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Non-Abelian Q-Stars

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ABSTRACT

We construct a new class of non-topological soliton stars which appears in global non-Abelian field theories coupled to classical Einstein gravity. It is the analogue of the Q-stars recently found in Abelian theories. If μ (of order 10^{-1} to 10^4 GeV) is a free-particle inverse Compton wavelength and m_{pl} is the Planck mass, these objects have energy densities $\varepsilon \sim \mu^4$, radii $\rho \sim m_{pl}/\mu^2$, global charges $q \sim m_{pl}^3/\mu^3$, and masses $M \sim m_{pl}^3/\mu^2$ obeying a generalized Chandrasekhar limit. We give an explicit $SO(3)$ example which demonstrates their very simple structure: the stellar surface reproduces the non-Abelian Q-ball, but within the star the fields are no longer constant.

1. Introduction

Q-stars, introduced in ref. 1, are unions of non-topological soliton stars [2] and Q-balls, [3–5] in which a thin surface shell of Q-ball connects an interior coherent field with the outside vacuum. The Q-star is stabilized by a conserved global charge. Assuming spherical symmetry

$$ds^2 = -B(\rho)^{-1}dt^2 + A(\rho)^{-1}d\rho^2 + \rho^2 d\Omega \tag{1.1}$$

and rigid rotation in internal group space leads to equations of motion within the surface shell analogous to the equations of Newtonian mechanics; these give boundary conditions for the Einstein equations in the Q-star interior.

This paper concerns non-Abelian Q-stars, a general class of objects independent of any particular theory. We choose a simple model here, however, in order to clearly show their structure.

2. Setting Up the Q-Star

Consider a global $SO(3)$-symmetric Lagrangian

$$\mathcal{L} = -\frac{1}{2}\, Tr\; g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - Tr\; U(\phi)\ , \tag{2.1}$$

with a general renormalizable potential

$$U = \frac{\mu^2}{2}\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \tag{2.2}$$

and ϕ in the $SO(3)$ **5** representation: real, symmetric, and traceless. Q-balls *without gravity* were studied in this model by Safian, Coleman, and Axenides, who showed that it includes the lowest energy solitons of the very similar $SU(3)$

theory.[6] Rescaling

$$
\begin{aligned}
g &= \mu\tilde{g} \\
\phi &= \frac{\mu}{\tilde{g}}\tilde{\phi} \\
\lambda &= \tilde{g}^2\tilde{\lambda}
\end{aligned}
\tag{2.3}
$$

gives the simple form

$$
\begin{aligned}
U &= \frac{\mu^4}{\tilde{g}^2}\,\tilde{U}\ , \\
\tilde{U} &= \frac{\tilde{\phi}^2}{2} + \frac{\tilde{\phi}^3}{3!} + \frac{\tilde{\lambda}}{4!}\tilde{\phi}^4\ ;
\end{aligned}
\tag{2.4}
$$

all energy densities will scale by $\mu^4/\tilde{g}^2$ if we also scale frequencies and lengths by $\omega = \mu\tilde{\omega}$ and $\rho = \tilde{\rho}\mu^{-1}$. However, since the natural length in Einstein's equations is $(\tilde{g}/\epsilon)\mu^{-1}$ with

$$
\epsilon \equiv \sqrt{8\pi\mu^2 G} = 1.03 \times 10^{-16}(\mu/250\ GeV)\ , \tag{2.5}
$$

we actually scale

$$
\rho = \frac{\tilde{g}S}{\epsilon\mu}x \tag{2.6}
$$

with $0 \leq x \leq 1$ inside the Q-star. S is a free parameter which merely sets the inner edge of the Q-star surface at $x = 1$. We will express all quantities in these units, drop terms of $\mathcal{O}(\epsilon)$, and use primes to denote x derivatives. We shall see that factors of $\tilde{g}$ are basically unimportant to Q-star structure.

The G_{tt} and $G_{\rho\rho}$ Einstein equations are then respectively

$$
\begin{aligned}
A - 1 + xA' &= -S^2x^2(\mathcal{V} + \mathcal{W} + \mathcal{U}) \\
A - 1 - x\frac{A}{B}B' &= S^2x^2(\mathcal{V} + \mathcal{W} - \mathcal{U})
\end{aligned}
\tag{2.7}
$$

with

$$
\begin{aligned}
\mathcal{U} &= Tr\ \tilde{U} = Tr\ \left(\frac{\tilde{\phi}^2}{2} + \frac{\tilde{\phi}^3}{3!} + \frac{\tilde{\lambda}}{4!}\tilde{\phi}^4\right) \\
\mathcal{V} &= Tr\ \frac{1}{2}A\left(\frac{\partial\tilde{\phi}}{\partial x}\right)^2 \frac{\epsilon^2}{\tilde{g}^2 S^2} \\
\mathcal{W} &= Tr\ \frac{1}{2}B\left(\frac{\partial\tilde{\phi}}{\partial(\mu t)}\right)^2 = -Tr\ \frac{1}{2}B\ [\tilde{\Omega},\tilde{\phi}]^2\ .
\end{aligned}
\tag{2.8}
$$

Note that $\mathcal{V}$ is of $\mathcal{O}(1)$ within the surface but of $\mathcal{O}(\epsilon^2)$ in the interior of the Q-star.

