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ANALYTIC ESTIMATES OF COUPLING IN DAMPING RINGS

T. O. RAUBENHEIMER AND R. D. RUTH

Stanford Linear Accelerator Center, Stanford University, Stanford, CA, 94309

ABSTRACT

In this paper we present analytic formulas to estimate the vertical emittance in weakly coupled electron/positron storage rings. We consider contributions from both the vertical dispersion and linear coupling of the betatron motions. In addition to simple expressions for random misalignments and rotations of the magnets, formulas are presented to calculate the vertical emittance blowup due to orbit distortions. The orbit distortions are assumed to be caused by random misalignments, but because the closed orbit is correlated from point to point, the effects must be treated differently. We consider only corrected orbits. Finally, the analytic expressions are compared with computer simulations of storage rings with random misalignments.

INTRODUCTION

In an ideal uncoupled ring there is no vertical dispersion or linear coupling. Thus the synchrotron radiation opening angle, which is very small, determines the vertical emittance. In practice, this is not the case. First, vertical bending fields and a non-zero vertical orbit in the quadrupole magnets will directly introduce some vertical dispersion. Second, a non-zero vertical orbit through the sextupole magnets, vertical sextupole misalignments, or rotational misalignments of the quadrupoles couple the horizontal and vertical planes. This coupling has two effects both of which increase the vertical emittance. It couples the horizontal dispersion to the vertical, causing an increase in the vertical, and it couples the x and y betatron motion so that energy is transferred between the two.

In this paper we analyze the effects of the coupling perturbatively, *i.e.* assuming a large aspect ratio ϵ_x/ϵ_y. We will first describe the closed orbit correlation function which we use to approximate a corrected closed orbit. Next we will calculate the dispersion resulting from both a distribution of random errors and a corrected closed orbit. We will then calculate results for the linear coupling in a similar manner. Because the contributions from the vertical dispersion and the linear coupling are statistically independent, these contributions to ϵ_y simply add.

CLOSED ORBIT

We only consider the closed orbit after substantial correction. When the orbit is corrected its Fourier spectrum tends towards that of white noise. The orbit correction reduces the dominant harmonics on either side of the tune while increasing the other modes. We approximate this by assuming that the correctors "randomize" the orbit and thus points on either side of a corrector are uncorrelated.

The closed orbit is found using the periodic Green function for the ring. The driving term is: $G(s) = G_{yc} + \Theta_B G_x - K_1 y_m$ where Θ_B are bend rotations, y_m are vertical misalignments, and G_{yc} is the inverse bending radius of any vertical bending fields. Thus, the correlation function between two points s and s' is

$$\langle y_c(s')y_c(s)\rangle = \frac{\sqrt{\beta_y(s)\beta_y(s')}}{4\sin^2\pi\nu_y}\int_{s'}^{s'+C} dz' \int_{s}^{s+C} dz\sqrt{\beta'\beta} \times \langle G(z')G(z)\cos'\cos\rangle \tag{1}$$

where $\beta' = \beta_y(z')$, $\cos = \cos(\psi_y(s) - \psi_y(z) + \pi\nu_y)$, $\cos' = \cos(\psi_y(s') - \psi_y(z') + \pi\nu_y)$, and C is the ring circumference.

Equation (1) can be written as the square of the closed orbit at s multiplied by both $\cos\Delta\psi$ and the square root of the beta functions, plus a term of order $\Delta s/C$. Since we assume that many dipole correctors are used, $\Delta s/C$ will be small. Thus we approximate the correlation function as

$$\langle y_c(s')y_c(s)\rangle = \sqrt{\beta_y(s)\beta_y(s')}\frac{\langle y_c^2\rangle}{\beta_y}\begin{cases}\cos\Delta\psi, & \text{No correctors between } s \text{ and } s'\\ 0, & \text{Otherwise}\end{cases} \tag{2}$$

where $\Delta\psi = \psi_y(s') - \psi_y(s)$ and the term $\langle y_c^2\rangle/\beta_y$ is the square of the residual orbit after correction. Note that although we could improve the approximation by assuming that the correlation decays in some manner, rather than just dropping to zero, our approximation should illustrate the essential behavior.

