int_0^1 dZ \; \vec{\epsilon} \cdot \vec{P}(z) \;\; \exp[i\Phi_{tot}(Z)] \; J(\Sigma_\perp) \,, \tag{5.8}$$

where the coherence form factor is given by

$$J(\Sigma_\perp) = \int^B d^2 b \; \exp\left[i\overrightarrow{\Sigma_\perp} \cdot \overrightarrow{b_\perp} \right] \;. \tag{5.9}$$

The square of the matrix element, averaged over the incident photon polarization, is

$$\tfrac{1}{2} \sum_{pol} M^* M = 2\pi\alpha |J(\Sigma_\perp)|^2 \int_0^1 dZ_1\, dZ_2 \; S \; \exp[i\Delta\Phi_{tot}(Z_1, Z_2)] \;. \tag{5.10}$$

The quantities that we need are evaluated in a straightforward manner. The polarization sum is

$$
\begin{aligned}
S(Boson) &= \sum_{pol} \overrightarrow{\epsilon} \cdot \overrightarrow{P}(Z_1) \times \overrightarrow{\epsilon} \cdot P(Z_2) \\
&= (\overrightarrow{\Delta'_\perp})^2 - L^2 E_0^2 w^2 \ .
\end{aligned}
\tag{5.11}
$$

The phase difference is

$$
-\Delta\Phi_{tot}(Z_1, Z_2))] \;=\; sw \;+\; \frac{1}{3} r^3 w^3 \ , \tag{5.12}
$$

where in this case of a constant field,

$$
\begin{aligned}
s &= \frac{L}{8kx(1-x)} \left[4\,m^2 + (\overrightarrow{\Delta'_\perp})^2 \right] \;=\; y\, C_x \left[1 + \frac{1}{4m^2} (\Delta'_\perp)^2 \right] \\
r^3 &= \frac{L}{8kx(1-x)} \left[E_0^2 L^2 \right] \;=\; (\epsilon\, y)^3 \left(\frac{C_x}{\epsilon} \right) \ .
\end{aligned}
\tag{5.13}
$$

<u>Final State Sum:</u>

The square of the matrix element averaged over the polarization and integrated over the transverse momentum of the photon is computed as before:

$$
\int\!\!\int M * M \;=\; A \int_0^1 dZ_1\, dZ_2 \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\, S\, \exp[i\Delta\Phi_{tot}(Z_1, Z_2)\,] \int \frac{d^2\Sigma_\perp}{(2\pi)^2} \frac{|J(\Sigma_\perp)|^2}{\pi B^2} \ , \tag{5.14}
$$

where now

$$
A \;=\; \tfrac{1}{2}\, \alpha\, \pi^2 L^2 B^2 \ . \tag{5.15}
$$

Finally, one achieves

$$
\int\!\!\int M * M \;=\; 2\pi A \int \frac{d^2\Delta'_\perp}{(2\pi)^2}\, S \left[-\frac{s}{r^3} \right] \frac{1}{r} Ai \left[\frac{s}{r} \right] \int \frac{d^2\Sigma_\perp}{(2\pi)^2} \frac{|J(\Sigma_\perp)|^2}{\pi B^2} \ , \tag{5.16}
$$

where, using the definition of r and s, S simplifies to

$$S\left[-\frac{s}{r^3}\right] = 4m^2\left[1 + \frac{1}{2m^2}(\Delta'_\perp)^2\right] . \tag{5.17}$$

Now we proceed to a series of variable changes to make this integral tractable. First define the familiar

$$v = \frac{s}{r} = \frac{yC_x}{r}\left[1 + \frac{1}{4m^2}(\Delta'_\perp)^2\right] , \tag{5.18}$$

and replace the integration over $\Delta'_\perp$ by an integral over v. Introducing

$$v_0 = yC_x/r = \left(\frac{C_x}{\epsilon}\right)^{2/3} \equiv u_c = \frac{u_p}{\epsilon^{2/3}} , \tag{5.19}$$

we find the form

$$\int\int M * M = \alpha \frac{\left(2\pi m^2 LB\right)^2}{yC_x} \int_{u_c}^{\infty} \frac{dv}{u_c} \left[2v - u_c\right] Ai(v) . \tag{5.20}$$

Collecting the above results, the production probability and cross section for the constant field case become

$$\frac{dP(pair)}{dx} = \frac{1}{\pi B^2}\frac{d\sigma}{dx} = \alpha\, y\, C_p\, G_c(u_c) , \tag{5.21}$$

where

$$G_c(u_c) = \frac{1}{u_c}\int_{u_c}^{\infty} dv\ \left[2\,v - u_c\right] Ai(v) . \tag{5.22}$$

The entire dependence on the overall strength of the electric field is through the dependence of the variable u_c on ϵ. In the next section we will relate the results of these two field configurations in a physical way.

