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A FORMALISM AND COMPUTER PROGRAM FOR COUPLED LATTICES

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ABSTRACT

In this paper, a formalism to calculate the lattice functions and emittances of a coupled electron/positron storage ring is presented. The lattice functions are calculated directly from the modal matrix of the betatron transport matrix for the ring. The emittances and damping rates are then calculated from the invariants found in the diagonalized representation. In addition, a computer program is described which uses the formalism to calculate the coupled lattice functions, emittances and damping rates. The program can either reconstruct the closed orbit from BPM data and dipole corrector strengths, or construct an orbit from misalignments entered into the lattice and then optionally correct the orbit with dipole correctors. The lattice functions, emittances, etc. are then calculated about the resulting closed orbit.

INTRODUCTION

Coupling in storage rings has two effects. First, the horizontal and vertical dispersion functions are coupled and thereby modified. Second, the normal modes of the betatron oscillations rotate from the x and y axes. In electron/positron rings, both of these effects change the equilibrium beam size. This paper discusses a procedure to calculate these effects is discussed. The procedure is a generalization of Refs. 1 and 5. In addition, it is very similar to the work described in Ref. 2. The approach of Ref. 2 uses a 6x6 representation whereas the procedure discussed here uses a 4x4 representation and the explicit form of the dispersion function. While the 6x6 approach provides an elegant tool for calculation, the changes in the dispersion function and the rotation of the normal mode axes are separated in the 4x4 procedure; this makes it easy to use one's one-dimensional intuition to understand the coupled case.

In the next section invariants of the single particle motion (ignoring radiation) are found by diagonalizing the betatron transport matrix. Here, the normalization and the advantages of using a symplectic form are also discussed. Next, the effects of synchrotron radiation are included and the normal mode emittances and damping rates are calculated. Throughout, analogy is made with the uncoupled case to gain an intuitive understanding for the procedure. Finally, a parameterization of the coupled lattice is discussed, and then a program that performs the coupled calculations is briefly described.

COUPLED LATTICES

In a storage ring the linear betatron motion in the transverse planes can be described by a 4x4 transport matrix,

$$\vec{x}(s) = \mathrm{P}(s,s_0)\vec{x}(s_0) \qquad \text{where} \qquad \vec{x} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \tag{1}$$

and $\mathrm{P}(s,s_0)$ transports the motion from point s_0 to point s. The transport matrix for one turn of the ring at point s_0, $\mathrm{M}(s_0)$, is given by $\mathrm{P}(s_0+C,s_0)$. When the lattice is uncoupled, M will be block diagonal and the normal modes of the betatron oscillations will be along the x and y axes. In contrast, when the lattice contains coupling elements, the normal modes rotate from the x and y axes. To calculate these normal modes, we will generalize a procedure discussed in Ref. 1.

To analyze the motion, we diagonalize the transport matrix M. The transport matrix is symplectic,[3] and assuming that the motion is stable, the eigenvalues will be complex conjugate pairs of the form $\lambda_j = e^{i\mu_j}$. Thus,

$$\mathrm{E}^{-1}(s_0)\mathrm{M}(s_0)\mathrm{E}(s_0) = \begin{pmatrix} e^{i\mu_1} & & 0 & \\ & e^{-i\mu_1} & & \\ & & e^{i\mu_3} & \\ 0 & & & e^{-i\mu_3} \end{pmatrix} . \tag{2}$$

Here, μ_1 and μ_3 are the tunes of the two eigenmodes, denoted 1 and 3, and $\mathrm{E}(s_0)$ is the modal matrix which is constructed from columns of the eigenvectors and is, in general, complex valued.

The modal matrix, $\mathrm{E}(s_0)$, defines a coordinate transformation to the eigenbasis at a point s_0

$$\vec{x}(s_0) = \mathrm{E}(s_0)\vec{\xi}(s_0) \qquad \vec{\xi}(s_0) = \mathrm{E}^{-1}(s_0)\vec{x}(s_0) . \tag{3}$$

Since the eigenvalues are constant about the ring, the modal matrix must transform in the same manner as a position vector

$$\mathrm{E}(s) = \mathrm{P}(s,s_0)\mathrm{E}(s_0) . \tag{4}$$

Thus, the vector $\vec{\xi}$ does not depend upon s; it only depends on the initial values and is a *constant* of the motion.

