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STRING CALCULATION OF FAYET-ILIOPOULOS D-TERMS IN ARBITRARY SUPERSYMMETRIC COMPACTIFICATIONS*

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ABSTRACT

We calculate the Fayet-Iliopoulos D-terms generated at one loop for the heterotic string in any background which preserves four-dimensional space-time supersymmetry and maintains (2,0) world-sheet superconformal invariance. Although our calculation is performed in the full string theory, the result can be evaluated entirely in terms of properties of the massless spectrum. Furthermore it agrees with the result of a computation in the four-dimensional effective field theory using a stringy regularization. We also check our general result through a more explicit calculation in an orbifold background.

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1. INTRODUCTION

The recent surge of interest in string theories has its origin in the hope that these theories might provide a realistic, consistent and fundamental theory of all interactions [1]. For any string theory to do so, however, it must among other things explain the vanishing of the cosmological constant, as well as the smallness of the weak interaction scale compared to the Planck scale. Within our current understanding of strings, space-time supersymmetry seems crucial to the ultimate resolution of these problems. It has been known for some time now that in a four-dimensional supersymmetric field theory [2] loop corrections cannot renormalize the superpotential [3]. In many cases this protects scalar masses from becoming large if they are small at the tree level. For superstring theories in space-time supersymmetric vacua it has been argued on general grounds [4, 5] that the superpotential for the massless fields is likewise not renormalized. Such a renormalization could generate in particular a dilaton tadpole, *i.e.* destablize the vacuum. In supersymmetric theories, however, there may be another type of loop effect which destabilizes the vacuum and generates a large cosmological constant and scalar masses. This is the Fayet-Iliopoulos D-term [6]. If the unbroken gauge group of the theory contains one or more $U(1)$ factors, then radiative corrections in the theory may generate a term of the form $\sum_a c^{(a)} D^{(a)}$ in the effective action [7], where $D^{(a)}$ is the auxiliary field associated with the a'th abelian factor $U^{(a)}(1)$ and $c^{(a)}$ are some arbitrary coefficients.

Using symmetry and effective lagrangian considerations it was argued in Ref. [8] that Fayet-Iliopoulos D-terms can indeed be generated in string loop perturbation even though they are absent at tree level. These terms arise if the trace of the $U^{(a)}(1)$ generator over the tree level massless chiral fermions is nonzero. We shall refer to such a $U(1)$ as 'anomalous', although in fact these apparent anomalies are cancelled by the presence of a Wess-Zumino term, precisely as in the Green-Schwarz anomaly cancellation mechanism [9]. The presence of these D-terms gives masses at one loop to scalars charged under this anomalous $U(1)$. Furthermore they will destabilize the vacuum by generating a dilaton tadpole at the two loop level [8].

The purpose of this paper is to calculate the D-terms in the full string theory via the scalar masses they generate. As we shall show these terms are exactly calculable in any background preserving supersymmetry. We start our analysis from the two point function on the torus for a $U(1)$-charged scalar in an arbitrary supersymmetric background. We show that in the limit $k^2 \to 0$, the calculation reduces to evaluating the expectation value of a single local operator of conformal dimension $(1,1)$. This operator can be interpreted as the vertex operator for an auxiliary D-field. (Since this operator is not the highest component of a $2d$ superfield, it cannot be interpreted as the vertex operator of a physical field [10].) More evidence for this interpretation follows from the space-time supersymmetry transformation properties of this operator: It transforms, as it should, into the gaugino vertex operator.

In calculating the expectation value of the D-term on the torus (which apriori receives contributions from all spin structures) we show that it is related to the expectation value of a dimension $(0,1)$ operator in the periodic-periodic sector for the right-handed fermions. Demanding that the final answer be modular invariant, we evaluate this term entirely in terms of properties of the massless spectrum of the theory. Thus the dependence of the D-terms on the background configuration enters entirely through the spectrum of massless states. This is as expected from the low energy effective field theory, since the anomaly in the corresponding $U(1)$ current is generated by the massless states of the theory.

An important question at this stage is whether a string vacuum state which is destabilized by D-terms can be shifted to a nearby (for small string coupling constant) stable vacuum by giving expectation values to various massless scalars. The answer depends on the sign of the D-terms generated as well as details of the massless spectrum and tree-level couplings [11]. The sign we find here indicates that the standard Calabi-Yau compactifications of the $Spin(32)/Z_2$ heterotic string—as well as various $(2,0)$ orbifolds—can always be stabilized.

The organization of this paper is as follows: In sec. 2 we give a brief review of space-time supersymmetry first in a flat background and then in an arbitrary background preserving $(2,0)$ world-sheet supersymmetry. In sec. 3 we calculate the D-terms for $Spin(32)/Z_2$ or $E_8 \times E_8$ string theories compactified on arbitrary backgrounds using the approach outlined above. Sec. 4 contains an explicit calculation of the masses of the $U(1)$ charged scalars for the $Spin(32)/Z_2$ heterotic string theory compactified on a Z_3 orbifold. More specifically, there we calculate these masses by looking at the two point function of the charged scalars and also at the four point function of an appropriate set of fermions that factor on these charged scalars in the s-channel. Even though our techniques developed in sec. 3 are general and explicit enough to encompass all backgrounds including orbifolds, the calculations presented in sec. 4 provide an independent check on our earlier results. Sec. 5 contains our conclusions including a brief discussion of the consequences of our results for "re-stabilization" of vacua which have been destablized by D-terms. Finally in appendix A we discuss the construction of vertex operators for the auxiliary F and D fields.

2. SPACE-TIME SUPERSYMMETRY IN ARBITRARY BACKGROUNDS

In this section we shall discuss space-time supersymmetry in string theories compactified on arbitrary backgrounds which preserve (2,0) world-sheet superconformal invariance [10, 12 − 14].* These include Calabi-Yau manifolds [15, 16], various (2,0) orbifolds [17, 14] and asymmetric orbifolds [18], as well as other schemes of compactification [19, 20]. We start by briefly reviewing space-time supersymmetry in string theories in a flat background [10]. In this case the ten right-handed Majorana-Weyl fermions ψ^{μ} are grouped into five complex fermions and bosonized as

$$\begin{aligned} \psi^{\mu} &\sim e^{i\phi_{\mu}}, \\ \psi^{\bar{\mu}} &\sim e^{-i\phi_{\mu}}. \end{aligned} \tag{2.1}$$

Similarly the superconformal ghosts β, γ are bosonized as

$$\gamma \sim e^{\phi}\eta, \quad \beta \sim e^{-\phi}\partial\xi, \tag{2.2}$$

where ϕ is a bosonic field and η, ξ are fermionic fields. In terms of the bosonized fields we may construct the $SO(10)$ spin operators,

$$S_{\alpha} \sim e^{\frac{i}{2}(\pm\phi_1\pm\phi_2\pm\phi_3\pm\phi_4\pm\phi_5)}, \tag{2.3}$$

and the ghost spin operators,

$$S_g^{\pm} \sim e^{\pm\frac{\phi}{2}}. \tag{2.4}$$

In (2.3), if we restrict the total number of − signs to be even, we get a positive chirality spinor, whereas if we restrict it to be odd, we get a negative chirality spinor.