6. Connections and Limiting Behaviors

Relation between Constant and Varying Fields

Aside from the dependence of the variable u_c on the relative field strength ϵ, which is to be expected, the main difference between these cases is the analytic form of the integral that defines the $G(u)$ function. This gives rise to a difference in the limiting behavior of the production probability in the classical limit of large C which can be understood physically. In the case of the uniform charge distribution, we have (see Eq. (4.45))

$$u_p \, G(u_p) = \int_0^1 6t^3 dt \int_{u_p/t}^{\infty} dv \, \left[2v - \frac{u_p}{t}\right] Ai(v) \, , \tag{6.1}$$

The variable t is related to $\sigma_\perp$ which, in turn, is related to the impact parameter via Eq. (4.28), $(\sigma_\perp/\sigma_{max}) = (b/B)$. Interchanging limits and changing variables we can write

$$u_p \, G(u_p) = \int_{u_p}^{\infty} dv \, Ai(v) \int_{B^2(u_p/v)^3}^{B^2} \frac{db^2}{B^2} \left[2v \, (\frac{b^2}{B^2})^{1/3} - u_p\right] \, , \tag{6.2}$$

whereas the corresponding function in the constant field case is (see Eqs. (5.16) and (5.22), and recall the factor J),

$$u_c \, G_c(u_c) = \int_{u_c}^{\infty} dv \, Ai(v) \, [2v - u_c] \int_0^{B^2} \frac{db^2}{B^2} \, . \tag{6.3}$$

As expected, the constant field case gets equal contributions from the full face of the pulse; the case of a uniform charge density on the other hand, has a distribution over the face that depends upon the field configuration.

It is possible to derive the uniform density case from the constant case by assuming that the field changes slowly. The position dependence of the field is introduced through ϵ; from Eq. (5.2) the varying field is introduced by writing $\epsilon = b/B$, so that $u_c = u_p(B/b)^{2/3}$, and then averaging G_c over the area of the pulse:

$$
\begin{aligned}
G(u_p) &= \int_0^{B^2} \frac{db^2}{B^2}\frac{1}{u_c} \int_{u_c}^{\infty} \frac{dv}{u_c}\; [2\,v - u_c]\, Ai(v)\,, \\
&= \frac{1}{2u_p} \int_{u_p}^{\infty} dv Ai(v) \left(2u_p \int_{B^2(u_p/v)^3}^{B^2} \frac{db^2}{B^2}\left[2\,\frac{v}{u_c} - 1\right]\right)\,,
\end{aligned}
\qquad (6.4)
$$

which upon integration agrees with the result in the previous section. This is a very important relation and allows one to treat the general case of pair production and beamstrahlung from a finite region of slowly varying field.

Incidentally, this formula also shows the main physical difference between the uniform charge density case and the constant field case. In the ultra-quantum limit of small C, both get finite contributions from the full frontal area of the pulse but with a distribution of $(b/B)^{2/3}$ in the former circumstance. In the classical limit, i.e. weak fields for which $C \to \infty$, pair production from a uniform charge occurs predominantly in a ring near the outer edge of the pulse where the field has its maximum strength (thus minimizing the effect of the exponential damping at weak fields). This ring has a radius of $\sim B$, and a width of $\Delta b \sim B\left(1 - \left\langle (u/v)^{3/2} \right\rangle\right) \sim B/C_x$ for large C_x (and $\Delta b \sim B$ for $C_x \to 0$). This latter estimate follows from the exponential falloff of the Airy function in the integrand of Eq. (6.4). In this outer region, the two scaling variables should be essentially equal since $\epsilon \sim 1$.

Limiting Behaviors

Let us now examine the limiting form of our main results. The production probability and cross section for the general case is written as

$$\frac{dP(pair)}{dx} = \frac{1}{\pi B^2}\frac{d\sigma}{dx} = \alpha\, y\, C_p\, G(u) \, , \tag{6.5}$$

where C_p is defined universally as $m^2L/(2ky)$. The function $G(u)$ depends upon the field configuration as does its scaling variable u. It does not depend upon x for scalar electrons. For the constant field case

$$G_c(u_c) = \frac{1}{u_c}\int_{u_c}^{\infty} dv\ [2\,v - u_c]\, Ai(v) \, , \tag{6.6}$$

where $u_p = u_p\epsilon^{-2/3}$, and for the uniform charge density pulse,

$$G(u_p) = \frac{1}{2u_p}\int_{u_p}^{\infty} dv\ \left[3\,v - 2\,u_p - \frac{u_p^4}{v^3}\right] Ai(v) \, , \tag{6.7}$$

Extreme Quantum Limit: In this limit in which $C \to 0$ both u_c and u_p vanish, and the integral over the Airy functions can be performed:

$$\begin{aligned} G_c(u_c) &\sim \frac{2}{u_c}\, g_1 \; = \frac{2}{u_p}\, g_1\, \epsilon^{2/3} \\ G(u_p) &\sim \frac{3}{2u_p}\, g_1 \; - g_3 \ , \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} g_1 &= \int_0^\infty dv\ v Ai(v) \; = \left[3^{1/3}\,\Gamma(\frac{1}{3})\right]^{-1} \; = 0.25887\ldots \\ g_3 &= \int_0^\infty dv\ Ai(v) \; = \frac{1}{3} \ . \end{aligned} \tag{6.9}$$

