# Stable Static Solitons in the Nonlinear Sigma Model with a Topological Term* 

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#### Abstract

Soliton solutions in the nonlinear sigma model in $2+1$ space-time dimensions are analyzed classically in the presence of the Chern-Simons term using a $C P^{1}$ map. Making an expansion in $\theta_{C P}$, the new contributions to the energy functional push the solitons to infinite size. It is further shown that including quantum zero-point fluctuations around the soliton vacuum in the long-wavelength limit is insufficient to stablize its size.


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## I. Motivation

It is an attractive though unproven conjecture that the two-dimensional quantum Heisenberg magnet in the continuum and large spin limit is the nonlinear sigma model with or without further topological terms [1]. Making the standard map of the nonlinear sigma model to its $C P^{1}$ variant [2], a unique topological term can be written, namely the $2+1$ dimensional Chern-Simons density. The origin of this term, though obscure physically [3], does have dramatic effects on the physics. The resulting spin-statistics transmutation is important in one scenario for the microscopic mechanism conjectured for high-Tc superconductivity [4]. Polyakov et al. [6] observed that since the spin-statistics effect is predicted in the long-wavelength limit, this same term may be sufficient to stablize the size of the solitons whereas arbitrary higher-derivative terms drop out of the theory in this limit.

The bare $C P^{1}$ theory is given by the Lagrangian density

$$
\mathcal{L}=f\left(D_{\mu} Z\right)^{\dagger}\left(D^{\mu} Z\right)-\eta\left(Z^{\dagger} Z-1\right)
$$

with $\mu=0,1,2, D_{\mu}=\partial_{\mu}+i A_{\mu}$ and Z a complex two-spinor $(\alpha, \beta)$. The theory has one dimensionful coupling constant $f$ and constraint fields $\eta(x)$ and $A_{\mu}(x)$. Finite energy solutions of the equations of motion are well-known [5] and are given by $Z_{\text {sol }}=(\omega /|\omega|)$ with $\omega=\omega\left(\xi_{+}\right)$or $\omega\left(\xi_{-}\right)$where $\xi_{+}=x_{1}+i x_{2}$ and $\xi_{-}=x_{1}-i x_{2}$. The simplest polynomial analytic function for $\omega=u+\left(\left(\xi-\xi_{0}\right) / \lambda\right)^{N} v$ has energy $2 \pi f N$ where $u$ and $v$ are any convenient basis vectors in spinor space and the $\lambda$ denotes the size of the soliton. The expression for the energy in the case of static solitons is $E=f \int d^{2} x\left(D_{i} Z\right)^{\dagger}\left(D_{i} Z\right)$ where $A_{0}$ is zero via its equation of motion. This expression is invariant with respect to rescalings $x^{\prime}=\kappa x$. This dilation invariance of the bare $C P^{1}$ theory makes it possible for solitons of finite size to simply shrink to zero size thus making them only marginally stable against any perturbations. The rescaling invariance also means that no stable minimum exists in the energy as a function of soliton size. In the long-wavelength limit,
higher derivative corrections to the theory drop out, and the remaining terms may collapse the solitons. For this reason, the Chern-Simons term, while introducing higher derivative corrections and being relevant in the long-wavelength limit (e.g., the spin-statistics effect), is thought to play a crucial role in stabilizing the solitons [6]. Stabilization via quantum corrections around the soliton background will be weaker than the classical effects without the Chern-Simons term [7] by $\mathcal{O}(\hbar)$.

Higher derivative corrections by themselves introduce inverse powers of $\lambda$ in the energy functional. Except in the case of a careful conspiracy of signs (as in the Skyrme model), these terms will push the soliton configurations to infinite size. The paper analyzes two important effects of the Chern-Simons term on solitons; namely, the classical shift in the energy of the solitons and its effect on quantum corrections in the semiclassical picture.

