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CONSTRUCTION AND FOURIER ANALYSIS OF INVARIANT SURFACES FROM TRACKING DATA*

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ABSTRACT

We study invariant surfaces in phase space by application of a symplectic tracking code. For motion in two degrees of freedom we use the code to compute $\mathbf{I}(s), \mathbf{\Phi}(s)$ for $s = 0, C, 2C \ldots nC$, where $\mathbf{I} = (I_1, I_2), \mathbf{\Phi} = (\phi_1, \phi_2)$ are action-angle coordinates of points on a single orbit, and C is the circumference of the reference orbit. As a test to see whether the orbit lies on an invariant surface (*i.e.*, to test for regular and nonresonant motion) we fit the points to a smooth, piece-wise polynomial surface $\mathbf{I} = \hat{\mathbf{I}}(\phi_1, \phi_2)$. We then compute additional points on the same orbit, and test for their closeness to $\hat{\mathbf{I}}$. We find that data from a few thousand turns are sufficient to construct accurate approximations to an invariant surface, even in cases with strong nonlinearities. Two-dimensional Fourier analysis of the surface leads to information on the strength of nonlinear resonances, and provides the generator of a canonical transformation as a Fourier series in angle variables. The generator can be used in a program to derive rigorous bounds on the motion for a finite time T.

1. INTRODUCTION

Tracking of single particles by numerical integration of Hamilton's equations has become a standard and indispensable tool in the design of new accelerators and storage rings. Nevertheless, the question of how to interpret tracking data has not received a fully satisfactory answer. One approach is to compute a quantity called the "smear," which is a measure of the departure from linear motion as observed in tracking.[1] Through experience and guesswork, one tries to set a safe upper limit for the smear in lattice design. Unfortunately, there is no general and reliable rule for choosing a limit; indeed, the issue may depend on the type of lattice. It seems unlikely that a single number, the smear, can adequately characterize the large variety of nonlinear effects that might occur.

We describe a simple analysis of tracking data that gives much more detailed information about nonlinear effects. In an elementary and direct way we find the effect of each nonlinear resonance on the orbit in question. This information might be used in lattice optimization. Moreover, the method yields a generator of a canonical transformation, which can be used in a more ambitious program to set bounds on the motion for a finite time T.

2. FOURIER ANALYSIS OF INVARIANT TORI AND NONLINEAR RESONANCES

We discuss betatron motion in two degrees of freedom, employing angle-action coordinates $\mathbf{\Phi} = (\phi_1, \phi_2), \mathbf{I} = (I_1, I_2)$. The Cartesian phase-space coordinates $\mathbf{x} = (x_1, x_2), \mathbf{p} = (p_1, p_2)$ are given in terms of the lattice functions $\beta_i(s)$ by

$$x_i = [2I_i\beta_i(s)]^{1/2} \cos\phi_i \quad , \tag{1}$$

$$p_i = x_i' = -[2I_i/\beta_i(s)]^{1/2}\left[\sin\phi_i - \frac{1}{2}\beta_i'(s)\cos\phi_i\right] \quad . \tag{2}$$

According to Hamilton-Jacobi theory,[2] an invariant surface in phase space has the form

$$\mathbf{I} = \mathbf{J} + G_{\mathbf{\Phi}}(\mathbf{J}, \mathbf{\Phi}, s) \quad . \tag{3}$$

On the surface, the action $\mathbf{I}$ is a function of angle $\mathbf{\Phi}$ at each orbital location s. The value of the constant parameter $\mathbf{J}$, the invariant action, distinguishes one surface from another. The generating function G satisfies the Hamilton-Jacobi equation

$$H(\mathbf{J} + G_{\mathbf{\Phi}}, \mathbf{\Phi}, s) + G_s = H^{(1)}(\mathbf{J}, s) \quad , \tag{4}$$

where H is the Hamiltonian, and subscripts to G indicate partial derivatives.

The generator is 2π-periodic in $\mathbf{\Phi}$, and C-periodic in s, where C is the circumference of the closed reference orbit. Accordingly, the invariant surface is a three-dimensional torus, a point on the surface being specified by the three coordinates (ϕ_1, ϕ_2, s). A section of this torus at fixed s may be plotted in three-dimensional space. For instance, one may plot $I_i(s = 0)$ as a function of both angles ϕ_1, ϕ_2. It is such a plot that we produce by fits to tracking data. The invariant associated with a surface $\mathbf{I}(\mathbf{\Phi}, s)$ is, owing to periodicity of G, the integral

$$\mathbf{J} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} d\phi_1 d\phi_2 \mathbf{I}(\mathbf{\Phi}, s) \quad . \tag{5}$$

In view of periodicity in $\mathbf{\Phi}$ it is natural to expand the generator in a Fourier series, so that $G_{\mathbf{\Phi}}$ has the form

$$G_{\mathbf{\Phi}} = \sum_{\mathbf{m}} i\mathbf{m} g_{\mathbf{m}}(\mathbf{J}, s) e^{i\mathbf{m}\cdot\mathbf{\Phi}} \quad . \tag{6}$$