Thus the differential production probabilities in this limit are

$$\begin{aligned} \frac{dP_c}{dx} &= 2\,\alpha\, y\, g_1\, \left(C\,\epsilon^2\right)^{1/3}\, [x(1-x)]^{2/3} \\ \frac{dP}{dx} &= \frac{3}{2}\,\alpha\, y\, g_1\, \left(C\right)^{1/3}\, [x(1-x)]^{2/3} \ , \end{aligned} \tag{6.10}$$

and the total production probabilities are then achieved by integration:

$$\begin{aligned} P_c &= 2\,\alpha\, y\, g_1\, g_2\, \left(C\,\epsilon^2\right)^{1/3} \\ P &= \frac{3}{2}\,\alpha\, y\, g_1\, g_2\, \left(C\right)^{1/3} \ , \end{aligned} \tag{6.11}$$

where

$$g_2 = \int_0^1 dx\ [x(1-x)]^{2/3} = \frac{\Gamma^2(\frac{2}{3})}{7\,\Gamma(\frac{4}{3})} \; = 0.29334\ldots \tag{6.12}$$

$$g_1 g_2 = 0.075937\ldots$$

Classical Limit: In this opposite limit of $C \to \infty$, the Airy functions can be replaced by their form for large argument. Since the exponential decrease insures that the important range of v is near the lower limit u, we write for large u,

$$v = u + \frac{y}{u^{1/2}} \qquad dv = \frac{dy}{u^{1/2}} ,$$

$$Ai(v) \sim \frac{1}{2\pi^{1/2} v^{1/4}} \exp\left[-\frac{2}{3} v^{3/2}\right] + \ldots \tag{6.13}$$

$$\sim \frac{1}{2\pi^{1/2} u^{1/4}} \exp\left[-\frac{2}{3} u^{3/2} - y\right] .$$

Thus

$$G_c(u_c) \sim \frac{1}{2\pi^{1/2}\, u_c^{3/4}} \exp\left[-\frac{2}{3} u_c^{3/2}\right]$$

$$G(u_p) \sim \frac{3}{2\pi^{1/2}\, u_p^{9/4}} \exp\left[-\frac{2}{3} u_p^{3/2}\right] , \tag{6.14}$$

or more accurately, for $u > 1$,

$$G(u_p) \sim \frac{3}{2\pi^{1/2}\, u_p^{9/4}} \left[1 + \frac{4}{u_p^{3/2}}\right]^{-1} \exp\left[-\frac{2}{3} u_p^{3/2}\right] . \tag{6.15}$$

These two cases are related by

$$G(u_p) = \left[\frac{3}{u_p^{3/2}}\right] G_c(u_p) = \left[\frac{3}{C_x}\right] G_c(u_p) . \tag{6.16}$$

Thus the width of the contributing ring for the uniform charge case is

$$\Delta b = \frac{3}{2C_x} B , \tag{6.17}$$

in agreement with the estimate given earlier for the classical limit.

For large C, the $x-$distribution is controlled by the exponential factor which forces x to be near one-half. Expanding about this value, the differential probabilities become

$$\frac{dP_c}{dx} \simeq \frac{\alpha\, y}{4} \left(\frac{C\,\epsilon}{\pi}\right)^{1/2} \exp\left[-\frac{8}{3\epsilon}C\right] \exp\left[-\frac{8}{3\epsilon}C\,(2x-1)^2\right]$$
$$\frac{dP}{dx} \simeq \frac{3\,\alpha\, y}{16} \left(\frac{1}{\pi C}\right)^{1/2} \exp\left[-\frac{8}{3}C\right] \exp\left[-\frac{8}{3}C\,(2x-1)^2\right] \qquad (6.18)$$

and thus (at $\epsilon = 1$)

$$\frac{dP}{dx} = \frac{3}{4\,C} \frac{dP_c(\epsilon=1)}{dx} \quad . \qquad (6.19)$$

The total production probabilities are

$$P_c = \frac{\alpha\, y}{16} \left(\frac{3}{2}\right)^{1/2} \epsilon \exp\left[-\frac{8}{3\epsilon}C\right]$$
$$P = \frac{3}{4\,C}\, P_c(\epsilon=1) \quad . \qquad (6.20)$$

More accurate formulas for the latter case are

$$\frac{dP}{dx} \simeq \frac{3\,\alpha\, y}{16\sqrt{\pi C}} [4x(1-x)]^{3/2}\; Q_s(x) \exp\left[-\frac{8}{3}C\,(1+(2x-1)^2\,)\right] \;, \qquad (6.21)$$

where

$$Q_s(x) = \frac{1}{\left[1+\frac{4x(1-x)}{C}\right]} \;, \qquad (6.22)$$

and

$$P = \frac{3\alpha\, y}{64} \left(\frac{3}{2}\right)^{1/2} \frac{1}{(C+1)[1+\frac{9}{32C}]} \exp[-\frac{8}{3}C] \quad . \qquad (6.23)$$

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7. Dirac Electrons

The extension of our analysis to Dirac electrons is straightforward. The general form of the matrix element for this case is

$$M = e \left\langle \phi_f^{(-)} \middle| \vec{A} \cdot \vec{\alpha} \middle| vac \right\rangle , \tag{7.1}$$

where A is the photon field. It is convenient to use a chiral basis for the Dirac spinors. The Hamiltonian and the Dirac equation take the form $H\,\Psi(r) = E\,\Psi(r)$, which, written in terms of the ordinary two-component Pauli matrices, is

$$H = \begin{pmatrix} -i\,\vec{\sigma} \cdot \vec{\nabla} + V & m \\ m & +i\,\vec{\sigma} \cdot \vec{\nabla} + V \end{pmatrix} . \tag{7.2}$$

The wave function will be written as

$$\Psi = \begin{pmatrix} \psi_u[V] \\ \psi_l[V] \end{pmatrix} \exp[i\Phi(z,b)] , \tag{7.3}$$

where $\Phi(z,b)$ is the Klein-Gordon phase function.