VERTICAL DISPERSION

In the limit of small coupling, the equation for the vertical dispersion is

$$\eta_y'' + K_1\eta_y \simeq K_2 y_c\eta_x - \widetilde{K}_1\eta_x - G_{yc} + K_1 y_c \tag{3}$$

where η_x is the unperturbed horizontal dispersion and K_1, $\widetilde{K}_1$, and K_2 are the normalized quadrupole, skew quadrupole, and sextupole strengths respectively. The solution for η_y is found in the same manner as is the closed orbit, namely, by using the periodic Green function for the ring. The solution is

$$\eta_y = \frac{\sqrt{\beta_y(s)}}{2\sin\pi\nu_y}\int_{s}^{s+C}\sqrt{\beta_y(z)}\cos(\phi_y(s) - \phi_y(z) + \pi\nu_y)F dz \tag{4}$$

where $F(z) = (K_1 + K_2\eta_x)y_c - \widetilde{K}_1\eta_x - G_{yc}$. Notice that the term $(K_1 + K_2\eta_x)y_c$ is proportional to the *local* chromaticity; the chromaticity is given by

$$\xi_y \equiv \frac{d\nu_y}{dp/p_0} = -\frac{1}{4\pi}\oint(K_1 + K_2\eta_x)\beta_y ds\ . \tag{5}$$

Thus the vertical dispersion can be reduced by using *local* chromatic correction which reduces the driving term $F(z)$.

We can now calculate the vertical dispersion due to an ensemble of random errors. First we will present well known results for random quadrupole misalignments and bend rotations. Then we discuss the effects of a corrected closed orbit. Because a closed orbit is correlated from point to point, the effects differ from those of random errors. In this paper we only consider a corrected closed orbit; the case of an uncorrected orbit is discussed in Refs. 1-3.

The equation for $\langle\eta_y{}^2\rangle/\beta_y$ is:

$$\frac{\langle\eta_y{}^2\rangle}{\beta_y} = \frac{1}{4\sin^2\pi\nu_y}\int_{s}^{s+C}\int dz dz'\sqrt{\beta\beta'}\langle\cos\cos' F^2(z,z')\rangle \tag{6}$$

where β', cos, and cos' are defined as they were in Eq. (1). The function $F^2(z,z')$ is

$$F^2(z,z') = 4K_1K_1'\Theta\Theta'\eta_x\eta_x' + K_2K_2'y_m y_m'\eta_x\eta_x' + GG' - 2Gy_c'f' + y_c y_c' f f'\ , \tag{7}$$

where the primed quantities are functions of z'. In addition, $f(z)$ is proportional to the local chromaticity: $f(z) = K_1 + K_2\eta_x$.

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Because the errors are uncorrelated, the first three terms of Eq. (6) are

$$\frac{\langle \eta_y{}^2\rangle_{\text{quad rotation}}}{\beta_y} = \frac{1}{2\sin^2\pi\nu}\sum_{\{\text{quad}\}}(K_1L)^2\Theta^2\beta_y\eta_x^2$$
$$\frac{\langle \eta_y{}^2\rangle_{\text{sext misalign}}}{\beta_y} = \frac{1}{8\sin^2\pi\nu}\sum_{\{\text{sext}\}}(K_2L)^2y_m^2\beta_y\eta_x^2 \tag{8}$$

$$\frac{\langle \eta_y{}^2\rangle_{\text{dipole kicks}}}{\beta_y} = \frac{\langle y_c^2\rangle}{\beta_y}\,, \tag{9}$$

where the term $\langle y_c^2\rangle/\beta_y$ in Eq. (9) is equal to the square of the *residual* of the corrected orbit.

