SLAC–PUB–4847
January 1989
(T/E)

The Mass Ratio m_c/m_b in Semi-Leptonic b-Decays*

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ABSTRACT

The calculation of the s_{23} mixing angle from semi-leptonic B-decay depends on the mass ratio m_c/m_b. The energy scales at which the running masses m_c and m_b should be taken are determined once terms of order $[\alpha_s \cdot (m_c/m_b)]$ are taken into account. We give an analytic expression for the QCD correction to the decay rate to all orders in the ratio between on-shell masses. We explain how it should be modified when both masses are taken at a single common scale.

Submitted to *Phys. Lett. B*

⋆ Work supported by the Department of Energy, contract DE–AC03–76SF00515.

1. INTRODUCTION

The exact determination of the quark sector parameters is most important both as a test of the three generation Standard Model and as a probe of physics beyond this model. With three generations, the quark mixing matrix is parametrized by three real mixing angles and one complex phase. Two of these mixing angles, $s_{23} = |V_{cb}|$ and $s_{13} = |V_{ub}|$, are extracted from B-meson decay measurements.

The best experimentally measured and theoretically understood decay modes are the inclusive semi-leptonic decays. The quark-level process is $b \to q\ell^- \bar{\nu}_\ell$, where q is either a c-quark or a u-quark and ℓ is a charged lepton. The masses of the e and μ leptons and of the u-quark can be safely neglected (we later comment on the neglect of m_u). However, the charmed semi-leptonic decay rate strongly depends on the ratio m_c/m_b [1, 2].

The values of the masses m_c and m_b that should be used seem ambiguous [3, 4]. Quark masses run with the energy scale and it is not obvious what the relevant energy scales are. In particular, we want to decide whether to use the ratio between on-shell masses or the ratio between the masses taken at a common energy scale. We show that the calculations for these two possibilities differ at order $[\alpha_s \cdot (m_c/m_b)]$.

Previous calculations of QCD corrections to the decay rate (beyond zeroth order in the mass ratio) used numerical integration. We argue that they correspond to the ratio between on-shell masses. To show that, we perform the integration analytically. We then modify the calculation for the use of the ratio between masses at a single energy scale, finding that in this case there is no term of the form $(m_c/m_b)\ln(m_c/m_b)$, as expected on general grounds [5].

We further remark on the implications of our calculation on QCD corrections to charmless B-decays and on QED corrections to lepton decays, *e.g.* $\tau \to \mu\bar{\nu}_\mu\nu_\tau$.

We note that our results cannot be directly applied to top decays: we assume that the fermionic masses involved are all small compared to the mass of the intermediate boson. This does not hold for $m_t \sim M_W$, and modifications are necessary [6].

2. SEMI-LEPTONIC QUARK DECAY

We take a general case of a heavy quark h of charge $-\frac{1}{3}$ and a lighter quark l of charge $+\frac{2}{3}$. The value of a mixing term $|V_{lh}|$ is extracted from the inclusive semi-leptonic decay rate $M_h \to X_l \ell \nu_\ell$, where M_h is a weakly-decaying meson that contains the quark h. At the quark level one uses the spectator quark model, assuming that the partial width is given by the h-quark W-mediated decay:

$$\Gamma(M_h \to X_l \ell \nu_\ell) = \Gamma(h \to l \ell \bar{\nu}_\ell) = \frac{G_F^2 m_h^5 |V_{lh}|^2}{192\pi^3} F_{ps} F_{QCD}, \tag{1}$$

where F_{ps} is a phase space factor and F_{QCD} is a QCD correction factor. Both F_{ps} and F_{QCD} depend on the mass ratio

$$\rho \equiv \frac{m_l^2}{m_h^2}. \tag{2}$$

The calculation of F_{ps} is well-known:

$$F_{ps}(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln(\rho). \tag{3}$$

The F_{QCD} parameter is of the form

$$F_{QCD}(\rho) = 1 - \frac{2\alpha_s}{3\pi} f(\rho). \tag{4}$$

The crucial point is that with different definitions of ρ the function $f(\rho)$ is modified beyond zeroth order in ρ. We now show this explicitly.