For zero external potential, $\Phi = p \cdot r$, and the solution for an electron of momentum p is

$$\begin{aligned} \psi_u[0] &= +N_c \left(1 + \frac{1}{E+m}\,\vec{\sigma} \cdot \vec{p}\right) w_\pm \\ \psi_l[0] &= +N_c \left(1 - \frac{1}{E+m}\,\vec{\sigma} \cdot \vec{p}\right) w_\pm \quad , \end{aligned} \tag{7.4}$$

while that for a positron of momentum p is

$$\begin{aligned} \psi_u[0] &= +N_c \left(1 + \frac{1}{E+m}\,\vec{\sigma} \cdot \vec{p}\right) w_\pm \\ \psi_l[0] &= -N_c \left(1 - \frac{1}{E+m}\,\vec{\sigma} \cdot \vec{p}\right) w_\pm \quad , \end{aligned} \tag{7.5}$$

where the basic two component spinors $w_\pm$ for both the electron and positron and the normalization are given by

$$w_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad w_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad N_c = \left(\frac{E+m}{4E}\right)^{1/2} . \tag{7.6}$$

The equations satisfied by the upper and lower components in the presence of the external potential are

$$\begin{aligned} m\psi_u[V] &= \left(E - V + \vec{\sigma}\cdot(\vec{\nabla}\Phi) - i\vec{\sigma}\cdot\vec{\nabla}\right)\psi_l[V] \\ m\psi_l[V] &= \left(E - V - \vec{\sigma}\cdot(\vec{\nabla}\Phi) + i\vec{\sigma}\cdot\vec{\nabla}\right)\psi_u[V] . \end{aligned} \tag{7.7}$$

Using the equation satisfied by the phase function, the second order equations satisfied by these components are

$$\begin{aligned} \left[2i(\vec{\nabla}\Phi)\cdot\vec{\nabla} + \vec{\nabla}^2 - i\vec{\sigma}\cdot e\vec{E}(r)\right] w_u[V] &= 0 \\ \left[2i(\vec{\nabla}\Phi)\cdot\vec{\nabla} + \vec{\nabla}^2 + i\vec{\sigma}\cdot e\vec{E}(r)\right] w_l[V] &= 0 . \end{aligned} \tag{7.8}$$

For both the electron and positron solution, we must demand continuity at $z = L$ with the final state plane wave. Thus the matrix element (7.1) splits into a phase factor that is the same as that found for the spin zero case, and a spinor factor which will be computed below.

The solutions correct to order $(1/p)$ are achieved by using $\frac{d\Phi}{dz} = p$ (leading order only); we find $(Z = 1 - z/L)$

$$w_u[V] = \left[1 - \frac{LZ}{2p}\vec{\sigma}\cdot e\vec{E}\right] w_u[0] \qquad w_l[V] = \left[1 + \frac{LZ}{2p}\vec{\sigma}\cdot e\vec{E}\right] w_l[0] . \tag{7.9}$$

These solutions can be simplified by expanding. To the order required we have:

Electron-positive helicity-

$$w_u[V] = \left[1 + \frac{1}{2p^-}\vec{\sigma}_\perp \cdot \overrightarrow{p^-_{loc}}\right] w_+ \qquad\qquad w_l[V] = + \left[\frac{m - \vec{\sigma}_\perp \cdot \overrightarrow{p^-_{loc}}}{2p^-}\right] w_+ \tag{7.10}$$

Electron-negative helicity-

$$w_u[V] = \left[\frac{m + \vec{\sigma}_\perp \cdot \overrightarrow{p^-_{loc}}}{2p^-}\right] w_- \qquad\qquad w_l[V] = + \left[1 - \frac{1}{2p^-}\vec{\sigma}_\perp \cdot \overrightarrow{p^-_{loc}}\right] w_- \tag{7.11}$$

Positron-positive helicity-

$$w_u[V] = \left[1 + \frac{1}{2p^+}\vec{\sigma}_\perp \cdot \overrightarrow{p^+_{loc}}\right] w_+ \qquad\qquad w_l[V] = - \left[\frac{m - \vec{\sigma}_\perp \cdot \overrightarrow{p^+_{loc}}}{2p^+}\right] w_+ \tag{7.12}$$

Positron-negative helicity-

$$w_u[V] = \left[\frac{m + \vec{\sigma}_\perp \cdot \overrightarrow{p^+_{loc}}}{2p^+}\right] w_- \qquad\qquad w_l[V] = - \left[1 - \frac{1}{2p^+}\vec{\sigma}_\perp \cdot \overrightarrow{p^+_{loc}}\right] w_- \ , \tag{7.13}$$

where the local momenta are given by Eq. (4.5) and (5.5),

$$(p^\pm_{loc})_\perp = p^\pm_\perp \pm \vec{E}_\perp LZ = \mp\frac{1}{4}\,\vec{\epsilon_\perp} \cdot \left[\overrightarrow{\Delta_\perp} - \frac{3}{aZ^2}\overrightarrow{\Sigma_\perp}\right] \ , \tag{7.14}$$

when evaluated at the stationary point.