## II. Classical Effects

The full theory with Chern-Simons term is given by

$$
\begin{equation*}
\mathcal{L}=f\left(D_{\mu} Z\right)^{\dagger}\left(D^{\mu} Z\right)-\eta\left(Z^{\dagger} Z-1\right)+\theta \varepsilon_{\mu \nu \rho} A^{\mu} \partial^{\nu} A^{\rho} \tag{1}
\end{equation*}
$$

The $\theta$ can also be written as $\left(\theta_{C P} / 4 \pi^{2}\right)$ with $\theta_{C P}=\pi$ responsible for bose-fermi transmutations [8]. The last term makes the $\vec{A}(x)$ fields dynamical thus changing the theory drastically [9]. The equations of motion that result from Eq. (1) are:

$$
\begin{align*}
\left(D_{\mu} D^{\mu}-\eta\right) Z & =0 \\
Z^{\dagger} Z & =1  \tag{2}\\
A_{\mu} Z^{\dagger} Z & =-\left(\frac{J_{\mu}}{2}\right)-\left(\frac{\theta}{f}\right) \varepsilon_{\mu \nu \lambda} \partial^{\nu} A^{\lambda}
\end{align*}
$$

where $J_{\mu}=i\left(\partial_{\mu} Z^{\dagger} Z-Z^{\dagger} \partial_{\mu} Z\right)$.
For static solutions $Z\left(x_{1}, x_{2}\right)$, the zero component of the last equation reads $A_{0}=-(\theta / f) \vec{\nabla} \times \vec{A}$. The equations in this subspace of solutions are only gauge
invariant with respect to time-independent gauge transformations. Therefore, one cannot set $A_{0}$ to zero by gauge fixing without changing the time dependence of the $Z(x)$ fields. This equation which is Gauss's law for this theory implies $A_{0}$ is entirely determined by the other $\vec{A}$ 's. If we further look for solutions where the $\vec{A}(x)$ 's are also static, the two other equations can be combined with this to give the equation

$$
\begin{equation*}
A_{i}=-\frac{J_{i}}{2}-\left(\frac{\theta}{f}\right)^{2} \epsilon_{i j} \partial_{j}(\vec{\nabla} \times \vec{A}) \tag{3}
\end{equation*}
$$

Solutions to the last equation, Gauss's law and the $Z$ equation constitute solitons with energy given by

$$
\begin{equation*}
E=f \int d^{2} x\left[A_{0}^{2}+\left(D_{i} Z\right)^{\dagger}\left(D_{i} Z\right)\right] \tag{4}
\end{equation*}
$$

Since $A_{0}$ is not related to the currents to leading order, we divide the cases as to whether $A_{0}$ is zero or not. For configurations with $A_{0}=0$, one gets $\vec{A}=-(1 / 2) \vec{J}$ and $\vec{\nabla} \times \vec{J}=0$. The solutions to the $Z$ equation are simply given by $Z_{\text {sol }}$ (no other nontrivial solutions to this equation are known). However, for these solutions, we cannot satisfy $\vec{\nabla} \times \vec{J}=0$. Therefore, for $A_{0}$ zero, there are no consistent solutions.

Finite energy solutions require $A_{0}$ to vanish at the boundary while $D_{i} Z>$ $(1 / r)$. Motivated by the ansatz for the vortex in Higgs theories, we try the following ansatz in circular coordinates $(r, \phi)$ for soliton solutions

$$
Z=\left[\begin{array}{c}
G(r) e^{i N \phi}  \tag{5}\\
H(r)
\end{array}\right]
$$

and require $A_{\phi}=A(r), A_{r}=0$ and $A_{0}=A_{0}(r)$. There is only one degree of freedom left in the $Z$ 's since $Z^{\dagger} Z=1$ implies $G^{2}+H^{2}=1$. Using this ansatz, it can be shown that the equations for the top and bottom component of $Z$ are related by a factor $(G / H)$. Therefore, the one independent equation will determine $G(r)$. For arbitrary $\theta$, the equations of motion for $G(r), A_{\phi}$ and $A_{0}$ become:

$$
\begin{align*}
\frac{d}{d r}\left(\frac{r G^{\prime}}{H}\right)-\frac{N^{2} H G}{r}\left(1+\frac{2 r A_{\phi}}{N}\right) & =0 \\
A_{\phi}-\left(\frac{\theta}{f}\right)^{2}\left(-\frac{A_{\phi}}{r^{2}}+\frac{A_{\phi}^{\prime}}{r}+A_{\phi}^{\prime \prime}\right)+\frac{N G^{2}}{r} & =0  \tag{6}\\
A_{0} & =-\frac{\theta}{f} \frac{1}{r} \frac{d}{d r}\left(r A_{\phi}\right)
\end{align*}
$$

The $A_{r}$ equation of motion consistently sets it to zero for our ansatz. For the case $\theta=0$, Gauss's law implies $A_{0}=0$, and the $A$ equation of motion gives $A_{\phi}=-(N / r) G^{2}$. The second order $G(r)$ equation is solved whenever the first order equation $G^{\prime}= \pm(N / r) G H^{2}$ is satisfied. Imposing the boundary condition $G=1$ at infinity gives the well-known solutions at $0=0$, namely, $G=\left(\tilde{r}^{N}\right) /\left(1+\tilde{r}^{2 N}\right)^{\frac{1}{2}}$ where $\tilde{r}=r / \lambda$ with $\lambda$ the soliton size. For more general theta we have two coupled second order differential equations. We note that the boundary configurations $Z=\left(e^{i N \phi}, 0\right), A_{\phi}=-(N / r)$ give $A_{0}=0$ at the boundary, winding number $N$ and satisfy the equations of motion in addition to regulating the energy functional at large radius. That is, barring any pathological singularities, the solutions to the two differential equations with the same boundary conditions as the lowest order soliton solutions constitute new finite energy configurations of the theory with non-zero $\theta_{C P}$. Instead of pursuing the complicated problem of the complete analytical solutions, we instead perform a perturbative analysis in $\theta$.

In the case where $A_{0}$ is fixed by the $\vec{A}$ 's, one can examine solutions through an expansion in $\theta$. An expansion in $\theta$ is also an expansion in $\hbar$, since in physical units, the true coefficient of the Chern-Simons term is $\hbar \theta$. Moreover, the equations of motion imply that higher powers of $\theta$ occur with higher derivatives of the fields. Thus, in the long-wavelength limit, in the classical limit, and in the small theta limit taken simultaneously, we have

$$
\begin{align*}
& A_{i}=-\frac{1}{2} J_{i}+\frac{1}{2}\left(\frac{\theta}{f}\right)^{2} \epsilon_{i j} \partial_{j}(\vec{\nabla} \times \vec{J})+\mathcal{O}\left(\theta^{4}\right)  \tag{7}\\
& A_{0}=\left(\frac{\theta}{2 f}\right) \vec{\nabla} \times \vec{J}+\mathcal{O}\left(\theta^{3}\right)
\end{align*}
$$

Equivalently, one can use the nonlocal expression for the $\vec{A}$ 's in terms of the $\vec{J}$ 's [10]. The only dynamical equation left to solve is $D_{i} D_{i} Z-\left(Z^{\dagger} D_{i} D_{i} Z\right) Z=0$ with $i=1,2$ where we note that all the $A_{0}$ pieces drop out. Because of the complicated expansion for $A_{i}$, this nonlinear equation has no obvious solutions. For solutions of the form $Z=Z_{\text {sol }}+\theta \tilde{Z}$, where $\tilde{Z}$ is not necessarily (anti-)analytic, one finds that the above equation and the constraint $Z^{\dagger} Z=1$ lead to an overdetermined system of equations. Instead of expanding around $Z_{\text {sol }}$, if one starts with a general (anti-) analytic $\omega$ with $Z=(\omega /|\omega|)$, the $Z$ equation of motion requires