An explicit form for $\vec{\xi}$ can be found by restricting the normalization of the eigenvectors so that they, like the eigenvalues, are complex conjugate pairs; the two eigenvectors in a pair then have the same magnitude. The modal matrix now has the form

$$\mathrm{E} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{e}_1 & \vec{e}_1{}^* & \vec{e}_3 & \vec{e}_3{}^* \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} , \tag{5}$$

where $\vec{e}_1$ and $\vec{e}_3$ are the eigenvectors associated with eigenvalues $e^{i\mu_1}$ and $e^{i\mu_3}$, and the * represents the complex conjugate. With this restriction, the components of $\vec{\xi}$ will also be complex conjugate pairs and can be written in terms of a modulus and a phase:

$$\vec{\xi} = \mathrm{E}^{-1}(s)\vec{x}(s) \qquad \text{where} \qquad \vec{\xi} = \begin{pmatrix} \sqrt{J_1}e^{i\theta_1} \\ \sqrt{J_1}e^{-i\theta_1} \\ \sqrt{J_3}e^{i\theta_3} \\ \sqrt{J_3}e^{-i\theta_3} \end{pmatrix} . \tag{6}$$

Here, J_1 and J_3 are single particle invariants and θ_1 and θ_3 are the respective phases. We will see that with an appropriate choice of normalization, these variables will be the action-angle coordinates for the two normal modes.

Since the motion is derived from a linear Hamiltonian system and is described by a canonical transformation, there exists a constraint on the eigenvectors,[3] namely,

$$\widetilde{\vec{e}_i(s)}\mathrm{S}\vec{e}_j(s) = \begin{cases} 0 & \text{if } \lambda_i \neq 1/\lambda_j \\ \text{const} & \text{if } \lambda_i = 1/\lambda_j \end{cases} \tag{7}$$

where $\vec{e}$ and λ are the eigenvectors and eigenvalues, and S is the

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anti-symmetric matrix

$$\mathrm{S} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{8}$$

Thus, with an appropriate choice of normalization, the coordinate transformation to the eigenbasis will be an extended canonical transformation,[4] *i.e.*

$$\widetilde{\mathrm{E}(s)}\mathrm{S}\mathrm{E}(s) = i\mathrm{S} . \tag{9}$$

There are two advantages to this choice of normalization. First, the inverse of the modal matrix, which will be needed later, is trivial to find:

$$\mathrm{E}^{-1}(s) = i\mathrm{S}\widetilde{\mathrm{E}(s)}\mathrm{S} . \tag{10}$$

Second, the invariants, J_1, J_3, θ_1, and θ_3, are the action-angle coordinates for the two normal modes. Thus, as in the uncoupled case, the normal mode emittances are just the ensemble averages of the actions:

$$\epsilon_1 = \langle J_1 \rangle \qquad \epsilon_3 = \langle J_3 \rangle . \tag{11}$$

Furthermore, as in the uncoupled case, the four-dimensional emittance is just the product of the two emittances.

NORMAL MODE EMITTANCES

In an electron/positron storage ring the emittance is determined by two competing processes: quantum excitation and radiation damping, both of which result from the synchrotron radiation. In this section, we will first consider the quantum excitation which is due to the discrete nature of the radiation and then discuss the radiation damping which results from the average properties of the radiation. Our treatments of both the excitation and the damping are simple generalizations of the treatments given in Ref. 5.

In the absence of synchrotron radiation, the single particle invariants, J_1 and J_3, can be calculated using Eq. (3),

$$J_1 = \widetilde{\vec{x}(s)}\mathrm{A}(s)\vec{x}(s) \qquad \text{and} \qquad J_3 = \widetilde{\vec{x}(s)}\mathrm{B}(s)\vec{x}(s) , \tag{12}$$

where the matrices A and B are

$$\begin{aligned} \mathrm{A}_{jk}(s) &\equiv \frac{1}{2}\left(\mathrm{E}^{-1}{}_{1j}\mathrm{E}^{-1}{}_{2k} + \mathrm{E}^{-1}{}_{2j}\mathrm{E}^{-1}{}_{1k}\right) \\ \mathrm{B}_{jk}(s) &\equiv \frac{1}{2}\left(\mathrm{E}^{-1}{}_{3j}\mathrm{E}^{-1}{}_{4k} + \mathrm{E}^{-1}{}_{4j}\mathrm{E}^{-1}{}_{3k}\right) . \end{aligned} \tag{13}$$

These two matrices, A and B, are both real symmetric matrices since the eigenvectors are complex conjugate pairs.