In the last step of (2.8) we use the rigid rotation condition $\partial\phi/\partial t = i[\Omega,\phi]$ where $\Omega = \mu\tilde{\Omega}$ is an $SO(3)$ frequency matrix; we can always diagonalize $\phi = e^{iR}\ \phi_{diag}\ e^{-iR}$ and minimize the kinetic energy by assuming rigid rotation, setting the matrix $R(\rho,t) = \Omega t + \mathbf{C}$ with ϕ_{diag} independent of t. A global $SO(3)$ rotation eliminates the constant $\mathbf{C}$. Working in that basis we follow ref. 6 and write

$$
\phi_{diag} = \mathrm{diag}(\phi_1,\phi_2,\phi_3) = -\frac{1}{2}\phi_2 \cdot \mathrm{diag}(1+y,\ -2,\ 1-y)\ . \tag{2.9}
$$

If the vacuum does not break the $SO(3)$ symmetry (*i.e.* $\tilde{\lambda} \geq \frac{1}{9}$), there is a conserved charge

$$
Q \equiv \int d^3\rho\sqrt{-g}\ J^0 = -i\int d^3\rho\sqrt{-g}\ B\ [\phi,\dot{\phi}] = i\tilde{g}\tilde{q}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{2.10}
$$

in the diagonal basis. Then

$$
\tilde{q} = \frac{4\pi S^3}{\epsilon^3}\int_0^1 x^2 dx\ \sqrt{B/A}\ \tilde{\phi}_2^2\ y^2\ \tilde{\omega} \tag{2.11}
$$

and

$$\tilde{\Omega} = -i\tilde{\omega} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} . \tag{2.12}$$

The remaining Einstein equations are identical and related[2] by the Bianchi identity to the ϕ-field Euler-Lagrange equations

$$-B\left[\tilde{\Omega}, [\tilde{\Omega}, \tilde{\phi}]\right] + \frac{\partial \mathcal{U}}{\partial \tilde{\phi}} - \frac{I}{3} Tr\left(\frac{\partial \mathcal{U}}{\partial \tilde{\phi}}\right) = \frac{\epsilon^2}{\tilde{g}^2 S^2} A \left[\phi'' + \phi' \left\{\frac{2}{x} + \frac{1}{2}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right\}\right] . \tag{2.13}$$

In the Q-star interior we can neglect the right hand side, obtaining a matrix equation almost identical to Eq. (2.28) of ref. 6. There are two novelties here: we have a *local* squared frequency $\omega^2 B(x)$, and we no longer have $\mathcal{U} = \mathcal{W}$ throughout the interior. To first order in ϵ we replace the latter condition by a first integral of Eq. (2.13) ,

$$(\mathcal{V} + \mathcal{W} - \mathcal{U})\Big|_{x=a}^{x=b} = \int_a^b dx \left(\frac{\mathcal{V}}{A} A' + \frac{\mathcal{W}}{B} B'\right) . \tag{2.14}$$

This shows that *the pressure vanishes within the entire surface*; for $a = 1$ and $(b - a)$ of $\mathcal{O}(\epsilon)$,

$$\tilde{p} \equiv \mathcal{V} + \mathcal{W} - \mathcal{U} = 0 . \tag{2.15}$$

At $x = 1$ we also obtain an implicit boundary condition for Eqs. (2.7),

$$[\mathcal{W} - \mathcal{U}]_{x=1} = 0 , \tag{2.16}$$

which indicates that the inner surface of the Q-star is the Q-ball of ref. 6. Integrating Eqs. (2.7) and derivatives across the surface layer where (2.15) holds shows that up to $\mathcal{O}(\epsilon)$ A, B, and B' are continuous across the surface, while $A'(1) + 2S^2\mathcal{U}(1)$ matches to the Schwarzschild $A'(1^+)$ just outside the surface region where $\mathcal{U}$ vanishes. The Schwarzschild mass measured by distant test particles is obtained by integrating the G_{tt} equation:

$$M \;=\; \tilde{g}\mu\frac{4\pi S^3}{\epsilon^3}\int_0^1 x^2 dx\,(\mathcal{V}+\mathcal{W}+\mathcal{U}) \;\equiv\; \tilde{g}\mu\tilde{M}, \tag{2.17}$$

neglecting the mass in the thin surface. Integrating only to $x=0$ gives the second boundary condition, $A(x=0)=1$.