The coefficients $g_{\mathbf{m}}$ are familiar objects in canonical perturbation theory. As is well-known, the term $i\mathbf{m}g_{\mathbf{m}}$ is relatively large when $\mathbf{m} \cdot \boldsymbol{\nu}$ is sufficiently close to p/q, where p and q are small integers and $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the tune. Beyond this qualitative statement, it is helpful to know rather accurately the coefficients $i\mathbf{m}g_{\mathbf{m}}$. For a linear lattice these coefficients are zero, and to optimize a nonlinear lattice one can try to diminish offensively large coefficients by adjusting sextupole schemes, tunes, *etc.* A calculation of the coefficients by perturbation theory is reliable only for very weak nonlinearities. Nonperturbative methods, based on iterative solution of the Hamilton-Jacobi equation,[2] can deal with strong nonlinear effects, but as implemented at present are costly in computer time for large lattices. Although there are good prospects for reducing expense through improvements of these methods, an immediate route to inexpensive evaluation of the coefficients follows from the present work.

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To find invariant tori in tracking data, we follow a single particle orbit through several thousand turns, recording the values of $\mathbf{I}, \mathbf{\Phi}$ after each turn. The values of $\mathbf{\Phi} = (\phi_1, \phi_2)$ are computed modulo 2π, and of course are scattered over the square $(0, 2\pi) \times (0, 2\pi)$ in a fairly random manner. To fit the values of $\mathbf{I}$ to a smooth surface $\hat{\mathbf{I}}(\phi_1, \phi_2)$ one needs a surface-fitting program that can handle scattered values of $\mathbf{\Phi}$. We have applied the IMSL[3] routine IQHSCV, which uses Akima's algorithm[4] to put a smooth surface through the data $\mathbf{I}(\mathbf{\Phi})$. It does not smooth or average the data; the surface passes through each data point, and has continuous first partial derivatives. It consists of separate fifth-degree polynomials joined smoothly, each polynomial defined on a triangle in the $\mathbf{\Phi}$ plane with vertices at data points.

The program IQHSCV does not respect the periodicity condition, so that to apply it effectively we first have to extend the data by periodicity from the square $(0, 2\pi) \times (0, 2\pi)$ to a slightly bigger square $(-\epsilon, 2\pi + \epsilon) \times (-\epsilon, 2\pi + \epsilon)$. Otherwise, spurious jumps in the surface appear at the edges of the square, because without periodicity the continuity conditions have no power at the edges. We first take $0 < \phi_2 < 2\pi$ and extend the data by periodicity in ϕ_1, and then extend the new set in ϕ_2. Fitting a surface to the extended data set, then restricting attention to the original square, we obtain a surface that is very nearly periodic and well-behaved at the edges. Next we compute the Fourier coefficients of the surface by an FFT program, and form the sum Eq. (6) as the final proposal for an approximate invariant surface.

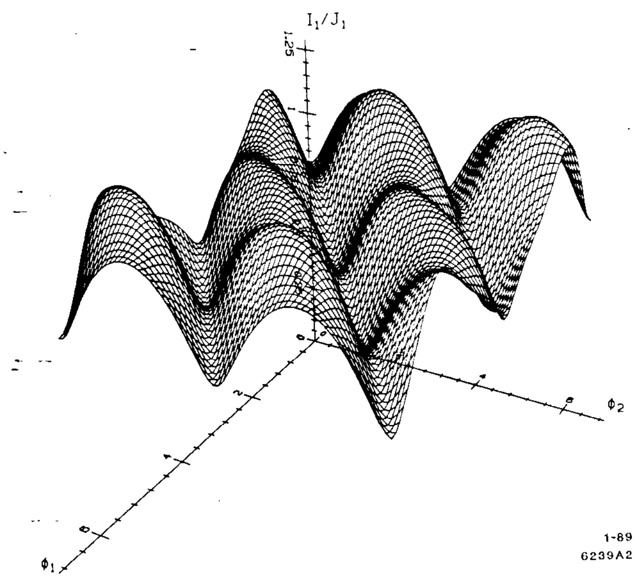

Fig. 1: Invariant surface for SLC North Damping Ring, obtained by a fit to tracking data. $I_1(\Phi_1, \Phi_2)/J_1$ *plotted as function of* (Φ_1, Φ_2). *Invariant action* $J_1 = 2.489 \cdot 10^{-6}$ m.