The matrix element of the current is straightforward to evaluate from these explicit solutions to the order required. The non-helicity flip current matrix elements have the form

$$\begin{aligned}
\left\langle | \vec{\alpha} \cdot \vec{\epsilon} | \right\rangle &= (w_u^-)^\dagger \, \vec{\sigma} \cdot \vec{\epsilon} \, w_u^+ - (w_l^-)^\dagger \, \vec{\sigma} \cdot \vec{\epsilon} \, w_l^+ \\
\left\langle \uparrow | \vec{\alpha} \cdot \vec{\epsilon} | \uparrow \right\rangle &= \frac{1}{2kx(1-x)} \left[+\epsilon \cdot A(Z) + i\hat{z} \cdot (\epsilon \times B(Z)) \right] \\
\left\langle \downarrow | \vec{\alpha} \cdot \vec{\epsilon} | \downarrow \right\rangle &= \frac{1}{2kx(1-x)} \left[-\epsilon \cdot A(Z) + i\hat{z} \cdot (\epsilon \times B(Z)) \right] ,
\end{aligned} \tag{7.15}$$

where

$$\begin{aligned}
A(Z) &= xp^+_{loc} + (1-x)p^-_{loc} = xp^+_\perp + (1-x)p^-_\perp - (1-2x)eELZ \\
B(Z) &= xp^+_{loc} - (1-x)p^-_{loc} = xp^+_\perp - (1-x)p^-_\perp + eELZ \, .
\end{aligned} \tag{7.16}$$

The electric field E may also depend implicitly on the coordinate Z. The helicity flip matrix elements are

$$\begin{aligned}
\left\langle \uparrow | \vec{\alpha} \cdot \vec{\epsilon} | \downarrow \right\rangle &= \frac{1}{2kx(1-x)} m(\epsilon_x - i\epsilon_y) \\
\left\langle \downarrow | \vec{\alpha} \cdot \vec{\epsilon} | \uparrow \right\rangle &= \frac{1}{2kx(1-x)} m(\epsilon_x + i\epsilon_y) \, .
\end{aligned} \tag{7.17}$$

Now the average over the photon polarization and the sum over the final helicity states can be carried out. One must also evaluate the field E at the stationary point in impact parameter, and in the square of the matrix element keep track of the two integration variables, Z_1 and Z_2. The form for the polarization average of the two non-flip amplitudes is

$$Sum(non - flip) =$$

$$\frac{1}{2}\sum_{pol}\left[\left\langle\uparrow|\vec{\alpha}\cdot\vec{\epsilon}|\uparrow\right\rangle_2^*\left\langle\uparrow|\vec{\alpha}\cdot\vec{\epsilon}|\uparrow\right\rangle_1+\left\langle\downarrow|\vec{\alpha}\cdot\vec{\epsilon}|\downarrow\right\rangle_2^*\left\langle\downarrow|\vec{\alpha}\cdot\vec{\epsilon}|\downarrow\right\rangle_1\right] , \tag{7.18}$$

and an explicit calculation yields

$$Sum(non - flip) = \frac{1}{N_s}\,[A(Z_2)\cdot A(Z_1) + B(Z_2)\cdot B(Z_1)] , \tag{7.19}$$

where $N_s = [2kx(1-x)]^2$.

Using the explicit expressions given earlier, Eq. (7.16), we find

$$Sum(non - flip) = \frac{1}{2N_s}\,[x^2 + (1-x)^2]\; S(boson) , \tag{7.20}$$

where $S(boson) = P(Z_2)\cdot P(Z_1)$ from Eq. (4.21). The two flip amplitudes yield

$$Sum(flip) =$$

$$\frac{1}{2}\sum_{pol}\left[\left\langle\uparrow|\vec{\alpha}\cdot\vec{\epsilon}|\downarrow\right\rangle_2^*\left\langle\uparrow|\vec{\alpha}\cdot\vec{\epsilon}|\downarrow\right\rangle_1+\left\langle\downarrow|\vec{\alpha}\cdot\vec{\epsilon}|\uparrow\right\rangle_2^*\left\langle\downarrow|\vec{\alpha}\cdot\vec{\epsilon}|\uparrow\right\rangle_1\right]$$

$$= \frac{1}{N_s}\,2m^2 . \tag{7.21}$$

Taking into account the differing normalization conventions between fermions and bosons (recall the wave function normalization factor of $[8k^3x(1-x)]$ in the rate calculation), the fermion result can be directly related to the boson result by the simple substitution

$$S(fermion) = S_{NF}(x)\; S(boson) + S_F(x)\, 4m^2 , \tag{7.22}$$

where

$$S_{NF}(x) = \frac{x^2 + (1-x)^2}{x(1-x)}$$
$$S_F(x) = \frac{1}{x(1-x)} \ . \qquad (7.23)$$

Spin Effects and Limiting Behaviors:

Using the above substitution, one now finds instead of Eq. (6.6) the simple result

$$\frac{dP(pair)}{dx} = \alpha\, y\, C_p\, G(u,x)\ , \qquad (7.24)$$

and for the constant field case,

$$G_c(u_c,x) = \frac{1}{u_c} \int_{u_c}^{\infty} dv\ \left[S_{NF}(x)(2\,v - u_c) + S_F(x) u_c\right] Ai(v)\ , \qquad (7.25)$$

where the relation $u_c = u_p \epsilon^{-2/3}$ exposes the dependence on the overall field strength. For the uniform charge density pulse, Eq. (6.7) becomes

$$G(u_p,x) = \frac{1}{2u_p} \int_{u_p}^{\infty} dv \left[S_{NF}(x) \left(3\,v - 2\,u_p - \frac{u_p^4}{v^3} \right) + 2\,S_F(x)\, u_p \left(1 - \frac{u_p^3}{v^3} \right) \right] Ai(v)\,, \qquad (7.26)$$