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* We use a convention where (m, n) supersymmetry denotes m right-handed and n left-handed supersymmetries. Also an operator of conformal dimension (h_R, h_L) has conformal dimension $h_R(h_L)$ with respect to the right (left)-handed component of the stress tensor $T(z)$ $(\bar{T}(\bar{z}))$.

Out of these spin fields one can construct local operators on the world-sheet. For example, the space-time supersymmetry charge is constructed by combining one of the positive chirality spinor fields in (2.3) with the ghost spin field S_g^-:

$$Q_\alpha = \oint S_g^-(z) S_\alpha(z) dz \equiv \oint J_\alpha(z) dz, \qquad (2.5)$$

while the right-handed part of a fermion emission vertex operator is given by [10, 21],

$$V_{-\frac{1}{2}}(z, u, k) = S_g^-(z) u^\alpha(k) S_\alpha(z) e^{ik\cdot X(z)}. \qquad (2.6)$$

The vertex $V_{-\frac{1}{2}}$ defined above carries a ghost charge $-\frac{1}{2}$. In practice, one may work with different equivalent vertex operators carrying different ghost charges. For example, an equivalent fermion emission vertex carrying ghost charge $\frac{1}{2}$ is given by

$$V_{\frac{1}{2}}(z, u, k) = S_g^+(z) \lim_{w\to z} \{(w - z)^{\frac{1}{2}} 2T_F(w) u^\alpha(k) S_\alpha(z) e^{ik\cdot X(z)}\}, \qquad (2.7)$$

where $T_F = \frac{1}{2}\psi_\mu \partial X^\mu$ is the fermionic component of the stress tensor for the matter fields.

A similar picture-changing operation may be carried out as well for the vertex operator for the emission of a space-time boson. The canonical bosonic vertex operator carries ghost charge 0 and is given by

$$V_0(z, \xi, k) = \xi_\mu [\partial X^\mu + ik_\nu \psi^\mu(z) \psi^\nu(z)] e^{ik\cdot X(z)}. \qquad (2.8)$$

However, an equivalent bosonic vertex operator carrying ghost charge -1 is given by

$$V_{-1}(z, \xi, k) = \xi_\mu e^{-\phi(z)} \psi^\mu(z) e^{ik\cdot X(z)}. \qquad (2.9)$$

Note that

$$e^{-\phi(z)} V_0(z) = \lim_{w\to z} \{(w - z)(T_F(w) V_{-1}(z))\}. \qquad (2.10)$$

Thus if we ignore the $e^{-\phi}$ factor in V_{-1}, the operators V_{-1} and V_0 may be considered as the lower and higher components of the same two dimensional superfield.

Let us now discuss the fate of the supersymmetry charges Q_α under compactification [10, 12 − 14]. Suppose we compactify the first six directions and denote these directions by $i, \bar{i}$ $(1 \leq i \leq 3)$, leaving the index μ for four dimensional Minkowski space-time. In this case ϕ_μ $(\mu = 4, 5)$ remain free fields, so one can still construct the $SO(4)$ spin fields $e^{\frac{i}{2}(\pm\phi_4 \pm \phi_5)}$ from them. On the other hand ϕ_i $(1 \leq i \leq 3)$ become interacting, and hence the supercharges Q_α are no longer conserved $(\partial_{\bar{z}} J_\alpha \neq 0)$. However if the σ-model describing the compactified theory has (2,0) world-sheet supersymmetry, it is possible to construct two conserved fields $\hat{S}^\pm$ which correspond to the fields $e^{\pm\frac{i}{2}(\phi_1+\phi_2+\phi_3)}$ in the free field case. As a result we may construct the conserved supercharges,

$$
\begin{aligned}
Q_\alpha &= \oint J_\alpha(z) dz = \oint S_g^-(z) \hat{S}^+(z) e^{\frac{i}{2}(\pm\phi_4 \pm \phi_5)} dz, \\
Q_{\dot{\alpha}} &= \oint J_{\dot{\alpha}}(z) dz = \oint S_g^-(z) \hat{S}^-(z) e^{\frac{i}{2}(\pm\phi_4 \mp \phi_5)} dz.
\end{aligned}
\tag{2.11}
$$

In J_α, ϕ_4 and ϕ_5 must have the same sign in the exponent, whereas in $J_{\dot{\alpha}}$ they must have opposite signs, in order to have the correct ten dimensional chirality. Thus the four dimensional spinors Q_α and $Q_{\dot{\alpha}}$ have positive and negative chirality, denoted by undotted and dotted indices respectively.

We now describe the construction of the fields $\hat{S}^\pm$. If the non-linear σ-model describing the dynamics of the internal (compactified) coordinates has a (2,0) supersymmetry, the right-moving $N = 2$ superconformal algebra is generated by the stress tensor $T(z)$, the two supercurrents $G^\pm(z)^\star$ and a $U(1)$ current $J(z)$.

⋆ The world-sheet supersymmetry current $T_F(z)$ which couples to the gravitino is just $G^+(z) + G^-(z)$.

The operator products of the various currents are given by,

$$
\begin{aligned}
T(z)T(w) \sim& \frac{3}{4}\frac{\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots, \\
T(z)G^{\pm}(w) \sim& \frac{3}{2}\frac{G^{\pm}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm}(w)}{z-w} + \cdots, \\
T(z)J(w) \sim& \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \cdots, \\
G^+(z)G^+(w) \sim& G^-(z)G^-(w) \sim O(1), \\
G^+(z)G^-(w) \sim& \frac{1}{8}\frac{\hat{c}}{(z-w)^3} + \frac{1}{4}\frac{J(w)}{(z-w)^2} + \frac{1}{4}\frac{T(z)}{z-w} + \cdots, \\
J(z)G^{\pm}(w) \sim& \pm\frac{G^{\pm}(z)}{z-w} + \cdots, \\
J(z)J(w) \sim& \frac{1}{2}\frac{\hat{c}}{(z-w)^2} + \cdots.
\end{aligned}
\tag{2.12}
$$