The next two terms can be written as a function of the closed orbit correlation function — Eq. (2). For the fourth term, we find

$$\left(\frac{\langle \eta_y{}^2\rangle}{\beta_y}\right)_4 \approx \frac{-1}{2\sin\pi\nu_y}\frac{\langle y_c^2\rangle}{\beta_y}J_0\frac{\cos\pi\nu_y}{N_{\text{cor}}}\,, \tag{10}$$

where N_{corr} is equal to the number of correctors and we have assumed that the correctors are uniformly distributed. Note that we have averaged over s since we are not interested in the explicit s dependance.

The function J_n was introduced in Ref. 4; it is the fourier transform of the chromaticity function $f\beta$:

$$J_n = \int_0^C ds(K_1 + K_2\eta_x)\beta_y e^{in\phi}\,, \tag{11}$$

where $\phi \equiv \int ds/\beta_y\nu_y$ so $\phi(C) - \phi(0) = 2\pi$. Notice that J_0 is proportional to the vertical chromaticity, Eq. (5): $J_0 = -4\pi\xi_y$.

Finally, the fifth term of Eq. (6) is

$$\left(\frac{\langle \eta_y{}^2\rangle}{\beta_y}\right)_5 \approx \frac{1}{16\sin^2\pi\nu_y}\frac{\langle y_c^2\rangle}{\beta_y}\frac{1}{N_{\text{corr}}}\Big[J_0^2 + |J_m|^2 + \sum_{j=1}^{\infty}(2|J_j|^2 + |J_{m+j}|^2 + |J_{m-j}|^2)\text{sinc}^2 j\pi/N_{\text{corr}}\Big]\,, \tag{12}$$

where m is the integer *nearest* to $2\nu_y$ and $\text{sinc}\,x = \sin x/x$. The sinc^2 function will decrease rapidly as j increases. The width of the function is roughly $j \sim N_{\text{corr}}/2$. This shows that as the orbit is corrected more components of $f\beta$ contribute to the vertical dispersion. As in the case of the uncorrected orbit, the components that are sampled are centered about the frequencies 0 and $2\nu_y$. If no large additional components of $f\beta$ are sampled, the dispersion of the corrected orbit should be smaller than that of an uncorrected orbit; orbit correction reduces the residual orbit $\langle y_c^2\rangle/\beta$ and the factor $1/N_{\text{corr}}$ further decreases the contribution.

Now, given the vertical dispersion, we can calculate the contribution to the vertical emittance. The Courant-Snyder dispersion invariant $\mathcal{H}_y$ is equal to $2\langle \eta_y{}^2\rangle/\beta_y$. Thus, the emittance is

$$\epsilon_{y\,\text{disp}} \approx 7.68\times10^{-13}\frac{\gamma^2}{J_y}\frac{\langle\eta_y^2\rangle}{\beta_y}\frac{\oint|G|^3ds}{\oint G^2ds} = \frac{2J_\epsilon}{J_y}\frac{\langle\eta_y^2\rangle}{\beta_y}\sigma_\epsilon^2\,, \tag{13}$$

where, for convenience, we have used the relative energy spread to simplify the expression. Here, J_ϵ is the longitudinal damping partition and σ_ϵ is the relative energy spread.

In Figure 1 we have plotted the vertical dispersion in the North damping ring (NDR) of the Stanford Linear Collider (SLC) vs. the vertical chromaticity. The points were found by simulating random quadrupole alignment errors $y_m = 150\,\mu$m. The resulting closed orbit was then corrected with 10 dipole correctors using an RMS minimization procedure. The data points were calculated from ten different error distributions and the dashed curve was calculated from Eqs. (9), (10), and (12). Notice that the analytic results agree quite well with the simulation.

