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LONGITUDINAL IMPEDANCE OF A SMOOTH TOROIDAL CHAMBER AT LOW AND INTERMEDIATE FREQUENCIES*

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ABSTRACT

We evaluate the longitudinal coupling impedance of a toroidal chamber with rectangular cross section in the frequency domain below the synchronous resonant modes. With infinite wall conductivity the impedance is purely reactive and consists of a "space charge" term, proportional to γ^{-2}, and a "curvature" term which survives at large γ. The curvature term is well represented as a quadratic function of frequency, namely

$$\frac{Z}{n} = iZ_o \left(\frac{h}{\pi R}\right)^2 \left[A - 3B\left(\frac{\nu}{\pi}\right)^2\right] ,$$

where h is the height of the chamber, R is the trajectory radius and $\nu = \omega h/c$. The constants A and B are of order 1. Thus, ImZ/n from curvature is typically a very small fraction of an ohm below the resonance domain, which begins where $\nu > (R/h)^{1/2}$. Consequences for beam stability, if any, arise from high frequency resonances, which can produce values of several ohms for Z/n.

1. INTRODUCTION

We consider a smooth toroidal vacuum chamber of rectangular cross section, as shown in Fig. 1. A beam circulating in such a chamber can excite resonant modes of the whole chamber that have phase velocity equal to the particle velocity. As is discussed in Refs. 1 and 2, these synchronous resonant modes are at frequencies greater than $\omega = n\omega_o$, where $\omega_o = \beta c/R$ is the revolution frequency for particles on a trajectory of radius R, and

$$n = \pi R^{3/2}/hw^{1/2} \quad . \tag{1.1}$$

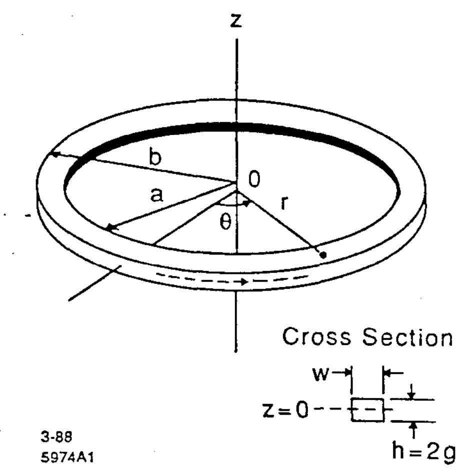

Fig. 1. Smooth toroidal vacuum chamber with rectangular cross section.

Here the beam is assumed to be centered in the chamber, which has height h and width w. Thus, the synchronous resonances are typically at rather high frequencies, for instance with $h = w = 3$ cm and $\beta = 1$, we have $f = \omega/2\pi > 100$ GHz for $R > 12$ m. Such frequencies are beyond the frequency spectrum of a typical bunch, if the charge distribution is smooth (a Gaussian or the like). Small ripples on a bunch could give high frequency components, however, and it is therefore not excluded that a very large vacuum chamber impedance at high frequency could be detrimental to beam stability. Resonant impedances in the present model are indeed large. In Fig. 2 we show a graph from Ref. 1 giving the real part of $Z(n, n\omega_o)/n$ for a chamber having roughly the dimensions of the SLC damping rings. Here $Z(n,\omega)$ is the longitudinal coupling impedance at longitudinal mode number n and circular frequency ω, for resistive chamber walls with the resistivity of aluminum. Since Z/n for the damping rings is thought to have a broad band value of around 2.5 ohms in the region of a few GHz, the value of 36 ohms at the first resonance is rather startling.

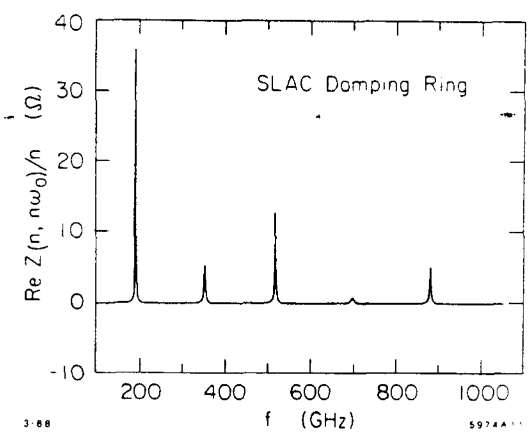

Fig. 2. The real part of $Z(n, n\omega_o)/n$ in the high frequency resonance region.

The present work is concerned with a question that was not treated thoroughly in Refs. 1 and 2; namely, the behavior of the longitudinal coupling impedance at frequencies from zero up to the first synchronous resonance. Since the problem is treated in great detail in a forthcoming paper[3] we shall merely summarize results.