Here $\hat{c}$ is the central charge of the $N = 1$ super-Virasoro algebra, normalized so that it is 6 for a 6 (real) dimensional internal space. The operator products in (2.12) which involve the $U(1)$ current $J(z)$ are consistent with representing $J(z)$ as the derivative of a free scalar field $H(z)$. That is, we write

$$
J(z) = +i\sqrt{3}\partial_z H(z), \tag{2.13}
$$

(for $\hat{c} = 6$) where

$$
\begin{aligned}
T(z)H(w) &\sim \frac{1}{z-w}\partial_w H(w), \\
H(z)H(w) &\sim -\ln(z-w),
\end{aligned}
\tag{2.14}
$$

so that the $H(z)$ propagator is conventionally normalized. Although H itself is not a conformal field, we may construct conformal fields from the derivatives and exponentials of H. Examples of such conformal fields are

$$\hat{S}^{\pm}(z) \sim e^{\pm i\frac{\sqrt{3}}{2}H(z)},$$
$$\epsilon^{\pm}(z) \sim e^{\pm i\sqrt{3}H(z)}. \tag{2.15}$$

The $\hat{S}^{\pm}(z)$ are precisely the operators needed to construct the space-time supercurrent in eq.(2.11). They also occur in the vertex operators for the gravitino and the gauginos in the four dimensional massless spectrum. The tensors $\epsilon^{\pm}(z)$ are known as holomorphic and anti-holomorphic three form fields [15, 12]. The analogs of these fields are required to construct the Lorentz generators in the Green-Schwarz [22] formulation of the theory [13]. As we shall see in appendix A, they also play a crucial role in the construction of vertex operators for the auxiliary F fields of a chiral supermultiplet.

The two dimensional theory describing the $E_8 \times E_8$ or the $Spin(32)/Z_2$ heterotic string theory [23] also contains thirty-two left-handed fermions which we shall denote by λ^M. In the standard compactification of these theories on a Calabi-Yau manifold with the spin connection identified with the gauge connection [15], twenty-six of these thirty-two fermions remain free, whereas the other six fermions naturally combine into three interacting complex fermions λ^i and their complex conjugates $\lambda^{\bar{i}}$. The σ-model describing the two dimensional theory involving the fields X^i, ψ^i, λ^i and their complex conjugates turns out to be left-right symmetric. As a result there must be a conserved $U(1)$ current $U(\bar{z})$ in the left-handed sector which is the analog of the current $J(z)$ in the right-handed sector. For the $E_8 \times E_8$ heterotic string theory compactified on Calabi-Yau manifolds $U(\bar{z})$ is one of the gauge currents for the unbroken E_6 gauge group, whereas for $Spin(32)/Z_2$ string theory $U(\bar{z})$ generates the $U(1)$ factor of the unbroken $SO(26) \times U(1)$ gauge group. In the latter theory the associated auxiliary D-field may acquire a vacuum expectation value, as we shall see in sec. 3. For more general models preserving (2,0) world-sheet supersymmetry, there may be, in general, several $U(1)$ currents in the left-handed sector, and the auxiliary field associated with some linear combination of them may develop a vacuum expectation value.

3. CALCULATION OF FAYET ILIOPOULOS D-TERMS IN ARBITRARY SUPERSYMMETRIC BACKGROUNDS

In this section we shall calculate the Fayet-Iliopoulos D-terms generated at the one loop order for the $Spin(32)/Z_2$ or $E_8 \times E_8$ heterotic string theory compactified on any six dimensional background which preserves $(2,0)$ world-sheet supersymmetry and space-time supersymmetry at tree level in the string coupling constant, and which has one or more $U(1)$ factors in the four dimensional gauge group at tree level. In fact no geometrical interpretation of the internal $(2,0)$ superconformal theory is necessary for our techniques to be valid.

In the presence of a linear term of the form $c^{(a)}D^{(a)}$ in the four dimensional effective action ($D^{(a)}$ denotes the auxiliary field associated with the a'th $U(1)$ factor $U^{(a)}(1)$), any scalar field of chirality h carrying charge $q^{(a)}$ under the group $U^{(a)}(1)$ gets a mass $\hat{g}h\sum_a q^{(a)}c^{(a)}$ [6,2]. Hence the coefficients $c^{(a)}$ may be obtained by calculating the mass of any scalar field that carries these $U(1)$ charges.*

Here $\hat{g}$ is the gauge coupling constant, and h is the chirality of the scalar field defined as follows: A scalar field belonging to a chiral supermultiplet commutes with one of the supersymmetry charges Q_α or $Q_{\dot{\alpha}}$, depending on the chirality of the fermion field in the multiplet. If it commutes with $Q_{\dot{\alpha}}(Q_\alpha)$ we shall define the chirality of the scalar field to be $-1(+1)$. Thus a negative (positive) chirality scalar transforms to an undotted (dotted) spinor field, under supersymmetry transformation by Q_α ($Q_{\dot{\alpha}}$), and belongs to the same supermultiplet as this field. Consequently we shall define the chirality of an undotted (dotted) spinor field to

* Note that the coefficients $c^{(a)}$ will depend on how we normalize the D fields. However, the scalar mass is unambiguous. We are using the normalization convention of Wess and Bagger [2] except that our gauge coupling constant ($\hat{g}$) is related to their (e) by $\hat{g} = e/2$.

be $-1(+1)$. As discussed in the appendix, the spinors appearing in the vertex operator of a state are related to the field describing the state through multiplication by $\not{k}$, consequently, a dotted (undotted) fermionic vertex operator will have chirality $-1(+1)$.

We start with the vertex operator $\frac{g}{2\pi}\int d^2 z V_0(z,k)$ for a general massless scalar field carrying $U^{(a)}(1)$ charge $q^{(a)}$ and the vertex operator $\frac{g}{2\pi}\int d^2 z\tilde{V}_0(z,k)$ for its CPT conjugate state.† The subscript 0 denotes that the vertex operators carry no net superconformal ghost charge; equivalently, they are the highest components of some superfields. Note that we have introduced an explicit factor of $\frac{g}{2\pi}$ multiplying the vertex operator where g is the coupling constant of the theory, whose relation with the four dimensional gauge coupling constant will be determined shortly. Let $W(z,k)$ and $\tilde{W}(z,k)$ be the lower components of the superfields of which V, $\tilde{V}$ are the highest components. Then W, $\tilde{W}$ are conformal fields of dimension $(\frac{1}{2},1)$ and are space-time scalars. These fields are of two different types. One type combines ψ^μ from the right-handed sector with $\bar{\partial}X^\nu$ from the left-handed sector in order to form a Lorentz invariant object. This gives the vertex operators for the dilaton and its associated axion, which carry no internal $U^{(a)}(1)$ charge and hence are not of immediate interest to us. The other type of $W,\tilde{W}$ field is completely constructed out of internal fields carrying no Lorentz index. These take the general form

$$
\begin{aligned}
W(z,k) &= f(\varphi^j)e^{ik_\mu X^\mu}, \\
\tilde{W}(z,k) &= \tilde{f}(\varphi^j)e^{ik_\mu X^\mu},
\end{aligned}
\tag{3.1}
$$

where f and $\tilde{f}$ are some functions of the interacting fields φ^j. For σ-model compactifications φ^j consist of X^i, ψ^i, their complex conjugates, and λ^M. By a picture-changing operation we may convert the vertex operator $V_0(\tilde{V}_0)$ to a

† In our convention $d^2z = 2idxdy$ if $z = x + iy$.

vertex operator $V_{-1}(\tilde{V}_{-1})$ carrying net ghost charge -1,

$$\begin{aligned} V_{-1}(z,\bar{z},k) &= e^{-\phi}W(z,\bar{z},k), \\ \tilde{V}_{-1}(z,\bar{z},k) &= e^{-\phi}\tilde{W}(z,\bar{z},k), \end{aligned} \tag{3.2}$$

where ϕ is a bosonized ghost field.