$$
\begin{equation*}
\partial_{i}(\nabla \times J) D_{i}^{0} Z=0 \tag{8}
\end{equation*}
$$

where $D_{i}^{0}=\partial_{i}-\left(Z^{\dagger} \partial_{i} Z\right)$, the lowest order part of $\vec{A}$. This equation is certainly not obeyed by the standard lowest order soliton configurations given in Section I. If for some solution, this equation were satisfied as $D_{i}^{0} Z=0$, the energy vanishes identically where the expression for the energy density to lowest order is given after partial integration:

$$
\frac{\mathcal{E}}{f}=A_{0}^{2}+\left(D_{i}^{0} Z\right)^{\dagger}\left(D_{i}^{0} Z\right)+\left(\frac{\theta}{f}\right)^{4}(\vec{\nabla} \times \vec{J}) \nabla^{2}(\vec{\nabla} \times \vec{J})
$$

since $\vec{\nabla} \times \vec{J} \sim\left(D_{i}^{0} Z\right)^{\dagger}\left(D_{j}^{0} Z\right)$. For solutions to this equation via $\vec{\nabla} \times \vec{J} \sim$ constant, the energy is not bounded. Since, we don't know what nontrivial solutions of finite energy do solve this equation, we analyze the $\vec{A}$ equations to order $\theta^{2}$ instead of order $\theta^{3}$. To this order, the solutions of Section I are still good since $\vec{A}$ is now simply $-(1 / 2) \vec{J}$. The energy is

$$
\begin{equation*}
E=f \int d^{2} x\left[A_{0}^{2}+\left(D_{i}^{0} Z\right)^{\dagger}\left(D_{i}^{0} Z\right)\right] \tag{9}
\end{equation*}
$$

Using the order $\theta$ expression for $A_{0}$, the energy of the soliton with winding number $N$ and size $\lambda$ is given by

$$
\begin{equation*}
E=2 \pi f N+4 \pi f N^{3}\left(\frac{\theta}{f \lambda}\right)^{2} B\left(2-\frac{1}{N}, 2+\frac{1}{N}\right) \tag{10}
\end{equation*}
$$

where $B(x, y)$ is the beta function [11]. Thus, to lowest order in $\theta$, the lowest
energy soliton is of infinite size.
If we consider the equations for static $Z$ but time-dependent $A$ 's, there are two cases again depending on whether $A_{0}$ is zero or not. For configurations with $A_{0}=0$, it is easy to show that the self-consistency of the equations of motion require the $A$ 's to be time-independent also for which we know there are no finite energy solutions. For the case $A_{0} \sim \vec{\nabla} \times \vec{A}$, one obtains again nonlocal expressions for all the $A$ 's in terms of the $J$ 's [12]. Instead, one can make an expansion in $\theta$ using the equations of motion as before and the lowest order terms in the current are exactly the same as Eq. (3). For this class of configurations, the $Z$ equation of motion and the energy functional is unchanged. There exist solutions then to the same order as before with large solitons preferred.

Before we conclude this section, we briefly review the case of nonstatic solutions to the equations of motion. If one restricts $A_{0}$ configurations to vanish at infinite radius [13], one can use the full gauge invariance of the equations of motion in Eq. (2) to set $A_{0}=0$. The $Z$ equation of motion is modified to $\ddot{Z}-\left(Z^{\dagger} \ddot{Z}\right) Z-$ $\left(D_{i} D_{i} Z\right)+\left(Z^{\dagger} D_{i} D_{i} Z\right) Z=0$. The $\vec{A}$ equations impose one more constraint on the $Z$ fields, namely,