The equations of motion within the Q-star now take a form independent of $\tilde{g}$,

$$\tilde{\phi}+\frac{\tilde{\phi}^2}{2}+\frac{\tilde{\lambda}}{6}\tilde{\phi}^3-\frac{I}{3}Tr\left(\frac{\tilde{\phi}^2}{2}+\frac{\tilde{\lambda}}{6}\tilde{\phi}^3\right)=2\tilde{\omega}^2B\left(\tilde{\phi}_1-\tilde{\phi}_3\right)\ \mathrm{diag}(1,0,-1)\ . \tag{2.18}$$

Then multiplying (2.13) by $\tilde{\phi}$ and tracing yields $2\mathcal{W}=Tr\ (\tilde{\phi}\cdot\partial\mathcal{U}/\partial\tilde{\phi})$. Applying (2.16) we obtain expressions for the fields at the inner edge of the surface region:

$$\tilde{\phi}_2(1) \;=\; \frac{12}{\tilde{\lambda}}\left.\frac{y^2-1}{(y^2+3)^2}\right|_{x=1} \tag{2.19}$$

$$\tilde{\omega}^2B(1) \;=\; \left[\frac{1}{4}\left(1+\frac{3}{y^2}\right)-\frac{3}{4\tilde{\lambda}}\frac{1}{y^2}\frac{(y^2-1)^2}{(y^2+3)^2}\right]_{x=1}, \tag{2.20}$$

correcting Eq. (2.37) of ref. 6. Taking the $(2,2)$ component of (2.18) yields

$$y^2(x) \;=\; \frac{3}{\tilde{\phi}_2}\frac{\tilde{\lambda}\tilde{\phi}_2^2+2\tilde{\phi}_2+8}{2-\tilde{\lambda}\tilde{\phi}_2}\ , \tag{2.21}$$

which combined with the above trace yields a quadratic equation for $\tilde{\phi}_2$. Taking the root that matches the surface Q-ball, we obtain

$$\tilde{\phi}_2(x) \;=\; \frac{1-4\tilde{\lambda}\tilde{\omega}^2B(x)}{2\tilde{\lambda}}+\frac{1}{2\tilde{\lambda}}\sqrt{\left\{1-4\tilde{\lambda}\tilde{\omega}^2B(x)\right\}^2\;+\;8\tilde{\lambda}\left\{4\tilde{\omega}^2B(x)-1\right\}}\ . \tag{2.22}$$

We may now substitute for $\tilde{\phi}_2(x)$ and $y(x)$ in Eqs. (2.7) and solve numerically for $A(x)$ and $\tilde{\omega}^2B(x)$. We require only an explicit form of the boundary condition on $\tilde{\omega}^2 B(x)$. Eq. (2.19) substituted into (2.21) at $x = 1$ yields

$$\left[(\tilde{\lambda} - 1)(y^2 + 3)^3 \; + \; 16(y^2 + 3)^2 \; - \; 72(y^2 + 3) \; + \; 96\right]_{x=1} = 0 \; , \tag{2.23}$$

and a real solution requires $\tilde{\lambda} < 1$.[6] In terms of $\alpha \equiv 16/(1 - \tilde{\lambda}) \geq 18$, the real solution is

$$\begin{aligned} y^2(1) = \; -3 \; + \; \frac{\alpha}{3} + &\left[\alpha\left(\frac{\alpha^2}{27} - \frac{3}{4}\alpha + 3\right) + 3\alpha\sqrt{\frac{5\alpha^2}{(6)^4} - \frac{\alpha}{8} + 1}\,\right]^{1/3} \\ + &\left[\alpha\left(\frac{\alpha^2}{27} - \frac{3}{4}\alpha + 3\right) - 3\alpha\sqrt{\frac{5\alpha^2}{(6)^4} - \frac{\alpha}{8} + 1}\,\right]^{1/3} \; . \end{aligned} \tag{2.24}$$

With $\tilde{\omega}^2 B(1)$ now known explicitly from (2.20), we have the required boundary condition. The external Schwarzschild form $B(x)^{-1} = (1 - \tilde{M}/4\pi S x)$ in addition leads to

$$\tilde{\omega}^2 = \left[\tilde{\omega}^2 B(1)\right]\left(1 - \frac{\tilde{M}}{4\pi S}\right) \; , \tag{2.25}$$

which completes our solution, given $\tilde{\lambda}$ and S.