$V_0(\tilde{V}_0)$ may be determined in terms of $W(\tilde{W})$ by applying the world-sheet supersymmetry generator:

$$\begin{aligned} V_0 &= \lim_{w\to z}(w-z)\{2T_F^{\text{int}}(w) + \psi_\mu(w)\partial X^\mu(w)\}W(z,\bar{z}). \\ &= \{g(z,\bar{z}) - ik_\mu\psi^\mu(z)f(z,\bar{z})\}e^{ik_\mu X^\mu}. \end{aligned} \tag{3.3}$$

Here T_F^{int} denotes the fermionic component of the super stress tensor involving the internal fields, and

$$g(z,\bar{z}) = \lim_{w\to z}\{(w-z)(2T_F^{\text{int}}(w)f(z,\bar{z}))\}. \tag{3.4}$$

Similarly we may express $\tilde{V}_0$ as,

$$\tilde{V}_0(z,\bar{z},k) = \{\tilde{g}(z,\bar{z}) - ik_\mu\psi^\mu(z)\tilde{f}(z,\bar{z})\}e^{ik\cdot X}. \tag{3.5}$$

The vertex operators are normalized so that acting on the $SL(2,C)$ invariant ground state $f(z,\bar{z})$ creates a normalized state, *i.e.* the leading term in the product $f(z,\bar{z})\tilde{f}(w,\bar{w})$ is normalized to be $(z-w)^{-1}(\bar{z}-\bar{w})^{-2}$. The vertex operators for the gauge bosons in the -1 picture are normalized in the same way, after taking out the $\frac{g}{2\pi}$ factor. With this normalization, if three vertex operators

$\frac{g}{2\pi}V^{(1)}(z,k_1)$, $\frac{g}{2\pi}V^{(2)}(z,k_2)$ and $\frac{g}{2\pi}V^{(3)}(z,k_3)$ satisfy the operator product,*

$$V^{(1)}(z,k_1)V^{(2)}(w,k_2) \sim \varepsilon\frac{1}{|z-w|^{2-2k_1\cdot k_2}}V^{(3)}(w,k_3), \tag{3.6}$$

then the tree level three point coupling between these states is given by $(-2i\varepsilon g)$. Using the vertex operator for the gauge field $(e^{-\phi}\psi^{\mu}(z)U^{(a)}(\bar{z})e^{ik\cdot X})$ in the -1 picture, it is not difficult to see that in this scheme the coupling of a $U^{(a)}(1)$ gauge boson to a scalar field of momentum k is given by $2gq^{(a)}k_{\mu}$. Thus g may be identified with the four dimensional gauge coupling constant $\hat{g}$.

At this point we shall give a definition of the chirality of a scalar field in terms of its operator product expansion with the current $J(z)$. In order to survive the GSO projection, the operators $W(z,\bar{z},k)$, $\tilde{W}(z,\bar{z},k)$ must carry odd two dimensional fermion number. Since the charge J_0 associated with the current $J(z)$ measures the fermion number in the internal dimension (up to a sign), the operators $f(z,\bar{z})$, $\tilde{f}(z,\bar{z})$ must carry odd J_0 charge. Equivalently, odd J_0 charge is required for V_{-1} and $\tilde{V}_{-1}$ to be local with respect to the supersymmetry current J_{α}. On the other hand f, $\tilde{f}$ have conformal dimension $(\frac{1}{2},1)$. It can be shown that the dimension of an operator carrying a charg e j_0 is bounded from below by $\frac{1}{6}j_0^2$ [12]. This constrains the J_0 eigenvalues of $f(\tilde{f})$ to be ± 1. This J_0 charge may be identified with the chirality $h(-h)$ of the corresponding space-time scalars. We then have the operator product expansion,

$$J(z)f(w,\bar{w}) \sim \frac{h}{z-w}f(w,\bar{w}), \quad J(z)\tilde{f}(w,\bar{w}) \sim -\frac{h}{z-w}\tilde{f}(w,\bar{w}). \tag{3.7}$$

In order to calculate the one loop contribution to the scalar mass, we need to evaluate the correlator of the operators $\frac{g}{2\pi}V(z_1,k)$ and $\frac{g}{2\pi}\tilde{V}(z_2,-k)$ on a torus characterized by the Teichmuller parameter τ, sum over all possible boundary conditions on the fermion fields (spin structures [24])‡ and then integrate over

* In our convention $\alpha' = 2$.

‡ Actually we should average over the possible boundary conditions in the τ direction, which gives rise to an extra factor of $\frac{1}{2}$. We absorb this in the definition of the correlator.

the location of the vertices on the world-sheet, as well as over τ. There are four possible boundary conditions on the torus for the right handed fermions. These may be expressed as (P,P), (P,A), (A,A) and (A,P), where, for example, (P,P) denotes periodic boundary condition on the fermions along both cycles on the torus. The spin structures (A,P), (A,A) and (P,A) are known as even spin structures, whereas the spin structure (P,P) is known as the odd spin structure. Neither the fermions ψ^μ nor the superconformal ghosts β, γ have any zero mode for any of the even spin structures, and so their contribution to the scalar mass squared has the form

$$-\frac{g^2}{4\pi^2}\int d^2\tau d^2z_1 d^2z_2 \big\langle V_0(z_1,k)\tilde{V}_0(z_2,-k)\big\rangle_e . \tag{3.8}$$

The subscript e in $\langle\ \rangle_e$ denotes a sum over even spin structures in the right-handed sector. The explicit minus sign reflects the fact that the two point function in a euclidean field theory is equal to $-m^2$. On the other hand, in the periodic-periodic (P,P) sector the superconformal ghosts β, γ as well as the free right-handed fermions ψ^μ have zero modes. The zero modes of β, γ signal the presence of a conformal Killing spinor, as well as a supermodulus in this sector. As a result, the contribution to m^2 from this sector has the form

$$\int d^2z_1 d^2z_2 \langle V_0(z_1,k)\tilde{V}_{-1}(z_2,-k)e^{\phi(z_0)}(T_F^{\rm int}(z_0)+\frac{1}{2}\psi_\mu\partial X^\mu(z_0))\rangle_{PP}. \tag{3.9}$$

This can be seen to vanish identically due to the zero modes of the free right-handed fermions ψ^μ, $\psi^{\bar\mu}$ $(\mu=4,5)$, using eqs. (3.1) – (3.3).