$$
\begin{equation*}
\left(\frac{f}{\theta}\right)(\vec{\nabla} \times \vec{J})-\left(\frac{f}{\theta}\right)^{2} J_{0}-\ddot{J}_{0}+\vec{\nabla} \cdot \dot{\vec{J}}=0 \tag{11}
\end{equation*}
$$

whose origin is essentially Gauss's Law. As usual, the $\vec{A}$ 's can be expressed nonlocally in terms of the $\vec{J}$ 's [12] and the perturbative formula is

$$
\begin{equation*}
A_{i}=-\frac{1}{2} J_{i}+\left(\frac{\theta}{2 f}\right) \epsilon_{i j} \dot{J}_{j}+\frac{1}{2}\left(\frac{0}{f}\right)^{2} \ddot{J}_{i}+\mathcal{O}\left(\theta^{3}\right) \tag{12}
\end{equation*}
$$

To order $\theta^{2}$, the constraint equation (9) reads $J_{0}=(\theta / f) \vec{\nabla} \times \vec{J}$. A solution of the form $Z=\exp (i \varphi(x, t)) Z_{\text {sol }}$ gives $\varphi=(i \theta / 2 f)(\vec{\nabla} \times \vec{J})+$ const. which makes $\vec{A}=-(1 / 2) \vec{J}$ to $\theta^{2}$ for which a static solution was found above. However, the $Z$ equation imposes the familiar equation $\partial_{i} \varphi\left(D_{i}^{0} Z\right)=0$ just as before but now to
order $\theta^{2}$. As discussed earlier, this severely restricts the hope of an obvious solution. Interestingly, though, the lowest order beyond the bare theory is problematic for the nonstatic case.

The $C P^{1}$ theory even without the Chern-Simons term induces higher order effective terms for the $A_{\mu}$ fields [14]. The lowest order gauge invariant term is $(1 / g) F_{\mu \nu} F^{\mu \nu}$. With this term present, $Z_{s o l}$ is a solution to the equations of motion only to order [15] $\theta^{2}$ and order ( $1 / g f$ ). For the case of static solitons, we were able to show that the bare solitons still satisfy the equations of motion but a nontrivial contribution to the energy functional from the Chern-Simons term drives these solitons to infinite size. The nonstatic case is a complex problem even to next to leading order in perturbation theory. Since the classical analysis shows that the static soliton configurations satisfy the equations of motion to order $\theta^{2}$, the role of quantum fluctuations needs to be examined.

## III. Quantum Fluctuations

In Section II, we made an expansion of the classical equations of motion in powers of $\theta$. To order $\theta^{2}$, static solitons solve the equations of motion. These configurations then are in the perturbative sense, points about which a saddle-point expansion can be made. By expanding the action around these classical configurations, we will obtain the spectrum of excitations in the soliton background [16]. Usually the spectrum to lowest order consists of noninteracting spin-waves having the energy $E=\sum_{i} \hbar \omega_{i}\left(n_{i}+(1 / 2)\right)$. The contribution of the zero point motion to the soliton energy is therefore $\sum_{i}(1 / 2) \hbar \omega_{i}$. Making a similar correction to the vacuum energy, one obtains the quantum soliton mass in this semiclassical picture [17]. The main effect of the soliton background is to shift the eigen-frequencies of the spin waves $\omega_{i}$. In the path-integral point of view, the object of interest is the transition amplitude between all field configurations $\Phi$ at time $t=0$ to the same field configurations at a later time $t=T$. The correct expression in the canonical language is $\operatorname{Tr}\left(e^{i H T}\right)$ which in the path integral language becomes $\sim \int[d \Phi] e^{i S}$ where $\Phi$ represents all the fields and $S$ the action for our model.