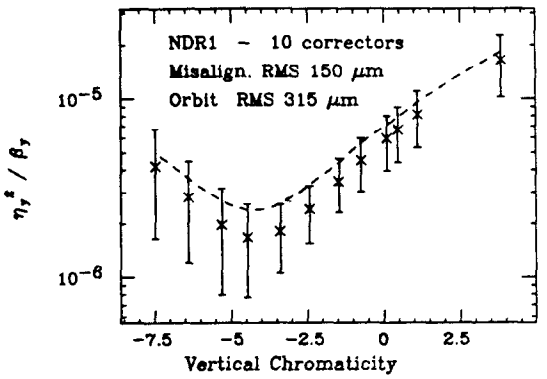

Fig. 1. Vertical dispersion vs. chromaticity.

In Figure 2, we have plotted the contribution from the vertical dispersion to the emittance for the chromatically corrected ring. Here, we have scaled the vertical emittance by $\langle y_c^2\rangle/\beta_y$ and we have plotted the contribution vs. the number of correctors used to correct the orbit. Notice that our approximations agree very well when $N_{\text{corr}} \gg 1$. Also notice that, initially, the scaled emittance increases; this occurs since J_n has a large mode at the ring superperiodicity, $N_s = 2$, which is sampled when the orbit is corrected. After the initial increase, the contribution decreases as $1/N_{\text{corr}}$ as expected.

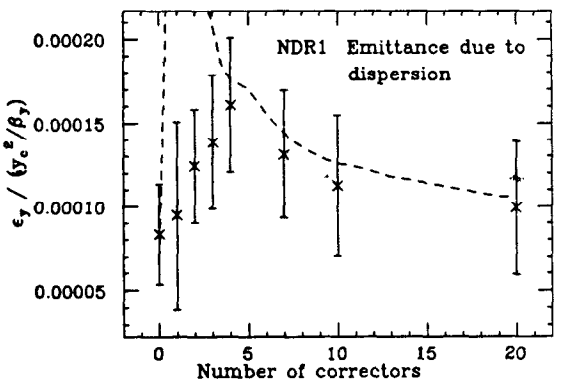

Fig. 2. Vertical emittance vs. N_{corr}.

LINEAR COUPLING

The equation for the vertical betatron motion is

$$y_\beta'' + K_1y_\beta \simeq (\widetilde{K}_1 + K_2y_c)x_\beta\,. \tag{14}$$

We use this to calculate the change in y_β. In addition to betatron oscillations, the motion damped due to radiation damping. Thus, after N turns, y_β is equal to

$$y_\beta(C) = y_0e^{-NT_0\alpha_y}\sin(2\pi N\nu_y + \psi_0) + \sum_{i=1}^{N\,\text{turns}}(gLx_\beta)_i\sqrt{\beta_i\beta(C)}e^{-(N-i)T_0\alpha_y}\sin(2\pi N\nu_y - \psi_i) \tag{15}$$

where y_0 is the initial amplitude, T_0 is the revolution period, and α_y is the vertical damping rate. Also, $g = \widetilde{K}_1 + K_2y_c$, L is the length of magnet i, and ψ_i is the vertical phase at element i.

The equilibrium emittance is found by averaging the equilibrium value of y_β^2/β_y over an ensemble of particles. Thus, we calculate the emittance by finding $\langle y_\beta^2\rangle/\beta_y$ as $N \to \infty$. This yields

$$\epsilon_y(s) \approx \lim_{N\to\infty}\frac{1}{4}\sum_{i,i'}^{N\,\text{turns}}e^{-(2N-i-i')T_0\alpha_y}\sum_{j,j'}^{\text{ring}}\langle g_jL_jg_{j'}L_{j'}\rangle \langle x_{\beta\,i,j}x_{\beta\,i',j'}\rangle\sqrt{\beta_j\beta_{j'}}\cos(\psi_{y\,i,j} - \psi_{y\,i',j'})\,. \tag{16}$$

Note that we have written the product of the two sines as the difference of two cosines and have only kept the $\psi_{y\,i,j} - \psi_{y\,i',j'}$ term. The $\cos(\psi_{y\,i,j} + \psi_{y\,i',j'})$ term will be small unless $2\nu_y = n$.