Since the expectation value of a single ψ^μ is zero by fermion number conservation, we may express (3.8) as,

$$\begin{aligned}\big\langle V_0(z_1,k)\tilde{V}_0(z_2,-k)\big\rangle_e =&\langle g(z_1)\tilde{g}(z_2)e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle_e\\ &+k_\mu k_\nu\big\langle\psi^\mu(z_1)f(z_1)\psi^\nu(z_2)\tilde{f}(z_2)e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\big\rangle_e .\end{aligned} \tag{3.10}$$

We shall now show that the first term on the right hand side of (3.10) vanishes identically. For this we need to know the supersymmetry transformation properties of $V_0(\tilde{V}_0)$. For V_0 of negative chirality (as defined in sec. 2) we have [10]

$$
\begin{aligned}
[Q_\alpha, V_0(z,k)] &= V_\psi((\not{k})_{\alpha\dot{\beta}} u^{\dot{\beta}}), \\
[Q_{\dot{\alpha}}, V_0(z,k)] &= 0, \\
[Q_\alpha, \tilde{V}_0(z,k)] &= 0, \\
[Q_{\dot{\alpha}}, \tilde{V}_0(z,k)] &= V_\psi((\not{k})_{\dot{\alpha}\beta} u^{\beta}),
\end{aligned}
\tag{3.11}
$$

where V_ψ is the vertex operator for the corresponding fermionic state. From (3.11) we see that,

$$
[Q_\alpha, V_0(z, k=0)] = [Q_{\dot{\alpha}}, \tilde{V}_0(z, k=0)] = 0. \tag{3.12}
$$

If V_0 had positive chirality, the roles of V_0 and $\tilde{V}_0$ would be interchanged.

Let us now introduce the fields

$$
P^+ = S_g^- \hat{S}^+ S_4^+ S_5^+ \tag{3.13}
$$

and

$$
P^- = S_g^+ \hat{S}^- S_4^- S_5^-, \tag{3.14}
$$

of conformal dimensions (1,0) and (0,0) respectively. The fields $\hat{S}^\pm$ have been defined in sec. 2. Since P^+ is the current for one of the supersymmetry charges Q_α, we get from eqs.(3.12), (3.3) and (3.5),

$$
\oint dz P^+(z) g(w,\bar{w}) = \oint dz P^+(z) \tilde{g}(w,\bar{w}) = 0. \tag{3.15}
$$

Now, $P^+(z)$ is a field of conformal dimension (1,0) and $g(w,\bar{w})$ ($\tilde{g}(w,\bar{w})$) is a field of conformal dimension (1,1). The fields $P^+(z)$, $g(w,\bar{w})$ and $\tilde{g}(w,\bar{w})$ are local with respect to each other. Thus the singularities in the operator product $P^+(z)g(w,\bar{w})$ ($P^+(z)\tilde{g}(w,\bar{w})$) can only be a double pole or a single pole. The double pole term is seen to be absent on dimensional grounds: It can occur if there is a field in the operator product with conformal dimension $h_R = 0$; however the right hand side of the operator product expansion must contain the field S_g^- of conformal dimension $h_R = \frac{3}{8}$ [10] coming from P^+. The single pole term is absent due to eq. (3.15). This show that $P^+(z)$ does not have any singularity near $g(w, \bar{w})$ or $\tilde{g}(w, \bar{w})$. On the other hand,

$$P^+(z)P^-(w) \sim \frac{1}{z-w}. \tag{3.16}$$

Let us now consider the correlator,

$$F(z, w, z_1, z_2) \equiv \langle P^+(z)P^-(w)g(z_1)\tilde{g}(z_2)e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle,$$

where in $\langle\ \rangle$ we have summed over all the spin structures. $F(z, w, z_1, z_2)$ must be doubly periodic as a function of z with periods 1 and τ, and has at most a single pole at $z = w$. But the only such function is a constant [26, 27], hence the residue of the pole at $z = w$ must vanish. We now get, using eq.(3.16),

$$\begin{aligned}\langle g(z_1)\tilde{g}(z_2)e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle_e = \lim_{z\to w}(z-w)\{F(z, w, z_1, z_2) \\ -\langle P^+(z)P^-(w)g(z_1)\tilde{g}(z_2)e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle_{PP}\} = 0.\end{aligned} \tag{3.17}$$

The contribution to the right-hand side of (3.17) from the periodic-periodic sector vanishes as follows: In this sector the correlator $\langle S_4^+(z)S_5^+(z)S_g^-(z)S_4^-(w)$ $S_5^-(w)S_g^+(w)\rangle_{PP}$ is proportional to $\vartheta_1(\frac{1}{2}z - \frac{1}{2}w)/(\vartheta_1(z-w))^{\frac{1}{4}}$ [28, 29]. Thus besides the singularity of order $(z-w)^{-\frac{1}{4}}$ determined by the operator product expansion, the correlator has an explicit factor of $(z-w)$. Combining this with the correlator $\langle \hat{S}^+\hat{S}^- g\tilde{g}\rangle$ in the internal space, which has at most a singularity of order $(z-w)^{-\frac{3}{4}}$, we see that the full correlator in the periodic-periodic sector is singularity free in the $z \to w$ limit.

Naively, the second term on the right hand side of eq. (3.10) also vanishes identically, since the free fermion correlator involving ψ^μ and ψ^ν must be proportional to $\delta^{\mu\nu}$, and hence the term is proportional to k^2 which vanishes on-shell. However, we shall set k^2 to zero only at the end of the calculation, after calculating the correlator and performing the integral over z_i. As we shall see, the integral over the world sheet coordinates gives a factor of $\frac{1}{k^2}$ which cancels the k^2 term in the numerator and gives a finite answer in the $k^2 \to 0$ limit.