Suppose we use for the ratio ρ:

$$\rho = \frac{[m_l(m_l)]^2}{[m_h(m_h)]^2}, \tag{5}$$

namely, each mass is taken on its mass shell. We should use certain functions $F_{ps}(\rho)$ and $f(\rho)$. Then we make the calculation using a different ratio ρ':

$$\rho' = \frac{[m_l(m_h)]^2}{[m_h(m_h)]^2}, \tag{6}$$

namely, both masses are taken on a single energy scale. Now we should use functions $F'_{ps}(\rho')$ and $f'(\rho')$. The ratios ρ and ρ' are related, to first order in α_s, by

$$\rho = \rho' \left[1 - \frac{2\alpha_s}{\pi} \ln(\rho')\right]. \tag{7}$$

As the difference is $O(\alpha_s)$ while the phase space factor is, by definition, zeroth order in α_s, clearly:

$$F'_{ps}(x) = F_{ps}(x). \tag{8}$$

However, modifications at order $(\alpha_s \rho)$ are required. To first order in ρ we have

$$\begin{aligned} F_{ps}(\rho) =& 1 - 8\rho \\ =& 1 - 8\rho' + \frac{16\alpha_s}{\pi} \rho' \ln(\rho') \\ =& F_{ps}(\rho') \cdot \left[1 + \frac{16\alpha_s}{\pi} \rho' \ln(\rho')\right]. \end{aligned} \tag{9}$$

The $O(\alpha_s)$ correction should be absorbed in a new function $F'_{QCD}(\rho')$:

$$F'_{QCD}(\rho') = 1 - \frac{2\alpha_s}{3\pi} f'(\rho') \tag{10}$$

where

$$f'(x) = f(x) - 24\ x\ \ln(x) + \cdots. \tag{11}$$

Indeed, the calculations with either ρ or ρ' differ at order $[\alpha_s\ (m_l/m_h)]$. Moreover, when we use ρ, the ratio between on-shell masses, we expect terms of the form

$\alpha_s \cdot \rho \ln(\rho)$ to appear in $F_{QCD}(\rho)$, as the gluon loops on external legs are calculated at different scales. However, when we use ρ', the ratio between masses at a single common scale, we expect no terms of the form $\alpha_s \cdot \rho' \ln(\rho')$ [5]. Eq. (11) then tells us that the coefficient of the $\rho \ln(\rho)$-term in $f(\rho)$ is 24. In the next section we give the analytic expression for $f(\rho)$ and show that this is indeed the case.

Previous calculations of the QCD corrections [1, 2, 7] are a modification of earlier QED calculations [8] of μ decay. In the QED calculation, the masses are by definition on-shell masses: lepton masses are experimentally measurable, and these physical masses identify with the on-shell masses. Consequently, *the existing calculations of QCD-corrected quark decays correspond to mass ratio between on-shell masses.*

3. AN ANALYTIC EXPRESSION FOR THE MASS-DEPENDENT CORRECTIONS

In previous calculations [1, 2, 7] the *differential* cross-section is given analytically. However, to derive the correction to the decay rate, the integration, being mathematically rather non-trivial, is carried out *numerically.* As we are interested in the coefficient of the $\rho \ln(\rho)$ term, we carried out an *analytic* integration to all orders in ρ. Our starting point was the differential cross section as given in ref. [7]. We now give the expression for the function $h(\rho) \equiv F_{ps}(\rho) f(\rho)$:

$$\begin{aligned} h(\rho) = & -(1-\rho^2)\left(\frac{25}{4} - \frac{239}{3}\rho + \frac{25}{4}\rho^2\right) + \rho \ln\rho \left(20 + 90\rho - \frac{4}{3}\rho^2 + \frac{17}{3}\rho^3\right) \\ & + \rho^2 \ln^2\rho \, (36 + \rho^2) + (1-\rho^2)\left(\frac{17}{3} - \frac{64}{3}\rho + \frac{17}{3}\rho^2\right) \ln(1-\rho) \\ & - 4(1 + 30\rho^2 + \rho^4) \ln\rho \ln(1-\rho) - (1 + 16\rho^2 + \rho^4)\left[6 Li_2(\rho) - \pi^2\right] \\ & - 32\rho^{3/2}(1+\rho)\left[\pi^2 - 4 Li_2(\sqrt{\rho}) + 4 Li_2(-\sqrt{\rho}) - 2\ln\rho \ln\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right] . \end{aligned} \tag{12}$$

The result is finite when $\rho \to 0$ as guaranteed by the Kinoshita theorem [9]. The dilogarithm function $Li_2(x)$ is defined as in ref. [10]:

$$Li_2(x) = -\int_0^x \frac{\ln(1-z)}{z} dz = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots \quad for \ |x| \leq 1. \tag{13}$$

Our result agrees with the numerical integration results given for specific cases in refs. [1,2].

For the cases of interest $\rho \leq 0.1$, for which an approximation to $O(\rho^3)$ works well:

$$\begin{aligned} h(\rho) =& \pi^2 - \frac{25}{4} + \rho(68 + 24\ln\rho) - \rho^{3/2} \cdot 32\pi^2 \\ &+\rho^2(16\pi^2 + 273 - 36\ln\rho + 36\ln^2\rho) - \rho^{5/2} \cdot 32\pi^2 \\ &+\rho^3\left(\frac{1052}{9} - \frac{152}{3}\ln\rho\right) + O(\rho^4). \end{aligned} \tag{14}$$

To find $f(\rho)$ one has to divide $h(\rho)$ by $F_{ps}(\rho)$ where $F_{ps}(\rho)$ is given in eq. (3). To first order in ρ we get:

$$f(\rho) = \pi^2 - \frac{25}{4} + \rho(18 + 8\pi^2 + 24\ln\rho). \tag{15}$$

The coefficient of the $\rho\ln(\rho)$ term is indeed 24. The function $f'(\rho')$ that should be used when the masses are taken at a single common scale can be derived from eq. (14) by applying eq. (11). To first order in ρ' we get:

$$f'(\rho') = \pi^2 - \frac{25}{4} + \rho'(18 + 8\pi^2). \tag{16}$$

There is no term of the form $\rho'\ln(\rho')$. Thus we proved our above statement: Previous numerical calculations, being in agreement with eq. (15) but not with eq. (16), correspond to the mass ratio between on-shell masses.

The function $h(\rho)$ can be used directly, by rewriting eq. (1) as:

$$\begin{aligned} \Gamma(h \to l\ell\bar{\nu}_\ell) &= \Gamma^{(0)} - \frac{2\alpha_s}{3\pi}\Gamma^{(1)} \\ \Gamma^{(0)} &= \frac{G_F^2 m_h^5 |V_{lh}|^2}{192\pi^3} F_{ps}(\rho) \\ \Gamma^{(1)} &= \frac{G_F^2 m_h^5 |V_{lh}|^2}{192\pi^3} h(\rho). \end{aligned} \tag{17}$$

4. THE s_{23} MIXING ANGLE

The above discussion is most relevant for the calculation of the $s_{23} = |V_{cb}|$ mixing angle (we use the parametrization of ref. [11]) from the charmed semi-leptonic B-decay [12]:

$$(s_{23})^2 = \left[\frac{192\pi^3}{G_F^2}\right]\left[\frac{BR(b \to c\ell\bar{\nu}_\ell)}{\tau_b}\right]\left[\frac{1}{\eta_0' F_{ps}^c}\right]\frac{1}{m_b^5} \tag{18}$$

with

$$\begin{aligned} \rho_c =& \frac{[m_c(m_c)]^2}{[m_b(m_b)]^2} \\ F_{ps}^c =& F_{ps}(\rho_c) \\ \eta_0' =& F_{QCD}(\rho_c) = 1 - \frac{2\alpha_s}{3\pi} f(\rho_c) \end{aligned} \tag{19}$$