Extreme Quantum Limit: In the limit of $C \to 0$, the differential production probabilities are (see Eq. (6.10))

$$\frac{dP_c}{dx} = 2\,\alpha\, y\, g_1\ \left(C\,\epsilon^2\right)^{1/3}\ [x(1-x)]^{2/3}\, S_{NF}(x)$$
$$\frac{dP}{dx} = \frac{3}{2}\,\alpha\, y\, g_1\ (C)^{1/3}\ [x(1-x)]^{2/3}\, S_{NF}(x)\ , \qquad (7.27)$$

and the total production probabilities are then achieved by integration:

$$\begin{aligned} P_c &= 10\,\alpha\,y\,g_1\,g_2\,\left(C\,\epsilon^2\right)^{1/3} \\ P &= \frac{15}{2}\,\alpha\,y\,g_1\,g_2\,\left(C\right)^{1/3}\,, \end{aligned} \tag{7.28}$$

which are a factor of 5 larger than the scalar boson case (compare Eq. (6.11)). This latter formula is shown in Fig. 1 and compared to an exact numerical integration.

Classical Limit: In the opposite limit of $C \to \infty$, the differential probabilities become (refer to Eq. (6.18)and expand the exponential dependence on C_x)

$$\begin{aligned} \frac{dP_c}{dx} &\simeq \frac{\alpha\,y}{4}\left(\frac{C\,\epsilon}{\pi}\right)^{1/2} S(x)\,\exp\left[-\frac{8}{3\epsilon}C\right]\,\exp\left[-\frac{8}{3\epsilon}C\,(2x-1)^2\right] \\ \frac{dP}{dx} &\simeq \frac{3\,\alpha\,y}{16}\left(\frac{1}{\pi C}\right)^{1/2} S(x)\,\exp\left[-\frac{8}{3}C\right]\,\exp\left[-\frac{8}{3}C\,(2x-1)^2\right]\,, \end{aligned} \tag{7.29}$$

where

$$S(x) = [4x(1-x)]^{3/2}\,[S_{NF}+S_F]\,. \tag{7.30}$$

Thus (at $\epsilon = 1$) we still have the relation

$$\frac{dP}{dx} = \frac{3}{4\,C}\,\frac{dP_c(\epsilon=1)}{dx}\,. \tag{7.31}$$

Using the fact that x is forced to one-half as C becomes large, we find that the spinor case is larger than the scalar case by the factor $S_{NF}(1/2) + S_F(1/2) = 6$. The total production probabilities are then (refer Eq. (6.20))

$$\begin{aligned} P_c &= \frac{3\alpha\,y}{8}\left(\frac{3}{2}\right)^{1/2}\epsilon\,\exp\left[-\frac{8}{3\epsilon}C\right] \\ P &= \frac{3}{4\,C}\,P_c(\epsilon=1)\,. \end{aligned} \tag{7.32}$$

More accurate formulas for the latter case of a uniform charge density are

$$\frac{dP}{dx} \simeq \frac{3\,\alpha\,y}{4\sqrt{\pi C}} \left[4x(1-x)\right]^{1/2}\; Q(x)\; \exp\left[-\frac{8}{3} C\,(1+(2x-1)^2)\right] \quad , \qquad (7.33)$$

where

$$Q(x) \;=\; \frac{x^2+(1-x)^2}{\left[1+\frac{4x(1-x)}{C}\right]} + \frac{1}{\left[1+\frac{6x(1-x)}{C}\right]} \quad , \qquad (7.34)$$

and

$$P \;=\; \frac{9\alpha\,y}{32}\left(\frac{3}{2}\right)^{1/2} \frac{1}{[C+\frac{3}{2}]} \exp[-\frac{8}{3}C] \quad . \qquad (7.35)$$

This latter formula is shown in Fig. 1 and compared to an exact numerical integration. In Fig. 2 a graph of the normalized x–distribution for several values of C are shown. The narrowing of the distribution for increasing C is evident. Finally, in Fig. 3, a comparison of the total production probability for spinor and scalar electrons are shown as a function of C.

A more complete discussion of the effects of pair production during pulse crossings will require a folding of the multi-photon spectrum into the production probabilities calculated here. If this folding process yields a pair production rate that is an important source of background, then one can change the collision geometry as has been suggested by Palmer,[23] and/or choose collider parameters as far into the classical regime as is necessary. For example, by using ribbon pulses to increase the effective value of $C \rightarrow D = C\,G$, in the notation of Ref. 8 , the beamstrahlung is decreased and in addition, the subsequent pair production is suppressed exponentially.

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FIGURE CAPTIONS

Figure 1. The total spinor pair production probability (divided by αy) as a function of C. The solid curve is the exact (numerical) result. The dashed curve is the approximation given in Eq. (7.35) relevant for large C, while the dotted curve is Eq. (7.28) which is a small C expansion. Typical values for the product αy lie in the range $1-5$.