A justification for this procedure may be given by considering the four point function $\langle \prod_i V_{(i)}(k_i, z_i) \rangle$ where the $V_{(i)}$ s factorize on $V_0, \tilde{V}_0$:

$$\lim_{z_2 \to z_1} V_{(1)}(z_1, k_1) V_{(2)}(z_2, k_2) \sim \varepsilon_1 \frac{1}{(z_1 - z_2)^{1-k_1 \cdot k_2}} \frac{1}{(\bar{z}_1 - \bar{z}_2)^{1-k_1 \cdot k_2}} V(z_1, k_1 + k_2),$$

$$\lim_{z_4 \to z_3} V_{(3)}(z_3, k_3) V_{(4)}(z_4, k_4) \sim \varepsilon_2 \frac{1}{(z_3 - z_4)^{1-k_3 \cdot k_4}} \frac{1}{(\bar{z}_3 - \bar{z}_4)^{1-k_3 \cdot k_4}} \tilde{V}(z_3, k_3 + k_4).$$

(3.18)

Thus at tree level there are three point couplings between $V_{(1)}$, $V_{(2)}$ and V, and between $V_{(3)}$, $V_{(4)}$ and $\tilde{V}$. One choice for these vertex operators is to take $V_{(1)}$ and $V_{(3)}$ as the vertex operators V and $\tilde{V}$ respectively, and $V_{(2)}$ and $V_{(4)}$ as the vertex operators of some $U(1)$ gauge bosons which couple to these scalar fields. Hence if we look at the one loop four point scattering, and look for poles in the s-channel, we should expect to see a double pole of the form

$$-(2ig)^2 \varepsilon_1 \varepsilon_2 \frac{m^2}{(2k_1 \cdot k_2)^2}, \tag{3.19}$$

where m^2 is the one loop mass square insertion for the scalar created by V. On the other hand, from (3.18) we get,

$$\frac{g^4}{(2\pi)^4} \int d^2\tau \int d^2z_1 d^2z_2 d^2z_3 d^2z_4 \langle V_{(1)}(z_1, k_1) V_{(2)}(z_2, k_2) V_{(3)}(z_3, k_3) V_{(4)}(z_4, k_4) \rangle$$

$$\sim (2ig)^2 \frac{\varepsilon_1}{2k_1 \cdot k_2} \frac{\varepsilon_2}{2k_3 \cdot k_4} \frac{(ig)^2}{(2\pi)^2} \int d^2\tau \int d^2z_1 d^2z_3 \langle V(z_1, k_1 + k_2) \tilde{V}(z_3, k_3 + k_4) \rangle.$$

(3.20)

The $(2k_1 \cdot k_2)^{-1}$ and $(2k_3 \cdot k_4)^{-1}$ appear from the z_3 and z_4 integrals respectively. Thus we see that the calculation of m^2 reduces to the calculation of $-\frac{g^2}{4\pi^2} \int d^2\tau \int d^2z_1 \int d^2z_3 \langle V(z_1, k_1 + k_2)\tilde{V}(z_3, k_3 + k_4)\rangle$ as before, however, now $(k_1 + k_2)^2$ need not be set to zero until the end of the calculation.

Thus we need to calculate the $\frac{1}{k^2}$ term in

$$\int d^2z_2 \langle \psi^\mu(z_1) f(z_1) \psi^\nu(z_2) \tilde{f}(z_2) e^{ik\cdot X(z_1)} e^{-ik\cdot X(z_2)} \rangle_e$$

$$= - \int d^2z_2 \left| \frac{\vartheta_1(z_1 - z_2)}{\vartheta_1'(0)} \right|^{-2k^2} \langle \psi^\mu(z_1)\psi^\nu(z_2) f(z_1) \tilde{f}(z_2)\rangle_e, \tag{3.21}$$

where we have explicitly calculated the relevant contribution from the X-correlator in the $k^2 \to 0$ limit [30].

The $\frac{1}{k^2}$ pole from the z_2 integral, comes from the $(z_1 - z_2)^{-1-k^2}(\bar{z}_1 - \bar{z}_2)^{-1-k^2}$ term in the integrand. Thus we must look at operator product expansions involving the ψ and the f fields. Now,

$$\psi^\mu(z_1)\psi^\nu(z_2) = \frac{\delta^{\mu\nu}}{(z_1 - z_2)} + O(z_1 - z_2), \tag{3.22}$$

$$f(z_1)\tilde{f}(z_2) = \sum_{0\le\alpha<1} \sum_{m=-\infty}^{1} \sum_{n=-\infty}^{2} \frac{A_{mn;\alpha}(z_1, \bar{z}_1)}{(z_1 - z_2)^{m-\alpha}(\bar{z}_1 - \bar{z}_2)^{n-\alpha}}. \tag{3.23}$$

Since f, $\tilde{f}$ are operators of conformal dimension $(\frac{1}{2},1)$, $A_{mn;\alpha}$ is an operator of conformal dimension $(1 - m + \alpha, 2 - n + \alpha)$. Thus $A_{12;0}$ must be proportional to the identity operator. As pointed out before, f, $\tilde{f}$ are normalized so that the coefficient of this operator is unity. Upon substituting (3.22) and (3.23) into (3.21) we see that only the term involving the operator $A_{01;0}$ from (3.23) and $\frac{\delta^{\mu\nu}}{z_1 - z_2}$ from eq. (3.22) can give the required singularity in the integrand. Let us denote the operator $A_{01;0}$ by N. Thus we get,

$$\int d^2 z_2 \big\langle V_0(z_1,k)\tilde{V}_0(z_2,-k)\big\rangle_e = -k^2 \int d^2 z_2 \\ (z_1 - z_2)^{-1-k^2}(\bar{z}_1 - \bar{z}_2)^{-1-k^2}\langle N(z_1,\bar{z}_1)\rangle_e. \tag{3.24}$$

So the coefficient of the linear term in D may be determined from the equation:

$$gh\sum_a q^{(a)}c^{(a)} = \frac{-ig^2}{2\pi}\int d^2\tau \int d^2 z_1 \langle N(z_1,\bar{z}_1)\rangle_e. \tag{3.25}$$

Let us now determine the operator N. First of all, note from eq.(3.23) that it is an operator of conformal dimension (1,1). Secondly, since N appears in the operator product of an operator with its CPT conjugate, it carries zero charge under the conserved $U(1)$ current J belonging to the right-handed $N=2$ superconformal algebra. Thus the operator product of J with N has at most a double pole, and no single pole:

$$J(z)N(w,\bar{w}) = \frac{1}{(z-w)^2}M(\bar{w}) + \text{ non-singular terms,} \tag{3.26}$$

where M is a field of conformal dimension (0,1). As a result, when $M(\bar{w})$ is combined with the operator $(\partial X^\mu + ik_\nu\psi^\mu\psi^\nu)$ from the right handed sector, it gives the vertex operator of a massless vector particle. So M must be a linear combination $(\sum_a \alpha^{(a)}U^{(a)}(\bar{w}))$ of the left-handed currents $U^{(a)}(\bar{w})$ associated with the generators of the four dimensional gauge group. Furthermore, since $M(\bar{w})$ appears in the product of a vertex operator f of a state with the vertex operator of its CPT conjugate state, it must carry zero charge under any element of the Cartan subalgebra of the four dimensional gauge group. As a result, only the currents associated with the generators of the Cartan subalgebra of the four dimensional gauge group may appear in the sum $\sum_a \alpha^{(a)} U^{(a)}(\bar{w})$. This of course includes all the $U(1)$ factors. We may now write

$$J(z)N(w,\bar{w}) = \frac{1}{(z-w)^2}\sum_a \alpha^{(a)} U^{(a)}(\bar{w}) + \text{ non-singular terms.} \qquad (3.27)$$