We use [13]:

$$\begin{aligned} m_c(m_c) &= 1.27 \pm 0.05\ GeV \\ m_b(m_b) &= 4.25 \pm 0.10\ GeV \end{aligned} \tag{20}$$

The mass ratio is then:

$$(\rho_c)^{1/2} = 0.30 \pm 0.02. \tag{21}$$

This gives [2]:

$$\begin{aligned} f(\rho_c) =& 2.51 \pm 0.06 \\ F_{ps}^c =& 0.52 \pm 0.04 \end{aligned} \tag{22}$$

To find η_0' one has to give a value to α_s. However, unlike ρ, the value of α_s (or equivalently the scale at which it should be taken) is *not* determined until we calculate to $O[(\alpha_s)^2]$. The best we can do is try to estimate the scale at which the $O[(\alpha_s)^2]$ corrections are smallest. We take $1.5\ GeV \leq \mu \leq 2.5\ GeV$ which, for $\Lambda_{QCD} = 150\ MeV$, gives [13] $\alpha_s = 0.20 \pm 0.02$. We get

$$\eta_0' = 0.89 \pm 0.01. \tag{23}$$

The value of $\eta_0' = 0.87$ given in ref. [3] corresponds to $\alpha_s = 0.24$. The η_0'-value is not sensitive to the uncertainty in ρ_c.

For the the semi-leptonic branching ratio we use the world average [14] of measurements both in the continuum and on the $\Upsilon(4S)$ peak:

$$BR(b \to e\nu_e X) = 0.115 \pm 0.004. \tag{24}$$

We assume for the present calculation $R \equiv \frac{\Gamma(b \to u\ell\nu_\ell)}{\Gamma(b \to c\ell\nu_\ell)} = 0$. This may give an s_{23}-value higher by up to 4% than the true value. The world average for the B lifetime is [15]

$$\tau_b = (1.18 \pm 0.14) \times 10^{-12}\ sec. \tag{25}$$

The largest uncertainty in the extraction of s_{23} from the semi-leptonic decay width comes from the m_b^5 dependence. A fit to the leptonic spectrum gives [4]

$$< m_b >= 4.95 \pm 0.05\ GeV. \tag{26}$$

This fit is based on the model by Altarelli *et al.* [16]. Theoretically, it is plausible to use $m_b(\mu)$ at a scale μ which corresponds to the average mass of the $\ell\nu$ system [4]. With $1.5\ GeV \leq \mu \leq 2.5\ GeV$ and $\Lambda_{QCD} = 150\ MeV$, the range is $4.6\ GeV \leq m_b(\mu) \leq 5.1\ GeV$, consistent with eq. (26). Thus, we choose the following range for m_b:

$$m_b = 4.9 \pm 0.3\ GeV. \tag{27}$$

The determination of s_{23} from the B-meson semileptonic decay (eq. (18)) is thus subject to both experimental and theoretical uncertainties. Due to the m_b^5-dependence, we cannot determine $(s_{23})^2$ to an accuracy better than 30%. Adding the errors in quadrature we get

$$(s_{23})^2 = (2.1 \pm 0.7) \times 10^{-3} \tag{28}$$

which gives

$$s_{23} = 0.046 \pm 0.008. \tag{29}$$

Calculations of s_{23} within models other than the free quark model usually give somewhat higher s_{23} values [16, 17, 18].