3. Discussion and Results

Our solution in rescaled units depends *only* on the scaled coupling $\tilde{\lambda}$ and the size S. Numerical solutions of the Einstein equations were obtained using the program COLSYS.[7] For small size ($S \ll 1$, the 'Q-ball regime'), gravity is negligible and we simply recover the Q-ball, exemplified in Figs. 1 and 2 where the fields are almost constant for $S \lesssim 0.05$. As S increases, all the fields start to vary and we enter the Q-star regime. The surface layer must however always be a Q-ball, requiring $\frac{1}{9} \leq \tilde{\lambda} < 1$.[6] The naive expectation that gravity could help stabilize a Q-star with $\tilde{\lambda} \geq 1$ against such self-repulsion is unjustified. Accordingly, in the Q-star interior as well as in the surface Q-ball, the boson field and charge density vanish when $\tilde{\lambda} \to 1^-$, while $\tilde{\omega}$ approaches the free-particle value of 1/2. Similarly, the entire Q-star smoothly approaches the new vacuum when $\tilde{\lambda} \to \frac{1}{9}^+$, breaking the $SO(3)$ symmetry. For $\tilde{\lambda} < \frac{1}{9}$ the new vacuum of course has a negative energy density, which produces the usual embarassment when coupled to gravity.

The existence of the Q-ball solution and thus the Q-star imposes a more general restriction on the form of $U(\phi)$;[5] $\mathcal{W} - \mathcal{U}$ must be an effective "one-particle potential" with x playing the role of time in a mechanical equation. The Q-ball is a solution which rolls in the effective potential, coming to rest at the vacuum value of ϕ at large radius.[3] A renormalizable potential compatible with this behavior must therefore include a cubic term, and indeed as $\tilde{g} \to 0$ the Q-star contracts to vanishing size, mass and charge. (The respective densities diverge as $\tilde{g}^{-2}$, but the volume vanishes as $\tilde{g}^3$.) From Eq. (2.17) , we also see that Q-stars obey a generalized Chandrasekhar relation, $M \sim \tilde{g}(m_{pl}^3/\mu^2)$.

Turning now to more detailed examination of the solutions, figs. 3 and 4 show the fields inside the star ($0 \leq x \leq 1$) for two values of the size S, with a typical coupling $\tilde{\lambda} = 0.6$. Fig. 1 confirms that at $x = 1$ all the solutions become the same Q-ball (with $\tilde{\lambda} = 0.6$); they also satisfy very accurately the Schwarzschild condition $A(1) = B^{-1}(1) = 1 - \tilde{M}/4\pi S$. The mass here increases faster with S than the volume (S^3); Fig. 2 shows that in the Q-ball regime, $S \lesssim 0.05$, the average density is constant, but as S increases the density does too and we lose the homogeneous 'Q-matter' limit. Gravity causes the fields to "sag" towards the center (compare Figs. 4 and 3) and increases the central density above the Q-ball value when S rises beyond the Q-ball range. In addition to this change of field *intensity*, the relative sizes of the local *eigenvalues* of ϕ also change inside, because y changes. Of course this is *not* equivalent to local $SO(3)$ rotations; gravity does not couple to global symmetries. For large S, the star becomes highly relativistic and approaches a black hole, $\tilde{M} = 4\pi S$. Eq. (2.25) shows that $\tilde{\omega}$ redshifts to zero in this limit, since $\tilde{\omega}^2 B(1)$ is fixed by $\tilde{\lambda}$.

The stability analysis of Q-stars is a straightforward extension of the Q-ball analysis.[6][8] Classically Q-stars are stable, supporting a spectrum of surface and acoustic waves similar to the Q-ball case.[9] Semi-classically they are stable against dispersal into free particles because $\omega < \mu/2$, the free-particle frequency; Fig. 5 shows the binding energy. Quantum mechanically with ϕ decaying only to fermions, gravity makes no essential difference to the surface evaporation scenario of ref. 8; the Q-star lifetime is in that case of order $10^{-7}(250\ GeV/\mu)^2$ seconds.[1]

In conclusion, we have seen that a global non-Abelian symmetry leads to no essential difficulties in constructing Q-stars.

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FIGURE CAPTIONS

1) $\tilde{\phi}_2$ and $y-3$ as functions of x, for $S=0.01$, 0.15, 0.25, 0.29, and 0.305.

2) Rescaled mass density, $\tilde{M}\epsilon^3/4\pi S^3$ as a function of S.

3) A, B^{-1}, $\tilde{\phi}_2$ and y as functions of x; $S=0.2$, $\tilde{\lambda}=0.6$.
Note that y is divided by 10 here and in the next figure.