In Figs. 1 and 2, we show results for the North Damping Ring of the SLC. Normalizing to the invariants, we plot I_1/J_1 and I_2/J_2 versus (ϕ_1, ϕ_2) at s = 0. The initial displacements of the orbit correspond to about 3.7 mm (horizontal) and 2.6 mm (vertical) at the septum magnet. A beam with emittance about 100 times larger than the final damped emittance would contain orbits with such offsets. Note that the origin of I_i/J_i in the plots is at zero. The deviation of the surfaces from planarity, *i.e.*, the deviation from linear motion, is impressively large. Each of the surfaces was fitted to 8000 points obtained from 8000 turns of a symplectic fourth-order tracking code. Because of the periodic

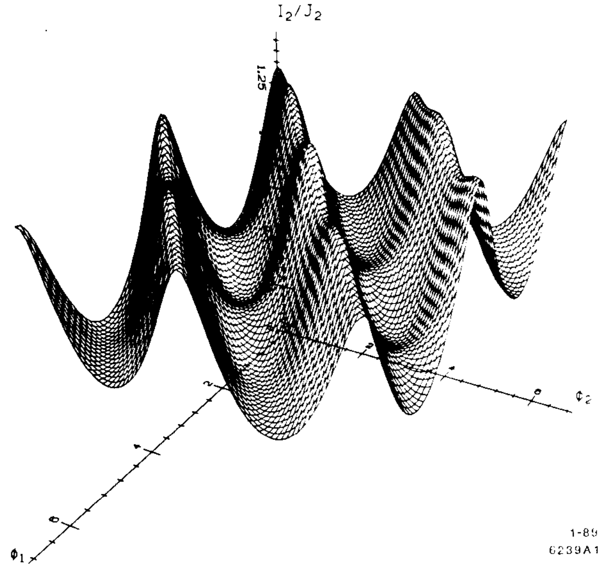

Fig. 2: Invariant surface for SLC North Damping Ring, obtained by a fit to tracking data. $I_2(\Phi_1, \Phi_2)/J_2$ *plotted as function of* (Φ_1, Φ_2). *Invariant action* $J_2 = 1.508 \cdot 10^{-6}$ m.

extension of data mentioned above, the fitting program actually fitted 9631 points, requiring 11.4 minutes on the IBM 3081 to make the fit.

Taking an FFT of the surface in Fig. 1 or Fig. 2, and retaining modes for $|m_1|, |m_2| \le 15$, we obtain a Fourier series representation Eq. (6) of the surface, which agrees well with the original piece-wise polynomial representation, and also agrees well with tracking beyond the original 8000 turns. Defining a metric ΔI for the difference between the tracked orbit and the Fourier series (the sum of deviations divided by the number of orbit points) we find $\Delta I_1/J_1 = 1.4 \cdot 10^{-4}, \Delta I_2/J_2 = 2.2 \cdot 10^{-4}$ for the first 8000 turns, and then very similar values for an additional 8000 turns. On using more than 8000 turns to find the piece-wise polynomial surface and more modes in Eq. (6), we did not see much decrease of $\Delta I_i/J_i$. We did, however, maintain values less than $2 \cdot 10^{-4}$ for $\Delta I_i/J_i$ over 32000 turns. In Table 1, we give the ten largest coefficients of I_1, measured in meters.

Table 1: Fourier Amplitudes of G_{Φ_1}

m_1	m_2	$\mathrm{Re}(im_1 g_m)$	$\mathrm{Im}(im_1 g_m)$
2	-2	$-2.06 \cdot 10^{-7}$	$-9.54 \cdot 10^{-9}$
1	0	$6.76 \cdot 10^{-8}$	$-1.91 \cdot 10^{-9}$
3	0	$-6.61 \cdot 10^{-8}$	$2.40 \cdot 10^{-10}$
4	-4	$-4.12 \cdot 10^{-8}$	$-3.77 \cdot 10^{-9}$
2	2	$1.21 \cdot 10^{-8}$	$-9.57 \cdot 10^{-11}$
5	-2	$1.17 \cdot 10^{-8}$	$5.09 \cdot 10^{-10}$
6	-6	$-1.16 \cdot 10^{-8}$	$-1.59 \cdot 10^{-9}$
1	2	$-9.34 \cdot 10^{-9}$	$4.48 \cdot 10^{-10}$
8	-8	$-4.15 \cdot 10^{-9}$	$-7.64 \cdot 10^{-10}$
7	-4	$2.98 \cdot 10^{-9}$	$2.62 \cdot 10^{-10}$

The approximate invariant surface in the form Eq. (6) for $s = 0$ can be extended to all s by integration of the Hamilton-Jacobi equation, Eq. (4). Further iterations of the Hamilton-Jacobi equation can refine the resultant generator's accuracy, possibly on a bigger set of Fourier modes. This could be a starting-point for a thorough stability study, as outlined in Ref. 5.

3. CONCLUSION

For motion in two transverse degrees of freedom, it is useful to plot tracking data in three dimensions. When an orbit lies on an invariant surface, it is not difficult to construct an accurate Fourier representation of the surface from data acquired in a few thousand turns of tracking. This provides a deeper view of the motion than the usual two-dimensional plots, since the effect of each potential nonlinear resonance is displayed.

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