5. CHARMLESS SEMI-LEPTONIC b DECAY

The $s_{13} = |V_{ub}|$ mixing angle is given by

$$(s_{13})^2 = \left[\frac{192\pi^3}{G_F^2}\right]\left[\frac{BR(b \to u\ell\bar{\nu}_\ell)}{\tau_b}\right]\left[\frac{1}{\eta_0'' F_{ps}^u}\right]\frac{1}{m_b^5} \tag{30}$$

with

$$\begin{aligned} \rho_u =& \frac{[m_u(m_u)]^2}{[m_b(m_b)]^2} \\ F_{ps}^u =& F_{ps}(\rho_u) \\ \eta_0'' =& F_{QCD}(\rho_u) = 1 - \frac{2\alpha_s}{3\pi} f(\rho_u) \end{aligned} \tag{31}$$

In all existing calculations the u-quark mass is neglected. However, as the u-quark is lighter than Λ_{QCD}, its mass is not well-defined on-shell. What we would really like to use is the ratio m_u/m_b at a common mass scale, as m_u at scales above Λ_{QCD} is well defined and known [13]:

$$\begin{aligned} m_u(1\ GeV) &= 5.1 \pm 1.5\ MeV \\ m_b(1\ GeV) &= 5.6 \pm 0.1\ GeV \\ \rho_u' &\approx (8^{+6}_{-4}) \times 10^{-7} \end{aligned} \tag{32}$$

The value of m_b given here corresponds to $\Lambda = 150\ MeV$. From eqs. (16) and (32) we can see that indeed ρ_u' can be safely put to 0:

$$\begin{aligned} F_{ps}^u =& F_{ps}(0) = 1 \\ f'(0) =& \pi^2 - \frac{25}{4} \approx 3.62. \end{aligned} \tag{33}$$

Using the same range for α_s as in the calculation of η_0' we get

$$\eta_0'' = 0.85 \pm 0.01. \tag{34}$$

Again, the value given in ref. [3], $\eta_0'' = 0.82$, corresponds to $\alpha_s = 0.24$. The *ratio* η_0'/η_0'' does not depend on our choice of α_s. We get:

$$\frac{s_{13}}{s_{23}} = 0.74\left[\frac{\Gamma(B \to X_u \ell \nu_\ell)}{\Gamma(B \to X_c \ell \nu_\ell)}\right]^{1/2}. \tag{35}$$

6. LEPTON DECAYS

The above calculation of QCD corrections to quark decays can be easily modified to calculate QED corrections to lepton decay, $\ell_i \to \ell_j \bar{\nu}_j \nu_i$, by the replacement

$$\alpha_s \to \frac{3}{4}\alpha_{EM}. \tag{36}$$

For the $\tau \to \mu \bar{\nu}_\mu \nu_\tau$ decay, we have

$$\begin{aligned} m_\mu =& 105.659\ MeV \\ m_\tau =& 1784.2\ MeV \\ \rho =& 3.5 \times 10^{-3}. \end{aligned} \tag{37}$$

We get:

$$\begin{aligned} F_{ps}(\rho = 3.5 \times 10^{-3}) =& 0.9728 \\ f(\rho = 3.5 \times 10^{-3}) =& 3.48. \end{aligned} \tag{38}$$

Although ρ is of order 10^{-3}, $f(\rho)$ is modified from its zeroth order value by as much as 4%. However, as α_{EM} is small, $F_{QED}(\rho)$ is modified from its zeroth order value by only 2 parts in 10^4:

$$\begin{aligned} F_{QED}(\rho = 0) = 0.9957 \\ F_{QED}(\rho = 3.5 \times 10^{-3}) = 0.9959. \end{aligned} \tag{39}$$

In ref. [19] QED corrections to τ decay rates are calculated with terms of order $[\alpha_{EM} \cdot (m_\mu^2/m_\tau^2)]$ neglected. They get $\frac{\Gamma(\tau\to\mu\nu\bar{\nu}(\gamma))}{\Gamma(\tau\to e\nu\bar{\nu}(\gamma))} = 0.9728$. We find a 0.02% correction to this value:

$$\frac{\Gamma(\tau \to \mu\nu\bar{\nu}(\gamma))}{\Gamma(\tau \to e\nu\bar{\nu}(\gamma))} = 0.9730. \tag{40}$$

7. CONCLUSIONS

The need for accuracy in the determination of the quark mixing angles necessitates a refinement of the ingredients involved in the calculation. We concern ourselves with one such aspect: the quark mass ratio that should be used in the calculation of semi-leptonic decay widths. We are interested in the difference between calculations using the ratio between the on-shell masses and those using the ratio between the masses taken at a common energy scale.