Since $J(z)J(w) \sim \frac{3}{(z-w)^2}$ + non-singular terms, we may express N as

$$N(w,\bar{w}) = \frac{1}{3}J(w)\sum_a \alpha^{(a)} U^{(a)}(\bar{w}) + \tilde{N}(w,\bar{w}), \qquad (3.28)$$

where

$$J(z)\tilde{N}(w,\bar{w}) = \text{ non-singular.} \qquad (3.29)$$

In order to determine the coefficients $\alpha^{(a)}$ in (3.28) let us consider the correlator,

$$\langle f(z_1,\bar{z}_1)\tilde{f}(z_2,\bar{z}_2)J(z_3)U^{(b)}(\bar{z}_3)\rangle_S, \qquad (3.30)$$

where the subscript S denotes that we are calculating the correlator on the sphere instead of a torus. By $SL(2,C)$ invariance this is proportional to [31]:

$$K^{(b)}(\bar{z}_1-\bar{z}_2)^{-1}(z_1-z_3)^{-1}(z_2-z_3)^{-1}(\bar{z}_1-\bar{z}_3)^{-1}(\bar{z}_2-\bar{z}_3)^{-1}. \qquad (3.31)$$

$K^{(b)}$ may be determined by first taking the limit $z_1 \to z_2$ in (3.30), using eq.(3.23), and then taking the $z_2 \to z_3$ limit. Only the term involving $A_{01;0} \equiv N$ in eq.(3.23) contributes in this limit, the contribution being given by,*

$$\alpha^{(b)}(\bar{z}_1-\bar{z}_2)^{-1}(\bar{z}_2-\bar{z}_3)^{-2}(z_2-z_3)^{-2}. \qquad (3.32)$$

Comparing with (3.31) we see that,

$$K^{(b)} = \alpha^{(b)}. \qquad (3.33)$$

On the other hand we may take the limit $z_3 \to z_1$ first, and then take the $z_1 \to z_2$

* We have chosen a normalization where $U^{(a)}(\bar{z})U^{(b)}(\bar{w}) \sim \frac{\delta^{ab}}{(\bar{z}-\bar{w})^2}$.

limit in (3.30). In the $z_3 \to z_1$ limit,

$$J(z_3)U^{(b)}(\bar{z}_3)f(z_1,\bar{z}_1) \sim \frac{h}{z_3 - z_1}\frac{q^{(b)}}{\bar{z}_3 - \bar{z}_1}f(z_1,\bar{z}_1), \tag{3.34}$$

h and $q^{(b)}$ being the (internal) chirality (± 1) and the $U^{(b)}(1)$ charges of the operator $f(z_1,\bar{z}_1)$. Since the internal and the four dimensional chiralities are correlated, h is also the four dimensional chirality of the field $V_0(z,k)$. Substituting (3.34) in (3.30) and taking the $z_1 \to z_2$ limit, we get for the leading singular term,

$$\frac{hq^{(b)}}{(z_3 - z_1)(\bar{z}_3 - \bar{z}_1)}\frac{1}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)^2}. \tag{3.35}$$

Comparing with (3.31) and (3.33) we get,

$$\alpha^{(b)} = K^{(b)} = hq^{(b)}. \tag{3.36}$$

Using eqs.(3.25), (3.28) and (3.36) we get,

$$h\sum_a q^{(a)}c^{(a)} = \frac{-ig}{2\pi}\int d^2\tau \int d^2 z_1 \Big[\frac{h}{3}\sum_a q^{(a)}\langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e + \langle \tilde{N}(z_1,\bar{z}_1)\rangle_e\Big]. \tag{3.37}$$

We shall now show that $\langle \tilde{N}(z_1,\bar{z}_1)\rangle_e$ vanishes identically. For this we need the representation of J, $\hat{S}^+$ and $\hat{S}^-$ in terms of the free field H introduced in sec. 2:

$$\begin{aligned} J(z) &= +i\sqrt{3}\partial_z H, \\ \hat{S}^+ &= e^{+i\frac{\sqrt{3}}{2}H(z)}. \end{aligned} \tag{3.38}$$

Using eq.(3.29) we see that $J(z)\tilde{N}(w)$ is free from any singularity, hence so must be the product $H(z)\tilde{N}(w)$. This, in turn, shows that $\hat{S}^+(z)\tilde{N}(w)$ is free from any singularity. In the same way we showed that $\langle P^+(z)P^-(w)g(z_1,\bar{z}_1)\tilde{g}(z_2,\bar{z}_2)$

$e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle$ is independent of z, we may show that $\langle P^+(z)P^-(w)\tilde{N}(z_1,\bar{z}_1)\rangle$ is a constant as a function of z, and so,[†]

$$\langle \tilde{N}(z_1,\bar{z}_1)\rangle_e = \lim_{z\to w}(z-w)\langle P^+(z)P^-(w)\tilde{N}(z_1,\bar{z}_1)\rangle_e = 0. \tag{3.39}$$

Thus,[*]

$$c^{(a)} = \frac{-ig}{2\pi}\frac{1}{3}\int d^2\tau \int d^2 z_1 \langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e. \tag{3.40}$$

From this it appears that $-\frac{i}{3}J(z)U^{(a)}(\bar{z})$ may be interpreted as the vertex operator for the auxiliary field $D^{(a)}$. More evidence for this interpretation will be presented in appendix A, along with a similar discussion for the vertex operators for the F-type auxiliary fields.[‡]

We now proceed to relate $\langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e$ to another correlator. We start from the correlator

$$\langle P^+(z)P^-(w)J(z_1)U^{(a)}(\bar{z}_1)\rangle. \tag{3.41}$$

Since

$$J(z_1)P^+(z) \sim \frac{3}{2}\frac{1}{z_1-z}P^+(z), \tag{3.42}$$

the correlator in eq. (3.41) has a pole at $z=w$ and at $z=z_1$ as a function of z. Since it is also periodic as a function of z it must vanish at two points on

† The contribution from the periodic-periodic sector may be subtracted from this correlator as was done in eq.(3.17), and may be shown to be singularity free in the $z\to w$ limit.

* On the right hand side of eq.(3.37) the sum over a runs over all the $U(1)$ factors of the four dimensional gauge group, as well as the Cartan subalgebras of non-Abelian factors in the gauge group. Hence it seems from eq.(3.40) that we generate D-terms associated with the non-Abelian part of the gauge group as well. However, $\langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e$ vanishes identically due to non-Abelian gauge symmetry if $U^{(a)}$ belongs to the Cartan subalgebra of a non-Abelian factor. This will be seen explicitly later.