4) A, B^{-1}, $\tilde{\phi}_2$ and y as functions of x; $S=0.3$, $\tilde{\lambda}=0.6$.

5) Solid line: rescaled mass, $\tilde{M}\epsilon^3$ as a function of rescaled charge $\tilde{q}\epsilon^3$.
Dashed line: $\tilde{M}=\tilde{q}$, to show the binding energy.

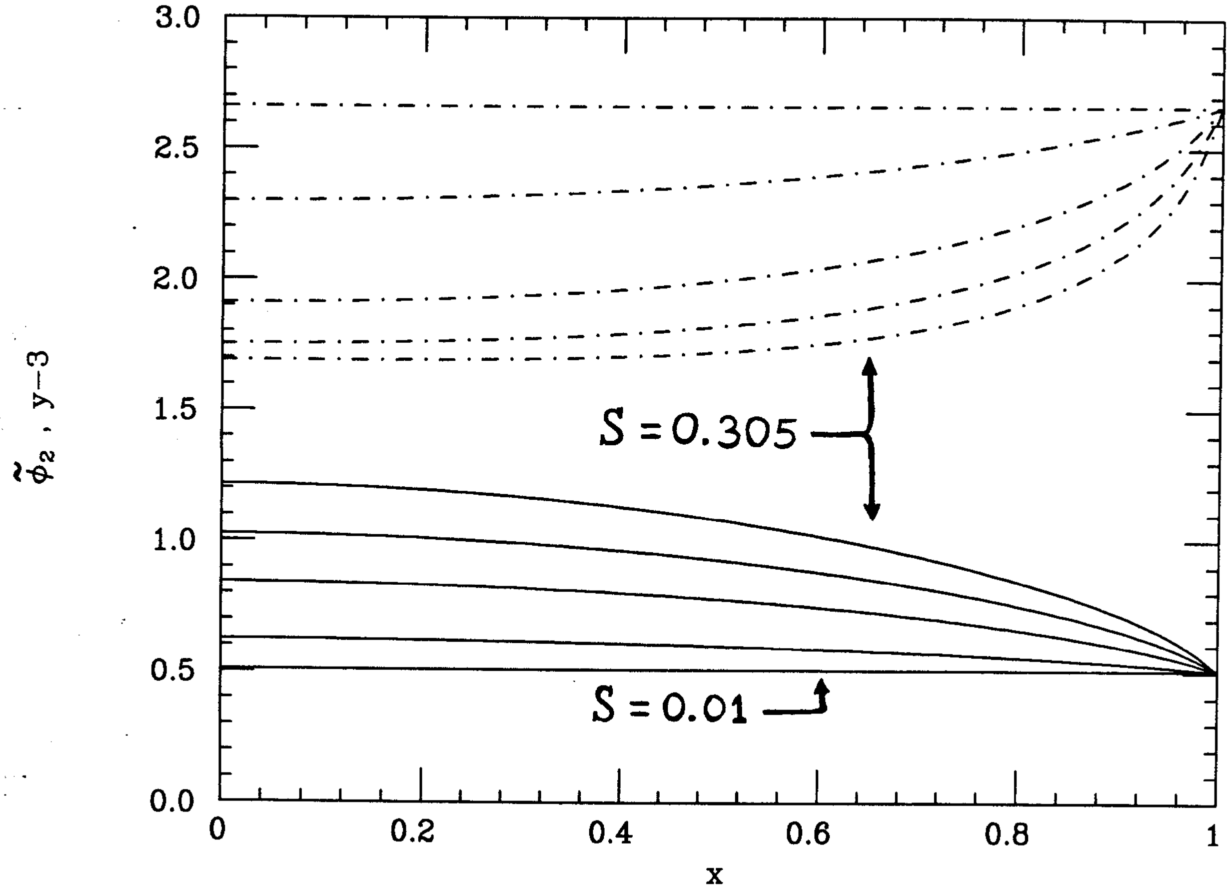


Fig. 1

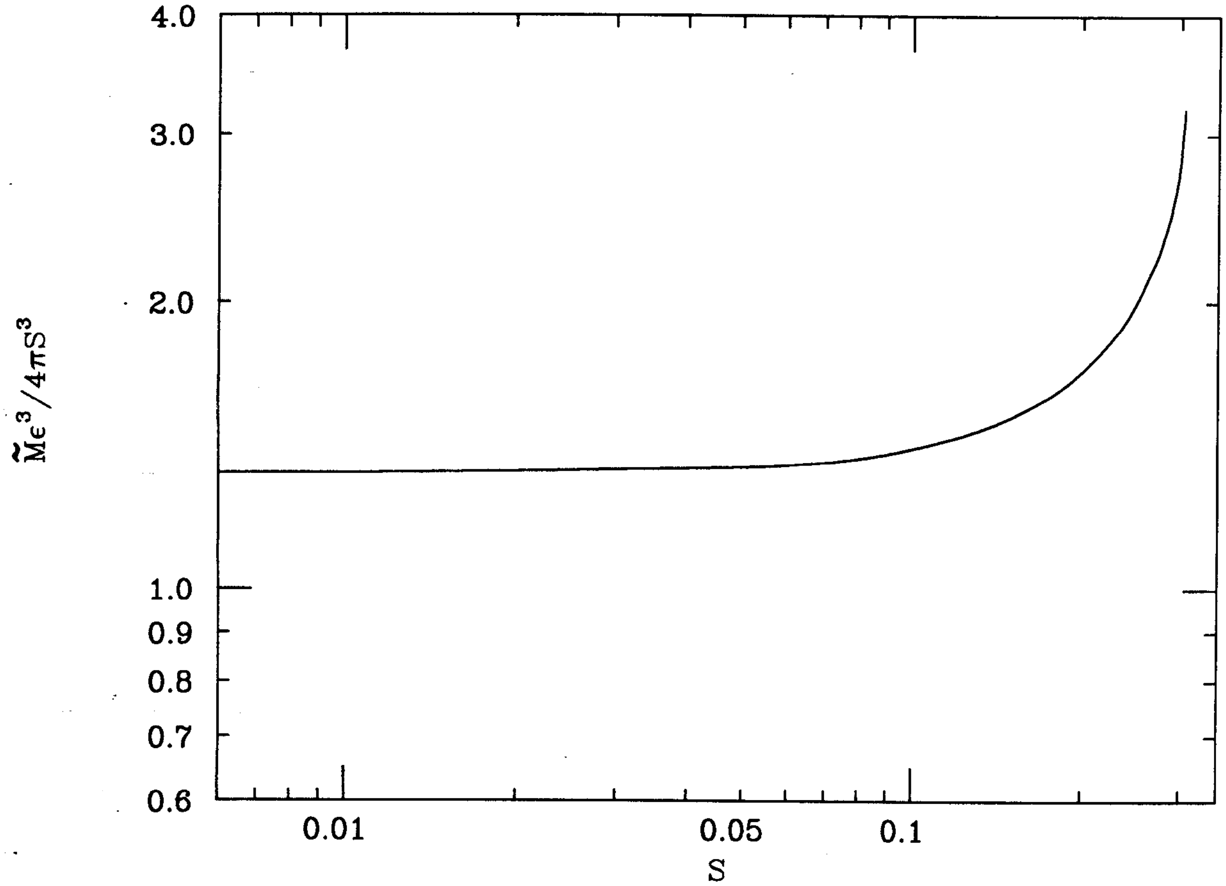
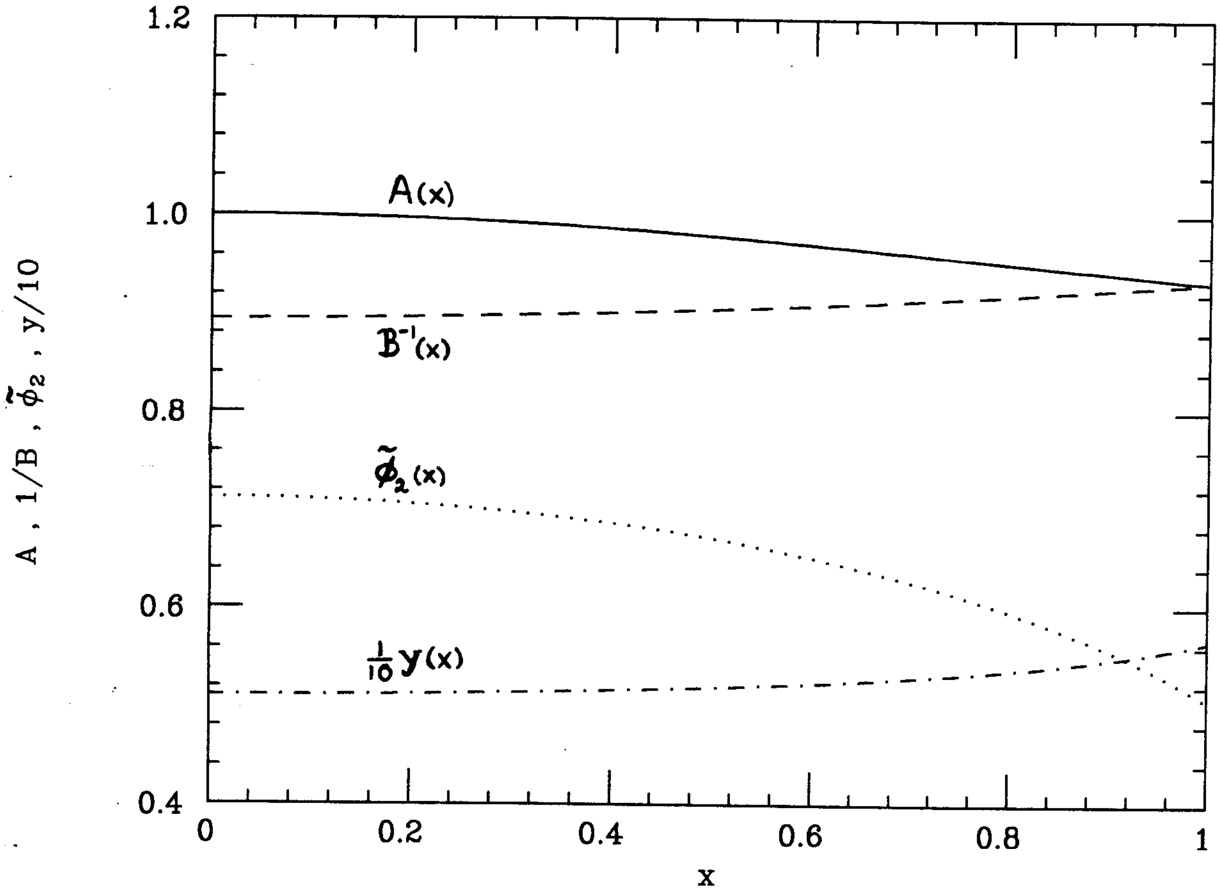