There are three possible cases:

a. The ratio m_l/m_h is close to 1. In this region the question is unimportant both in principle, as the two possible mass ratios are very close to each other, and in practice, as nature has not provided us yet with such a case.

b. The ratio m_l/m_h is close to 0. Here the question is interesting in principle, as for light quarks there is no well-defined on-shell mass. However, in practice the question is, again, unimportant because the mass ratio can be approximated to zero, and the calculations identify to zeroth order.

c. The ratio m_l/m_h is non-negligible, but not too close to 1. Here the question is important both in principle and in practice. We find that previous calculations, which were all numerical, correspond to the ratio between the on-shell masses.

We give an analytic expression for the QCD correction factor to all orders in the ratio between on-shell masses. We also give useful approximations to the QCD correction when either the ratio between on-shell masses or the ratio between the masses at a single scale is used.

ACKNOWLEDGEMENTS: I thank Fred Gilman, Howard Haber and Michael Peskin for their important advice, and Isi Dunietz and Russel Kauffman for comments on the manuscript.

REFERENCES

1. N. Cabibbo and L. Maiani, Phys. Lett. 79B (1978) 109;
   G. Corbó, Phys. Lett. 116B (1982) 298; Nucl. Phys. B212 (1983) 99.
2. A. Ali and E. Pietarinen, Nucl. Phys. B154 (1979) 519;
   N. Cabibbo, G. Corbo and L. Maiani, Nucl. Phys. B155 (1979) 83.
3. A.J. Buras, W. Slominski and H. Steger, Nucl. Phys. B238 (1984) 529; B245 (1984) 369.
4. E.H. Thorndike, *in* Proc. of the 1985 International Symposium on Lepton and Photon Interactions at High Energies, eds. M. Konuma and K. Takahashi, (Kyoto, 1985).
5. M. Peskin, private communication.
6. M. Jeżabek and J.H. Kühn, Phys. Lett. 207B (1988) 91.
7. Q. Hokim and X.-Y. Pham, Ann. Phys. 155 (1984) 202.
8. R.E. Behrends, R.J. Finkelstein and A. Sirlin, Phys. Rev. 101 (1956) 866;
   S.M. Berman,Phys. Rev. 112 (1958) 267;
   T. Kinoshita and A. Sirlin, Phys. Rev. 113 (1959) 1652.
9. T. Kinoshita, J. Math. Phys. 3 (1962) 650.
10. L. Lewin, Dilogarithms and associated functions, (Macdonald, London, 1958).
11. H. Harari and M. Leurer, Phys. Lett. 181B (1986) 123.
12. See *e.g.* J.S. Hagelin, Nucl. Phys. B193 (1981) 123;
    F.J. Gilman and J.S. Hagelin, Phys. Lett. B133 (1983) 443.
13. J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.
14. C.R. Ng *et al.*, HRS collaboration, Argonne preprint ANL-HEP-PR-88-11 (1988).
15. S.L. Wu, Nucl. Phys. B (Proc. Suppl.) 3 (1988) 39.

16. G. Altarelli, N. Cabibbo, G. Corbo, L. Maiani and G. Martinelli, Nucl. Phys. B208 (1982) 365.

17. M. Wirbel, B. Stech and M. Bauer, Z. Phys. C29 (1985) 637.

18. B. Grinstein, M.B. Wise and N. Isgur, Phys. Rev. Lett. 56 (1986) 298; Toronto preprint, UTPT-88-12 (1988).

19. W.J. Marciano and A. Sirlin, Phys. Rev. Lett. 61 (1988) 1815.