To calculate the change of J_1 and J_3 in the presence of radiation, two effects need to be considered: the radiation of photons and the replacement of the radiated longitudinal energy in RF cavities. First, when a photon is radiated, the closed orbit is displaced by an amount proportional to the dispersion function. Thus the change in the betatron motion is

$$\delta\vec{x} = \frac{u}{E_0}\vec{\eta} , \tag{14}$$

where u is the photon energy and $\vec{\eta}$ are the four components of the dispersion function. Second, when the particle passes through an RF cavity, the particle gains energy in the longitudinal direction, changing the slope of the trajectory

$$\delta x' = -x'\frac{\delta E^+}{E_0} ; \qquad \delta y' = -y'\frac{\delta E^+}{E_0} . \tag{15}$$

Here, δE^+ is the energy gain in the cavity. Notice that the effect of dispersion in the cavity has not been included; this effect is very small unless one is close to a synchrobetatron resonance.

Combining these two expressions, Eqs. (14) and (15), and using Eq. (12), the change in J_1, and similarly for J_3, is

$$\delta J_1 = \frac{u^2}{E_0^2}\widetilde{\vec{\eta}}\mathrm{A}\vec{\eta} + 2\frac{u}{E_0}\widetilde{\vec{x}}\mathrm{A}\vec{\eta} - 2\frac{\delta E^+}{E_0}\left(x'A_{2k}x_k + y'A_{4k}x_k\right) , \tag{16}$$

where the expression includes an implicit sum over the subscript k. Note that the term proportional to $(\delta E^+/E_0)^2$ has been ignored; this term will be much smaller than the others. Also notice that the first term does not directly depend upon x while the others do; thus, the first term, which is the quantum excitation term, is a random walk diffusion term while the others lead to exponential damping or anti-damping.

Initially, consider the first term of Eq. (16). Using radiation formulas for the emission rate of photons and Eq. (11) for the emittances, we calculate the average rate of change of the normal mode emittances due to the quantum diffusion. The result is

$$\begin{aligned} \frac{d\epsilon_1}{dt} &= C_q\gamma^2\frac{U_0}{T_0E_0}\frac{2\oint|G^3|\widetilde{\vec{\eta}}\mathrm{A}\vec{\eta}ds}{\oint G^2ds} \\ \frac{d\epsilon_3}{dt} &= C_q\gamma^2\frac{U_0}{T_0E_0}\frac{2\oint|G^3|\widetilde{\vec{\eta}}\mathrm{B}\vec{\eta}ds}{\oint G^2ds} . \end{aligned} \tag{17}$$

Here, $C_q = 3.84 \times 10^{-13}\,\mathrm{m}$, and U_0, T_0, and E_0 are the energy radiated per turn, the revolution time, and the energy of the particle, respectively. Finally, $G(s)$ is the reciprocal of the local bending radius, $G = 1/\rho = \sqrt{G_x^2 + G_y^2}$, and is proportional to the total bending field.

The other contributions to Eq. (16) are a bit more complicated to calculate. Since we are considering average effects, the photon energy u can be replaced by the power radiated $P_\gamma\delta t$. Unfortunately, P_γ depends upon $\vec{x}$; note that this was ignored in the quantum excitation calculation, but it has a small effect there. To include this dependance, we expand P_γ in a power series:

$$P_\gamma(\vec{x}) = P_\gamma(0)\left[1 + \frac{2K_1}{G^2}(yG_x + xG_y) + \frac{2\tilde{K}}{G^2}(xG_x - yG_y)\right] , \tag{18}$$

where G is defined above, and G_x and G_y are equal to the inverse of the horizontal and vertical projections of the local bending radius. In addition, K_1 and $\tilde{K}$ are the normalized quadrupole and skew quadrupole gradients: $K_1 = e/p_0c\,\partial B_y/\partial x$ and $\tilde{K} = e/p_0c\,\partial B_x/\partial x$.