Fig. 2



1.2
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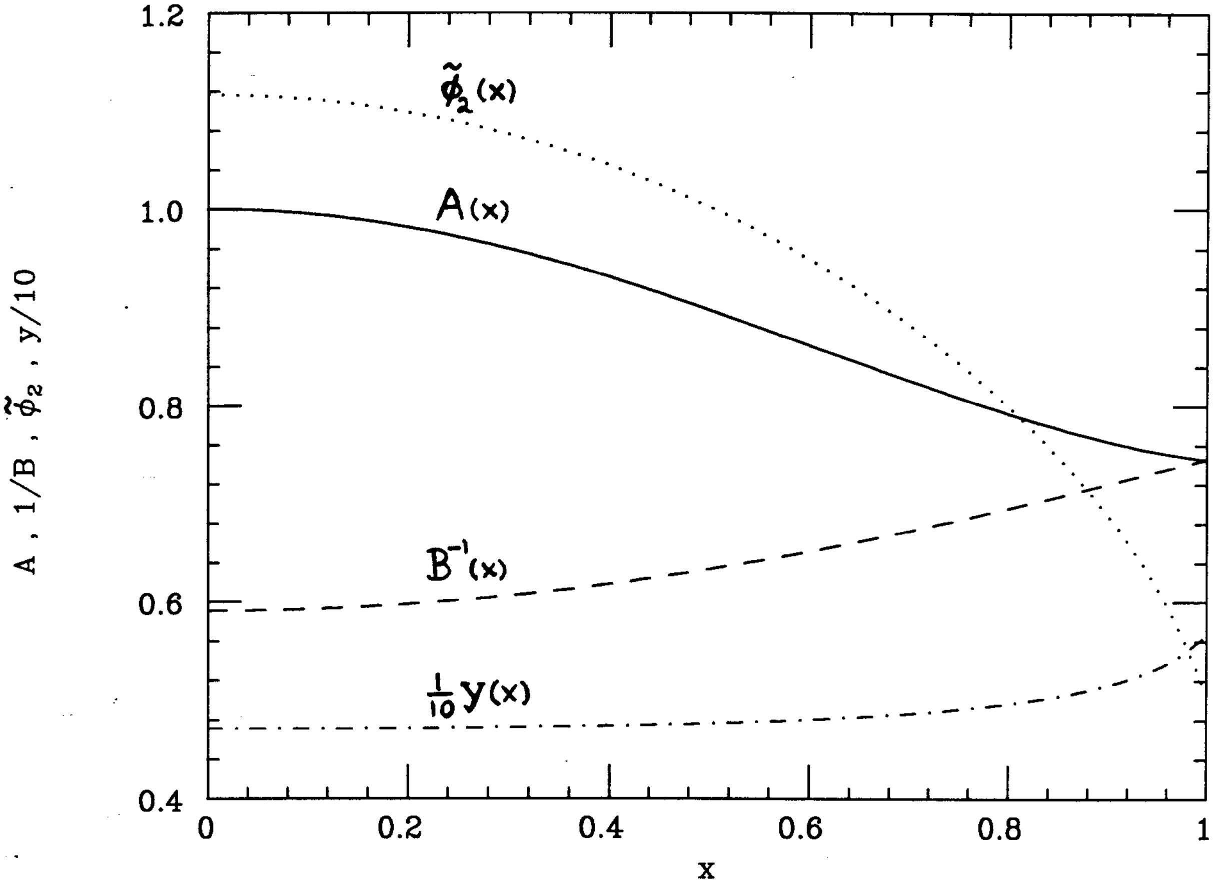
0.2
0.4
0.6
0.8
1
x
A , 1/B , $\tilde{\phi}_2$, y/10
A(x)
$B^{-1}(x)$
$\tilde{\phi}_2(x)$
$\frac{1}{10}y(x)$