2. LONGITUDINAL IMPEDANCE AT LOW AND INTERMEDIATE FREQUENCIES

It is convenient to work with a dimensionless variable proportional to frequency,

$$\nu = \frac{\omega h}{c} = \frac{2\pi h}{\lambda} \quad , \tag{2.1}$$

where h is the height of the chamber and λ is the wavelength. The previous papers[1,2] treated the region $\nu > (R/h)^{1/2}$ and also a small region near $\nu = 0$. We are presently concerned with the interval

$$0 < \nu < (R/h)^{1/2} \quad . \tag{2.2}$$

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The electromagnetic fields are expressed in terms of Bessel functions, in the cylindrical coordinates shown in Fig. 1. The impedance has a fairly involved expression in terms of the cross products of Bessel functions. In order to make that expression understandable, or even to compute it numerically, one has to invoke asymptotic expansions of Bessel functions. The matter of choosing appropriate expansions has previously caused some confusion. We have found that Olver's uniform asymptotic expansions are appropriate for $\nu > (R/h)^{1/2}$, while the Debye expansions suffice for $\nu < (R/h)^{1/2}$. The manipulations and estimates required to pick out the significant terms of the impedance are somewhat complicated, but the result is remarkably simple.

For the present study, we took the chamber to be perfectly conducting, since it seems likely that resistive wall effects have the same order of magnitude as in the case of a straight beam tube. Consequently, the impedance in the subresonant region is purely imaginary. It consists of a part which vanishes as γ^{-2} in the high energy limit $\gamma \to \infty$, usually called the "space charge" term, and a part that survives at large γ, which we call the "curvature" term. The space charge term, defined through Eqs. (4.14) and (4.23) of Ref. 3, has the same order of magnitude as that for straight beam tube. As usual, it diverges logarithmically when the beam approaches an ideal line charge. The curvature term is finite in the limit of a line charge. Its value in that limit is, to a good approximation,

$$\begin{aligned}\frac{Z(n,n\omega_o)}{n} &= \\ & iZ_o\left(\frac{h}{\pi R}\right)^2 \Big\{\Big[1 - \exp\{-2\pi(b-R)/h\} \\ & - \exp\{-2\pi(R-a)/h\}\Big] \\ & \times\left[1-3\left(\frac{\nu}{\pi}\right)^2\right] + 0.05179 - 0.01355\left(\frac{\nu}{\pi}\right)^2\Big\} + \rho \\ &= iZ_o\left(\frac{h}{\pi R}\right)^2\left[\hat{A} - 3\hat{B}\left(\frac{\nu}{\pi}\right)^2\right] + \rho \quad , \qquad (2.3)\end{aligned}$$

where $Z_o = 120\pi$ ohms is the impedance of free space, and the geometric parameters are as defined in Fig. 1. The constants $\hat{A}$ and $\hat{B}$ are nearly equal to 1. The term ρ, defined in Eq.(4.24) of Ref. 3, vanishes exponentially as the aspect ratio w/h increases, and is negligible for $w/h \geq 2$.

By numerical evaluation, we find that even when ρ is not negligible, it is very nearly a quadratic function of ν. It may then be combined with the first term of Eq. (2.3) to give the simple general result

$$\frac{Z(n,n\omega_o)}{n} = iZ_0\left(\frac{h}{\pi R}\right)^2\left[A - 3B\left(\frac{\nu}{\pi}\right)^2\right] , \quad \nu < (R/h)^{1/2} . \qquad (2.4)$$

For typical parameters, the dimensionless coefficients A, B are of order unity. For $R = 5.7$ m and $\beta = 1$ we find the following values:

$$\begin{aligned} A &= 0.7153 \ , \ B = 0.6714 \ , \ (w = 0.02\ m h = 0.02\ m) \\ A &= 1.009 \quad , \ B = 0.9766 \ , \ (w = 0.02\ m h = 0.01\ m) \\ A &= 0.2531 \ , \ B = 0.2117 \ , \ (w = 0.01\ m h = 0.02\ m) \qquad (2.5)\end{aligned}$$

The first example of Eq. (2.5) corresponds roughly to the parameters of the SLAC damping rings. For this example, we plot $Im Z/n$ in Fig. 3. The dashed curve is an essentially exact evaluation of the impedance using high order asymptotic expansions, while the solid curve is from Eq. (2.5). The quadratic formula in Eq. (2.4) fits the exact evaluation to three digits.

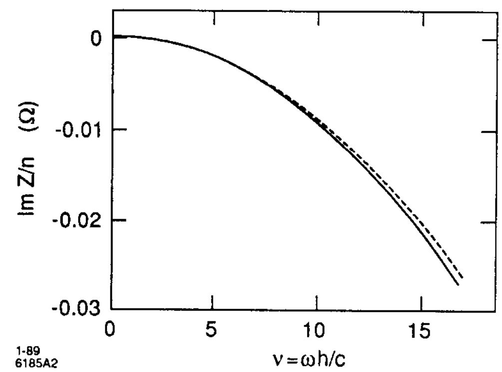

Fig. 3. Reactive longitudinal impedance due to curvature in the subresonant domain. Parameters are for the first example of Eq. (2.5).

Since A/B is close to 1, the reactive impedance has a zero near $\nu = \pi/3^{1/2}$. This is to be compared with the lowest TE cutoff, which for a straight rectangular tube lies at

$$\nu = \begin{cases} \pi \ , & h > w \ , \\ \pi h/w \ , & h < w \ . \end{cases} \qquad (2.6)$$

Since Z is positive imaginary at $\nu = 0$, a zero is expected; Z must be negative imaginary just before the first resonance.

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