Since the integral has an extremum at the soliton configurations to order $\theta^{2}$, we shift the fields $Z=Z^{c l}+\tilde{Z}, A_{\mu}=A_{\mu}^{c l}+\tilde{A}_{\mu}$, and $\eta=\eta^{c l}+\tilde{\lambda}$, the transition amplitude becomes:

$$
\begin{equation*}
\operatorname{Tr}\left(e^{i H T}\right)=N e^{i S^{c l}} \int[d \Phi] \delta\left(\tilde{Z}^{\dagger} Z^{c l}+Z_{c l}^{\dagger} \tilde{Z}\right) e^{i \tilde{S}} \tag{13}
\end{equation*}
$$

where $N$ is a normalization constant. It is straightforward to show that $S^{c l}=-T E$ where E is given in Eq. (4) and is evaluated exactly in Eq. (8). Corrections to this arising from $\tilde{S}$ will correspond to the zero point shifts in the Hamiltonian picture. The expression for $\tilde{S}$ to quadratic order in the quantum fields is given by

$$
\begin{align*}
\tilde{S}= & f \int d^{3} x \tilde{Z}^{\dagger}\left[-\left(D_{\mu} D^{\mu}\right)^{c l}-\eta^{c l}\right] \tilde{Z}-2 i \tilde{A}_{\mu}\left(Z_{c l}^{\dagger} \partial^{\mu} \tilde{Z}+\tilde{Z}^{\dagger} \partial^{\mu} Z^{c l}\right)  \tag{14}\\
& +\tilde{A}_{\mu} \tilde{A}^{\mu}+\theta \varepsilon^{\mu \nu \lambda} \tilde{A}_{\mu} \partial_{\nu} \tilde{A}_{\lambda}
\end{align*}
$$

where $\eta^{c l}=Z_{c l}^{\dagger} D_{\mu} D^{\mu} Z_{c l}$ The first term represents the action for spin-waves in the soliton background. Only the last term is specific to the $C P^{1}$ model with ChernSimons term. This term is entirely independent of the classical background unlike the spin-wave term. Were this term not present, the $\tilde{A}$ integration is a simple gaussian integration and the new effective action $\tilde{S}_{\text {eff }}$ would give the appropriate zero-point contribution of the spin-waves to the soliton energy. We can obtain a similar effective action for the theory with the Chern-Simons term if we use instead the equations of motion for the $\tilde{A}$ that emerge from $\tilde{S}$. From $\tilde{S}$, the variation principle gives $\tilde{A}_{\mu}=-\tilde{J}_{\mu}-(\theta / f) \varepsilon_{\mu \nu \lambda} \partial^{\nu} \tilde{A}^{\lambda}$, with $\tilde{J}_{\mu}=i\left(Z_{c l}^{\dagger} \partial_{\mu} \tilde{Z}+\tilde{Z}^{\dagger} \partial_{\mu} Z_{c l}\right)$. In this way, we are performing a second saddle-point approximation. We can similarly perform the same operation had we included cubic and quartic terms in $\tilde{S}$. The resulting effective action to order $\theta^{2}$ is

$$
\begin{equation*}
\tilde{S}_{e f f}=f \int d^{3} x \tilde{Z}^{\dagger}\left[-\left(D_{\mu} D^{\mu}\right)^{c l}+\eta^{c l}\right] \tilde{Z}-f \tilde{J}_{\mu} \tilde{J}^{\mu}+\theta \tilde{J}_{\mu} \partial_{\nu} \tilde{J}_{\lambda} \varepsilon^{\mu \nu \lambda} \tag{15}
\end{equation*}
$$