Figure 2. The x-distribution of the produced pair as a function of the scaling variable C for larger C values. Each curve is normalized to unity. Note the narrowing of the curve as C increases. The alternative scaling variable χ $(=1/C)$ used in Ref. 11 is also given.

Figure 3. The total pair production probability (divided by αy) as a function of C for spin one-half electrons (solid) and for scalar electrons (dashed). These curves were computed by numerical integration of the exact formulae.

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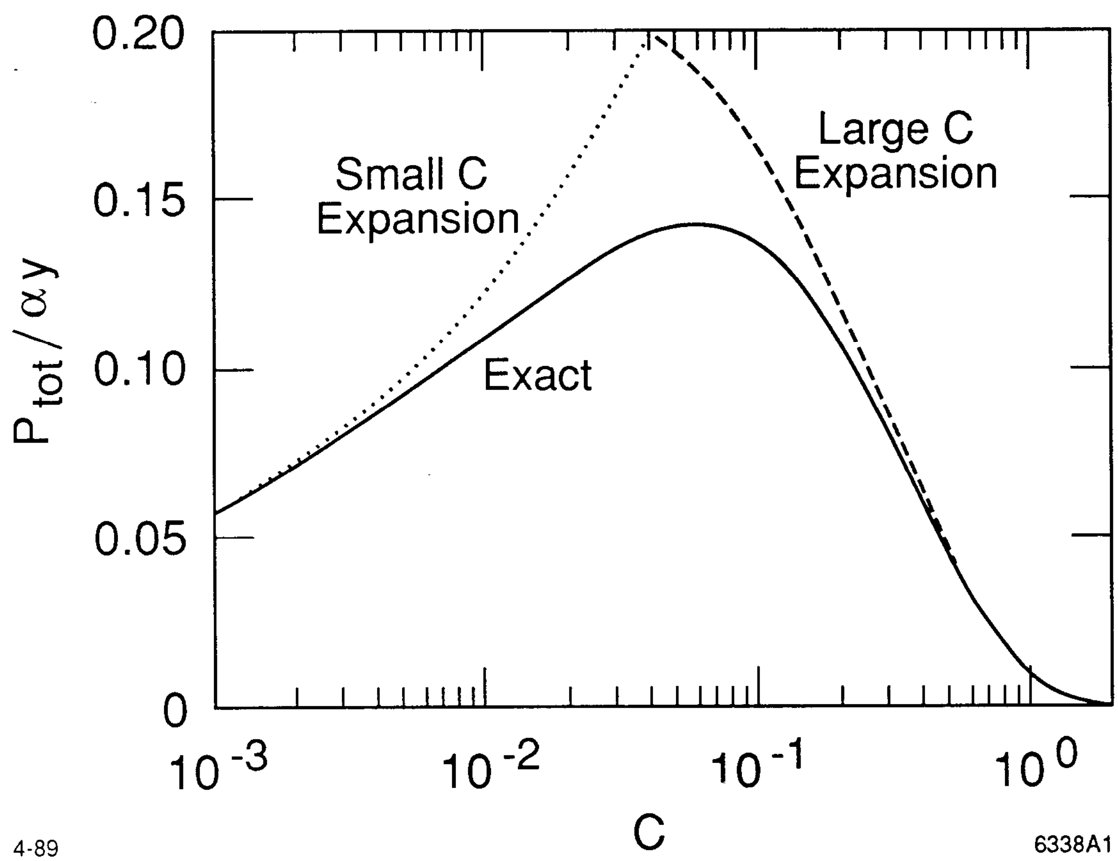