This can be further simplified by using the correlation function for the x betatron motion. If one ignores synchrotron radiation, the correlation function is equal to the emittance multiplied by both the square root of the beta functions and the cosine of the phase difference. The radiation, which is a stochastic process, causes the correlation function to decay exponentially with Δs. Thus,

$$\langle x_\beta(s) x_\beta(s')\rangle \approx \epsilon_x \sqrt{\beta(s)\beta(s')} \cos \Delta\psi_x e^{-|s-s'|\alpha_x/c} , \qquad (17)$$

where α_x is the horizontal damping rate.

We calculate the sum over turns in Eq. (16) as $N \to \infty$ and with the assumption that we are far from the coupling resonances. The contribution from both the sum and difference resonances is then

$$\epsilon_y(s) \approx \frac{\epsilon_x}{16}\frac{\alpha_x+\alpha_y}{\alpha_y} \int_s^{s+C} dz \int_s^{s+C} dz' g(z)g(z')\sqrt{\beta_x(z)\beta_x(z')}$$
$$\times \sqrt{\beta_y(z)\beta_y(z')}\left(\frac{\cos(\Delta\psi_x+\Delta\psi_y)}{\sin^2\pi(\nu_x+\nu_y)} + \frac{\cos(\Delta\psi_x-\Delta\psi_y)}{\sin^2\pi(\nu_x-\nu_y)}\right) \qquad (18)$$

where $g = (\widetilde{K}_1 + K_2 y_c)$, $\Delta\psi = \psi(z) - \psi(z')$, and we will denote the function of the damping rates as $\overline{\alpha}$, $\overline{\alpha} \equiv (\alpha_x + \alpha_y)/\alpha_y$. Finally, notice that we have dropped the averaging brackets from $g(z)g(z')$; all beam particles will have the same closed orbit provided that $y_c \gg \eta_y \delta E/E$.

This final relation is similar to the result found in Ref. 5. There the expression was derived by solving the Fokker-Planck equation when close to the difference coupling resonance. We keep both terms in Eq. (18) since we have found that contributions from the sum resonance can be significant even when the tunes are closer to the difference resonance.

Now we include the effects of both random errors and a corrected closed orbit. The quadrupole rotational errors, sextupole misalignments, and the closed orbit all contribute independently. For random errors, we find

$$\epsilon_{y\,\text{quad rot.}} \approx \epsilon_x\overline{\alpha}\frac{1-\cos 2\pi\nu_x \cos 2\pi\nu_y}{(\cos 2\pi\nu_x - \cos 2\pi\nu_y)^2} \sum_{\{\text{quad}\}} (K_1 L)^2 \Theta^2 \beta_x \beta_y$$
$$\epsilon_{y\,\text{sext mis.}} \approx \frac{\epsilon_x\overline{\alpha}}{4}\frac{1-\cos 2\pi\nu_x \cos 2\pi\nu_y}{(\cos 2\pi\nu_x - \cos 2\pi\nu_y)^2} \sum_{\{\text{sext}\}} (K_2 L)^2 y_m^2 \beta_x \beta_y . \qquad (19)$$

These relations are similar to equations used in Ref. 6; the relations differ in the form of $\overline{\alpha}$ which is determined by the damping.

Using Eq. (2) for the closed orbit, we find for a corrected orbit with $N_{\text{corr}} \gg 1$,

$$\epsilon_{y\,\text{cor orbit}} \approx \sum_{\Delta\nu,\psi} \frac{\epsilon_x\overline{\alpha}}{32\sin^2\pi\Delta\nu}\frac{\langle y_c^2\rangle}{\beta_y}$$
$$\times \sum_{n_c}^{N_{\text{corr}}} \left| \int_{n_c}^{n_c+1} dz K_2(z)\beta_y(z)\sqrt{\beta_x(z)} e^{i\psi}\right|^2 , \qquad (20)$$

where the sum over $\Delta\nu$ and ψ is a sum over the two values of $\Delta\nu = \{\nu_x - \nu_y, \nu_x + \nu_y\}$ and the two values of ψ associated with each value for $\Delta\nu$. The values of ψ are

$$\psi = \begin{cases} \psi_x + 2\psi_y & \text{and} \quad \psi_x, \quad \text{if } \Delta\nu = \nu_x + \nu_y \\ \psi_x - 2\psi_y & \text{and} \quad \psi_x, \quad \text{if } \Delta\nu = \nu_x - \nu_y \end{cases} . \qquad (21)$$

Note that the integral is calculated between correctors rather than over the entire ring.