Now the field integrations are to be done over $\tilde{Z}$ and $\tilde{Z}^{\dagger}$ only. The last term has the pieces quadratic in $\tilde{Z}$ which when combined with the higher order terms is equal
to the Hopf term that comes from the Chern-Simons term in a $\theta$ expansion [18]. The entire Hopf term represents the index $\Pi_{S^{3}}\left(S^{2}\right)$ and just represents the spinstatistics factor [19] for the configuration $Z_{c l}+\tilde{Z}$. The remaining terms are exactly the same as the bare $C P^{1}$ theory except that now $A_{0}^{c l} \sim(\theta / f) \vec{\nabla} \times J_{c l}$. This new potential term in the wave-equation for the spin-waves is of order $\left(\theta / f \lambda^{2}\right)$ apart from numerical factors less than one. Since the quantum fluctuations are $\mathcal{O}(\hbar)$, the change in the quantum fluctuations via the new $A_{0}$ will be smaller by the factors $\hbar \theta$. Even for $\theta_{C P}=\pi, \theta$ is small enough that we omit this new potential term in order to study the leading quantum effects, that is to zeroth order in $\theta$. The quantum corrections for the bare $C P^{1}$ are usually treated in the nonlinear sigma model language. The quantum fluctuations of the bare nonlinear sigma model have been fully addressed in both two and three dimensions [7,17].

Now we can combine the known results of the quantum fluctuations of the solitons in the nonlinear sigma model with the shift of the classical energy of these solitons in the presence of the Chern-Simons term to derive the new energy functional. Quantum fluctuations in the nonlinear sigma model break the dilation symmetry of the energy functional. When the zero point motions around the vacuum and the one-soliton background are taken into account, the result is to lower the classical energy of the soliton through a sum of phase shifts for spinwaves in the soliton background, or $E_{q u a n}=E_{\text {class }}-(\epsilon) \int d k \operatorname{tr}[\delta(k)]$ where $\epsilon$ is a constant of order $\hbar$. The minus sign in this equation is consistent with Quantum Mechanics, which predicts that the energy of the lowest state, in this case the soliton at rest, is decreased by second-order stationary perturbation theory. Using a large momentum cutoff corresponding to the inverse of the lattice spacing ( $k_{d}$ ) in computing the zero point motion, the energy of the soliton to order $\hbar \theta$ is [7]

$$
\begin{equation*}
E=2 \pi f+\frac{4 \pi f}{3}\left(\frac{\theta}{f \lambda}\right)^{2}-\frac{\zeta(2)}{3} \hbar c k_{d}{ }^{3} \lambda^{2} \tag{16}
\end{equation*}
$$

valid only for $k_{d} \lambda<1$ and where by $\lambda$, we mean $|\lambda|$. The quantum effects by themselves make larger solitons have lower energy. Larger solitons present a larger
disordered area relative to the Néel state and spin waves cost less energy on such a background compared to the ordered background. Varying with respect to $\lambda$, we find no stable minimum satisfying $k_{d} \lambda<1$.

Unlike the spin-statistics effect which holds in the long-wavelength limit, stable solitons do not emerge in the same limit even after the leading quantum corrections are included. Examining the last two equalities in Eq. (6), we see that the scalebreaking parameter is $\mathcal{O}(\theta / f)$, which is then the size of the exact solutions to the equations of motion. It is not clear whether perturbation theory to $\theta^{3}$ is sufficient to see that this scale arises, since to $\theta^{2}$, we have shown it does not.

The spin coupling constant $f$ is crucial to determining the final size as is the value of $\theta_{C P}$. Presumably the underlying dynamics of holes in the Heisenberg lattice determines both. The entire analysis above could also have been done by first eliminating the $A$ 's in terms of the $Z$ 's using the momentum expansion and then performing the saddle point approximation.

## IV. Conclusions

The addition of the Chern-Simons term introduces an additional contribution to the energy functional for the nonlinear sigma model. In the continuum, this term destablizes the soliton. In fact, full solutions to the equations of motion either in the continuum or the lattice with the topological term are unknown. Solving the equations to next to lowest order and including quantum corrections to $\mathcal{O}(\hbar \theta)$, stable finite-size soliton configurations do not emerge. The effect of the ChernSimons term in changing the wave equation for the spin-waves via the additional $A_{0}$ potential terms needs to be examined. These corrections will appear with a minus sign and be of $\mathcal{O}(\hbar \theta)$.