Fig. 3



1.2
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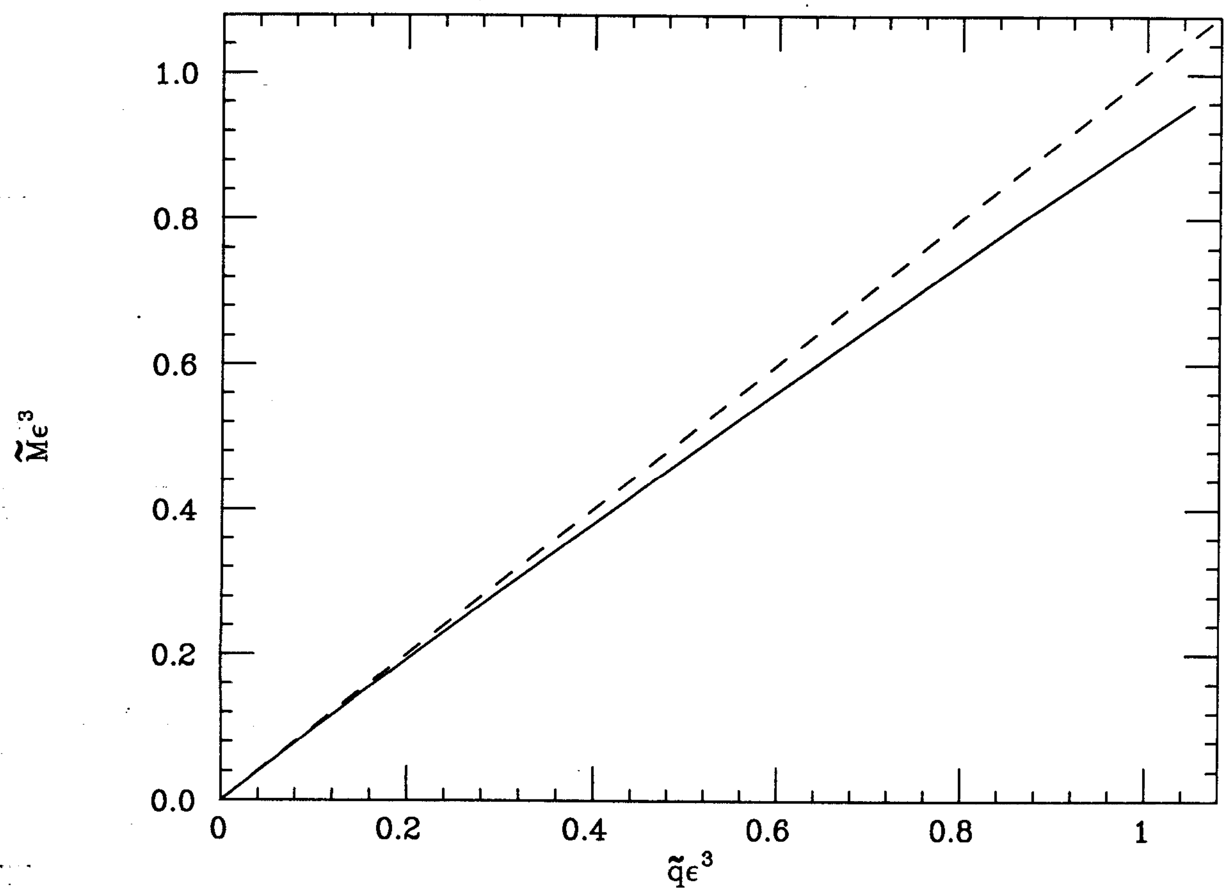
A , 1/B , $\tilde{\phi}_2$, y/10
$\tilde{\phi}_2(x)$
A(x)
$B^{-1}(x)$
$\frac{1}{10}y(x)$

Fig. 4


1.0
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$\tilde{M}\epsilon^3$
0
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0.4
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0.8
1
$\tilde{q}\epsilon^3$

Fig. 5