To complete this calculation, Eq. (3) is used to write the $\vec{x}$ coordinates in terms of $\vec{\xi}$. Then, one averages over particles, *i.e.*, over the phases θ_1 and θ_3. Finally, the result can be simplified using Eqs. (9) and (10). One finds

$$\frac{d\epsilon_{1,3}}{dt} = -2\alpha_{1,3}\epsilon_{1,3} \qquad \text{where} \qquad \alpha_{1,3} = \mathcal{J}_{1,3}\frac{U_0}{2E_0T_0} , \tag{19}$$

and where the damping rates, $\alpha_{1,3}$, have been expressed in terms of the normal mode damping partition functions $\mathcal{J}_1$ and $\mathcal{J}_3$:

$$\begin{aligned} \mathcal{J}_1 &= 1 + 2\,\mathrm{Imag}\left\{\frac{\oint\eta_j\mathrm{E}^{-1}{}_{1j}\left[C_x\mathrm{E}^{-1}{}_{22} + C_y\mathrm{E}^{-1}{}_{24}\right]ds}{\oint G^2ds}\right\} \\ \mathcal{J}_3 &= 1 + 2\,\mathrm{Imag}\left\{\frac{\oint\eta_j\mathrm{E}^{-1}{}_{3j}\left[C_x\mathrm{E}^{-1}{}_{42} + C_y\mathrm{E}^{-1}{}_{44}\right]ds}{\oint G^2ds}\right\} . \end{aligned} \tag{20}$$

The coefficients C_x and C_y are

$$C_x = G^2G_x + 2G_yK_1 ; \qquad C_y = G^2G_y + 2G_xK_1 . \tag{21}$$

Notice that the skew quadrupole terms have not been included in these coefficients, but they are trivial to add.

At this point, the equilibrium emittance is calculated by simply equating the rates of these two processes, quantum excitation and damping — Eqs. (17) and (19), respectively. The final results for the normal mode emittances are

$$\epsilon_1 = C_q \frac{\gamma^2}{\mathcal{J}_1} \frac{2\oint |G^3| \widetilde{\vec{\eta}} \mathrm{A} \vec{\eta} ds}{\oint G^2 ds} \qquad (22)$$

$$\epsilon_3 = C_q \frac{\gamma^2}{\mathcal{J}_3} \frac{2\oint |G^3| \widetilde{\vec{\eta}} \mathrm{B} \vec{\eta} ds}{\oint G^2 ds} \ .$$

BEAM SIZES and LATTICE FUNCTIONS

In this section the previous results are used to calculate both the action-angle representation for the beam position and the beam sizes. In doing so, a parameterization of coupled lattices will be discussed; this parameterization was originally introduced in Ref. 6. While this choice of parameters is intuitive, a few of the pitfalls will also be mentioned.

The particle position can be expressed in action-angle coordinates using Eqs. (3) and (6). To parameterize the coupled lattice, one simply makes an analogy with an uncoupled lattice and defines β functions and phases. Thus, Eq. (3) becomes

$$\begin{aligned} x(s) &= \sqrt{2J_1\beta_{x1}}\cos(\phi_{x1}+\theta_1) + \sqrt{2J_3\beta_{x3}}\cos(\phi_{x3}+\theta_3) \\ y(s) &= \sqrt{2J_1\beta_{y1}}\cos(\phi_{y1}+\theta_1) + \sqrt{2J_3\beta_{y3}}\cos(\phi_{y3}+\theta_3) \ , \end{aligned} \qquad (23)$$

where the β's and phases depend upon s. Likewise, we can solve for the beam sizes:

$$\begin{aligned} \langle x^2(s)\rangle &= \epsilon_1\beta_{x1}(s) + \epsilon_3\beta_{x3}(s) \\ \langle xy(s)\rangle &= \epsilon_1\sqrt{\beta_{x1}\beta_{y1}}\cos(\phi_{x1}-\phi_{y1}) \\ &\quad + \epsilon_3\sqrt{\beta_{x3}\beta_{y3}}\cos(\phi_{x3}-\phi_{y3}) \\ \langle y^2(s)\rangle &= \epsilon_1\beta_{y1}(s) + \epsilon_3\beta_{y3}(s) \ , \end{aligned} \qquad (24)$$

or, in general,

$$\langle x_j x_k\rangle = \epsilon_1 2\mathrm{Re}(\mathrm{E}_{j1}\mathrm{E}_{k1}^*) + \epsilon_3 2\mathrm{Re}(\mathrm{E}_{j3}\mathrm{E}_{k3}^*) \ . \qquad (25)$$

In terms of this parameterization, the elements of the modal matrix are

$$\begin{aligned} \mathrm{E}_{11} &= \sqrt{\beta_{x1}/2}\, e^{i\phi_{x1}} & \mathrm{E}_{13} &= \sqrt{\beta_{x3}/2}\, e^{i\phi_{x3}} \\ \mathrm{E}_{31} &= \sqrt{\beta_{y1}/2}\, e^{i\phi_{y1}} & \mathrm{E}_{33} &= \sqrt{\beta_{y3}/2}\, e^{i\phi_{y3}} \end{aligned} \qquad (26)$$

and the components E_{2j} and E_{4j} are found by taking the derivatives of E_{1j} and E_{3j} with respect to s. Notice that there are 16 individual parameters: four β's, four $d\beta/ds$'s, four ϕ's, and four $d\phi/ds$'s. This is cumbersome; only 10 independent parameters are needed to specify the matrix since the transfer matrix is symplectic. The dependances between parameters are found from the symplectic condition Eq. (9), but unfortunately, some of the dependances are complex and do not simplify the relations.