To find finite-size soliton solutions in perturbation theory then requires us to go to one higher order in $\theta$. Remarkably, however, the full solutions to the equations of motion may be tractable. Using the ansatz we made in section II and writing the radial coordinate as $r=\tan \chi$, the function $G(r)$ in the case $\theta=0$ and $\lambda=1$ becomes $\operatorname{simply} \sin \chi$. Therefore, a Fourier expansion in $\chi$ is an excellent basis set
for the full solutions for nonzero $\theta$. The scale-breaking parameter sets the size of the resulting solitons to $(\theta / f)($ constant $)$. Given the Fourier decomposition, it is straightforward to determine the unknown constant determining the physical size. This work is currently under progress.

Once solutions of reasonable accuracy have been found, a greater understanding of the role of solitons in the onset of superconductivity is needed. Their stablility is dependent on the values of $f$ and $\theta_{C P}$. If these topological structures are observable, the above calculation could predict $\theta_{C P}$. Yet these parameters should be derivable somehow from the lattice Heisenberg model or Hubbard model. The role of the holes in stabilizing the solitons energetically is the subject of our current investigation.

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10. If we write $\vec{A}_{i}=\partial_{i} \phi+\epsilon_{i j} \partial_{j} \psi$, we have

$$
\begin{aligned}
\nabla^{2} \phi & =-\left[\frac{(\nabla \cdot J)}{2}\right], \\
\nabla^{2} \psi & =\left(\frac{f}{\theta}\right) A_{0}, \quad \text { and } \\
{\left[\nabla^{2}-\left(\frac{f^{2}}{\theta^{2}}\right)\right] A_{0} } & =-\left[\frac{(\nabla \times J)}{2}\right]
\end{aligned}
$$

11. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, (Academic Press, 1980) 948.
12. For this case we get the nonlocal expressions

$$
\begin{aligned}
{\left[\square+\left(\frac{f}{\theta}\right)^{2}\right] A_{0} } & =\left(\frac{f}{\theta}\right)\left[\frac{(\vec{\nabla} \times \vec{J})}{2}\right] \\
\vec{\nabla} \cdot \vec{A} & =-\left[\frac{(\vec{\nabla} \cdot \vec{J})}{2}\right]+\dot{A}_{0} \\
\vec{\nabla} \times \vec{A} & =-\left(\frac{f}{\theta}\right) A_{0}
\end{aligned}
$$

13. The action is in fact only invariant with respect to gauge transformations that vanish at infinity. So, clearly, $A_{0}$ 's that do not vanish at infinity cannot be set to zero everywhere by a gauge transformation. For solitons that have vanishing $J_{0}$ and vanishing derivatives of $\vec{J}$ at infinity, $A_{0}$ also vanishes.
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15. For the simplest case of static $Z$ fields, the $A_{\mu}$ 's have the perturbative expansions

$$
\begin{aligned}
A_{0} & =-\left(\frac{\theta}{f}\right)(\vec{\nabla} \times \vec{J})+\mathcal{O}\left(\theta^{2}, \frac{1}{g^{2}}, \frac{\theta}{g}\right) \quad, \quad \text { and } \\
A_{i} & =-\left(\frac{J_{i}}{2}\right)-\left(\frac{1}{4 g f}\right)\left[\nabla^{2} J_{i}-\partial_{i}(\vec{\nabla} \cdot \vec{J})\right]+\mathcal{O}\left(\theta^{2}, \frac{1}{g^{2}}, \frac{\theta}{g}\right) \\
& \equiv-\left(\frac{J_{i}}{2}\right)-\left(\frac{1}{4 g f}\right) \tilde{A}_{i} .
\end{aligned}
$$

The $Z$ equation of motion in turn, then, implies to lowest order for (anti-) analytic solutions the equation $\tilde{A}_{i}\left(D_{i}^{0} Z\right)=0$, as before, but now to order $\theta^{2}$ !
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