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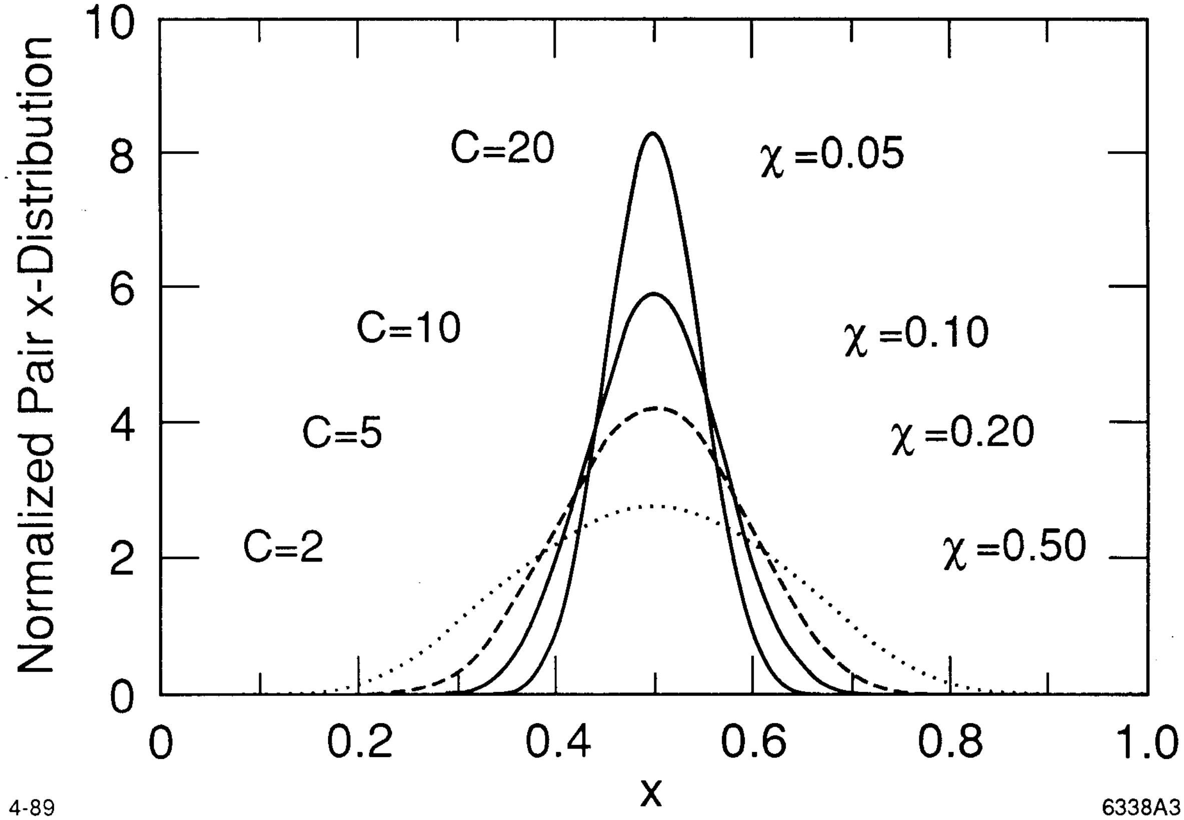
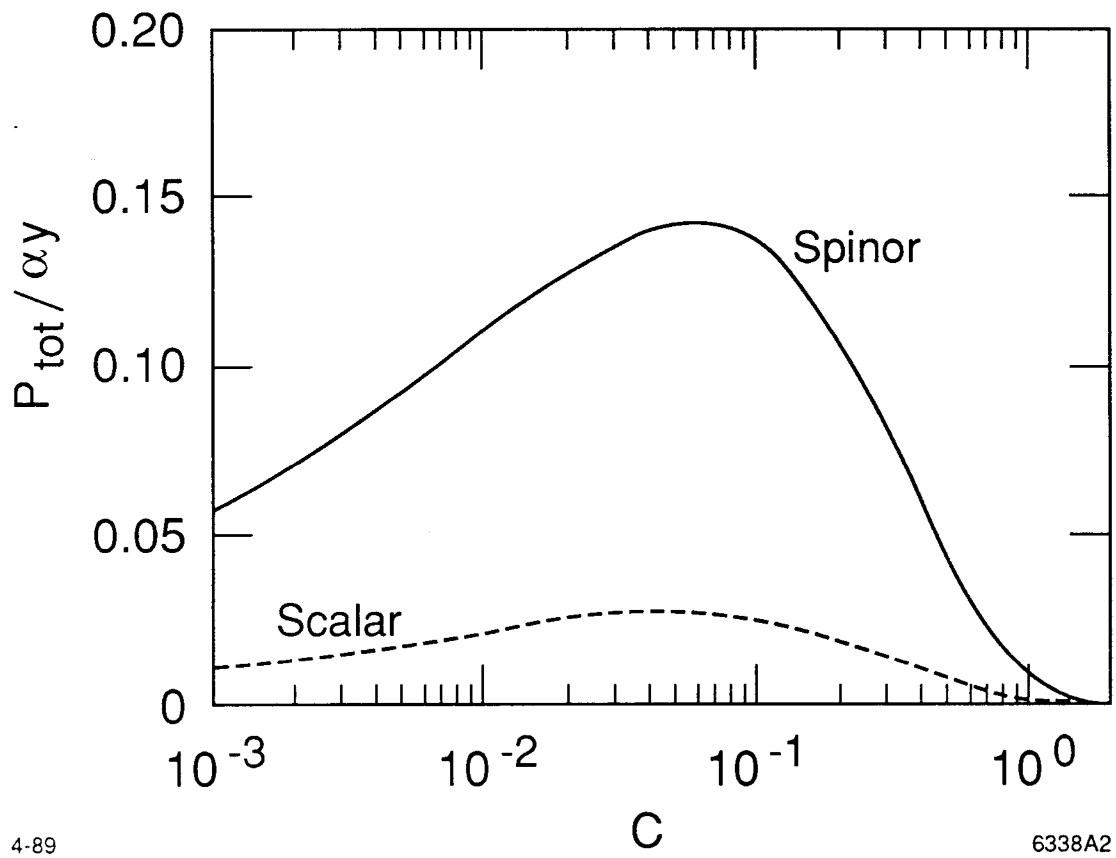
0.15
0.10
0.05
0
$P_{tot} / \alpha y$
Small C
Expansion
Large C
Expansion
Exact
10^{-3}
10^{-2}
10^{-1}
10^{0}
C
4-89
6338A1

Fig. 1



Normalized Pair x-Distribution
10
8
6
4
2
0
C=20
C=10
C=5
C=2
χ =0.05
χ =0.10
χ =0.20
χ =0.50
0
0.2
0.4
0.6
0.8
1.0
x
4-89
6338A3

Fig. 2



0.20
0.15
0.10
0.05
0
$P_{tot}/\alpha y$
Spinor
Scalar
10^{-3}
10^{-2}
10^{-1}
10^{0}
C
4-89
6338A2

Fig. 3