Another point of interest occurs when the ring is on a coupling resonance. When the ring is completely uncoupled, the coupled parameters reduce to the normal uncoupled parameterization:

$$\beta_{x1} = \beta_x \ , \quad \beta_{y3} = \beta_y \ , \quad \beta_{x3} = \beta_{y1} = 0 \ , \qquad (27)$$

where β_x and β_y are the normal β functions. Assuming that the ring is far from a coupling resonance, our parameters change slowly as the coupling is increased. This allows for an intuitive understanding; a kick in the x plane will lead to some small amplitude vertical motion. Unfortunately, when one is on a coupling resonance, the eigenvectors are no longer orientated near the x and y axes, even in the limit of small coupling. In this case

$$\beta_{x1} = \beta_{x3} = \frac{1}{2}\beta_x \ , \qquad \beta_{y1} = \beta_{y3} = \frac{1}{2}\beta_y \ , \qquad (28)$$

and thus, it is not as simple to gain a feel for the motion.

CEMIT

The formalism described above has been implemented in a computer program called CEMIT (Coupled EMITtances). The program will input an arbitrary storage ring lattice. Then, a number of random errors can be simulated, including magnet misalignments and power supply variations. Next, the closed orbit must be specified. Either the orbit is reconstructed from BPM (Beam Position Monitor) measurements which are input, or the orbit is calculated from the bending fields and errors. In a lattice that contains nonlinear elements, the closed orbit calculation is iterated until the desired convergence is attained. Finally, the orbit can be corrected with dipole correctors. Currently, the program uses a simple RMS correction procedure, but other methods can easily be implemented.

After calculating the closed orbit, the dispersion function associated with the orbit is found. Then, using the formalism described above, the program calculates the normal mode emittances and damping times. In addition, the coupled lattice functions and the beam sizes can be plotted.

The procedure that has been described for calculating the emittances uses the symplecticity of the transport matrix M. At this time, the symplecticity of M is ensured by approximating the nonlinear fields in a magnet with a delta-function element located at the center. In the future, it is planned to improve the program by using a more detailed canonical integration scheme.

Currently, the program has been used to study alignment tolerances in a preliminary damping ring design for the TLC (TeV Linear Collider).[7] In addition, the program was used to find vertical orbit bumps to increase the width of the coupling difference resonance in the SLC North Damping Ring, and it is also being used to study methods of using vertical separation to increase the number of bunches when colliding beams in the PEP ring at SLAC.

SUMMARY

A formalism for calculating the beam emittances and lattice functions in an arbitrarily coupled storage ring has been described. Throughout, analogy has been made with the uncoupled case to give intuition. Finally, the computer program CEMIT, which performs these calculations on an arbitrary storage ring lattice, has briefly been described.

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REFERENCES

1. P. L. Morton and N. C. Spencer, "Effects of Transverse Coupling in the SLAC Storage Ring", IEEE Trans. Nucl. Sci., NS-18, 1099 (1971).
2. A. W. Chao, "Evaluation of Beam Distribution Parameters in an Electron Storage Ring", *J. Appl. Phy.*, 50, 595 (1979).
3. E. D. Courant and H. S. Snyder, "Theory of the Alternating-Gradient Synchrotron", Annals of Phys., 3, 1 (1958).
4. H. Goldstein, Classical Mechanics, Addison-Wesley, Menlo Park, CA. (1981).
5. M. Sands, "The Physics of Electron Storage Rings", SLAC-121 (1971).
6. G. Ripken, "Untersuchungen zur Strahlfuhrung und Stabilitat der Teilchenbewegung in Beschleunigern und Storage-Ringen unter strenger Berucksichtigung einer Kopplung der Betatronschwingungen", DESY R1-70/4 (1970).
7. T. O. Raubenheimer *et al.*, "A Damping Ring Design for Future Linear Colliders", these proceedings and SLAC-PUB-4912 (1989).