The integrals in Eq. (20) are the same integrals one finds when using time dependant perturbation theory to calculate the effect of sextupoles on the betatron motion. The similarity arises because, over a short segment, the closed orbit oscillates like a free betatron oscillation. It is important to emphasize that Equation (20) describes an effect due to linear coupling — notice the resonant denominator in Eq. (20); it is not an effect of the third order difference resonances. Specifically, Eq. (20) is only valid when the closed orbit is broken into short segments (by correctors).

Typically, when correcting the dynamic aperture, one adjusts the sextupole strength and placement so that the first order aberrations will cancel over the ring. For example, in the NDR, the cell phase advances are $\nu_{x\,\text{cell}} \approx 0.37$ and $\nu_{y\,\text{cell}} \approx 0.12$. This causes the first order geometric aberrations due to the sextupoles to cancel over an arc of roughly 8.5 cells. Unfortunately, when correcting the orbit, we break this cancellation scheme, and thus the vertical emittance to grows.

In Figure 3 we plot the contribution to the vertical emittance from linear coupling vs. the number of orbit correctors used. The points plotted are generated by simulating random misalignments in the NDR as was done in Figures 1 and 2. Notice that there is a large variation in ϵ_y for different error distributions. This indicates that the coupling is very sensitive to the actual closed orbit. The dashed line is an approximation of Eq. (20). Notice that the contribution increases roughly linearly with the number of correctors. Also notice that the magnitude of this effect is greater than the contribution due to vertical dispersion. This has been true in all of the rings we have examined.

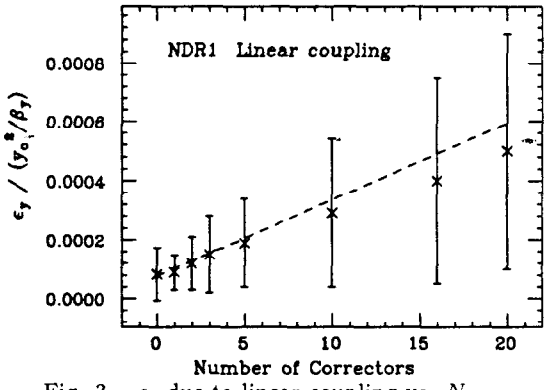

Fig. 3. ϵ_y due to linear coupling vs. N_{corr}.

SUMMARY

In this paper we have presented simple formulas to estimate the vertical emittance in weakly coupled storage rings. In particular, we consider the effect of a corrected closed orbit in generating vertical dispersion and linear coupling. The resulting formulas compare well with simulations of corrected closed orbits due to random alignment errors.

REFERENCES

1. T. Suzuki, "Vertical Dispersion Produced by Random Closed Orbit Distortions and Sextupoles", PEP-259 (1977).
2. J. Kewisch, *et. al.*, "Vertical Dispersion Generated by Correlated Closed Orbit Deviations", DESY 86-020 (1986).
3. T. O. Raubenheimer, "Analytic Estimates of the Vertical Emittance in Damping Rings", SLAC-PUB-4935, work in progress.
4. E. D. Courant and H. S. Snyder, "Theory of the Alternating-Gradient Synchrotron", Annals of Phys., **3**, 1-48 (1958).
5. A. W. Chao and M. J. Lee, "Vertical Beam Size due to Orbit and Alignment Errors", SLAC-PUB-1915 (1977).
6. A. Piwinski, "Beam Height in PETRA", DESY M-83-19 (1983).