‡ The existence of F-type auxiliary field vertex operators was noted by E. Martinec in collaboration with one of the authors (L.D.) and of D-type vertex operators by V. Kaplunovsky and L. D..

the torus, whose sum must be equal to $z_1 + w$. Let us denote these points by $z_1 + R(z_1, w)$ and $w - R(z_1, w)$. Then we may write,

$$\langle P^+(z)P^-(w)J(z_1)U^{(a)}(\bar{z}_1)\rangle$$

$$\equiv F(z_1, w)\frac{\vartheta_1(z - z_1 - R(z_1, w))\vartheta_1(z - w + R(z_1, w))}{\vartheta_1(z - w)\vartheta_1(z - z_1)}, \tag{3.43}$$

where $F(z_1, w)$ is a function of z_1 and w. It may be determined by noting that as $z \to z_1$, the residue at the pole in (3.43) must be given by $-\frac{3}{2}\langle P^+(z_1)P^-(w)U^{(a)}(\bar{z}_1)\rangle$. This gives

$$F(z_1, w) = -\frac{3}{2}\vartheta_1'(0)\frac{\vartheta_1(z_1 - w)\langle P^+(z_1)P^-(w)U^{(a)}(\bar{z}_1)\rangle}{\vartheta_1(-R(z_1, w))\vartheta_1(z_1 - w + R(z_1, w))}. \tag{3.44}$$

We now use

$$\langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e = \lim_{z\to w}\Big[(z - w)\{\langle P^+(z)P^-(w)J(z_1)U^{(a)}(\bar{z}_1)\rangle - \langle P^+(z)P^-(w)J(z_1)U^{(a)}(\bar{z}_1)\rangle_{PP}\}\Big]. \tag{3.45}$$

The first term may be determined using eqs.(3.43) and (3.44). The second term vanishes due to the extra $\vartheta_1(\frac{1}{2}z - \frac{1}{2}w)$ in the correlator involving $S_4^\pm$, $S_5^\pm$ and $S_g^\pm$. Thus we get,

$$\langle J(z_1)U^{(a)}(\bar{z}_1)\rangle_e = \frac{3}{2}\langle P^+(z_1)P^-(w)U^{(a)}(\bar{z}_1)\rangle. \tag{3.46}$$

Note that in the correlator on the right hand side of eq.(3.46) we must sum over all spin structures. We shall show that this correlator is related to the expectation value of $U^{(a)}(\bar{z}_1)$ in the periodic-periodic sector. In order to do this let us introduce the correlator $\langle P^+(z)P^-(w)U^{(a)}(\bar{z}_1)\rangle_\nu$ for a fixed spin structure ν in the right handed sector. This correlator is no longer periodic in z with period

1 and τ, instead the correlator in a given spin structure gets transformed to a correlator of a different spin structure under translation of z by 1 or τ. However, it must be doubly periodic as a function of z with period 2 and 2τ. Inside the parallelogram described by the period 2 and 2τ the correlator has four poles, at $z = w$, $w + \tau$, $w + 1$ and $w + 1 + \tau$. It is more convenient to use the variable $\frac{z}{2}$ in terms of which the periods become 1 and τ again. Hence the function must also have four zeros as a function of $\frac{z}{2}$, whose sum must be $2w + 1 + \tau$ (up to shifts by 1 or τ). Let us take the positions of these zeros to be at $\frac{z}{2} = \frac{w}{2} + A_\nu$, $\frac{w}{2} + B_\nu$, $\frac{w}{2} + C_\nu$ and $\frac{w}{2} + D_\nu$ respectively (mod 1 and τ). We shall take

$$A_\nu + B_\nu + C_\nu + D_\nu = 0. \tag{3.47}$$

Now, a function is completely determined by the positions of its zeros and poles up to a multiplicative constant (independent of z) [26, 27]. In the periodic-periodic sector ($\nu = 1$) we get*

$$\begin{aligned}
&\langle P^+(z) P^-(w) U^{(a)}(\bar{z}_1)\rangle_{PP} \\
&= K \frac{\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - A_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - B_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - C_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - D_1)}{\vartheta_1(\frac{1}{2}z - \frac{1}{2}w)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - \frac{1}{2})\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - \frac{1}{2}\tau)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w + \frac{1}{2}(1+\tau))} \\
&= \tilde{K} \frac{\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - A_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - B_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - C_1)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - D_1)}{\vartheta_1(z - w)},
\end{aligned} \tag{3.48}$$

where K, $\tilde{K}$ are unknown constants. (In going from the first to the second equality we have used the fact that $\vartheta_1(z - w)$ and $\vartheta_1(\frac{1}{2}z - \frac{1}{2}w)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - \frac{1}{2})$ $\vartheta_1(\frac{1}{2}z - \frac{1}{2}w - \frac{1}{2}\tau)\vartheta_1(\frac{1}{2}z - \frac{1}{2}w + \frac{1}{2}(1+\tau))$ have the same periodicity properties and same zeros in the z-plane, hence they must be proportional to each other.) Furthermore, since the part of the correlator involving $\langle S_4^+(z) S_5^+(z) S_g^-(w) S_4^-(w)$ $S_5^-(w) S_g^+(w)\rangle$ may be calculated explicitly and shown to be proportional to

* See ref. [29] for our ϑ-function conventions.

$\vartheta_1(\frac{z-w}{2})/(\vartheta_1(z-w))^{\frac{1}{4}}$ [28,29], we see from (3.48) that D_1 must vanish. The contribution to $\langle P^+(z)P^-(w)U^{(a)}(\bar z_1)\rangle_\nu$ for other spin structures ν may be determined by shifting z by 1 or τ in (3.48). We finally get,

$$\begin{aligned}&\langle P^+(z)P^-(w)U^{(a)}(\bar z_1)\rangle_\nu\\&=\tilde K\delta_\nu\frac{\vartheta_\nu(\frac12 z-\frac12 w-A_1)\vartheta_\nu(\frac12 z-\frac12 w-B_1)\vartheta_\nu(\frac12 z-\frac12 w-C_1)\vartheta_\nu(\frac12 z-\frac12 w)}{\vartheta_1(z-w)},\end{aligned} \tag{3.49}$$

where $\nu=1,2,3,4$ denotes spin structures (P,P), (P,A), (A,A) and (A,P) respectively. Here $\delta_1=\delta_3=1$ and $\delta_2=\delta_4=-1$. Summing over spin structures and using the Riemann theta identity [28,29] and eq.(3.47) with $D_1=0$ we get,

$$\langle P^+(z)P^-(w)U^{(a)}(\bar z_1)\rangle=2\tilde K\vartheta_1(-A_1)\vartheta_1(B_1)\vartheta_1(C_1). \tag{3.50}$$

On the other hand, in the periodic-periodic sector, $\langle P^+(z)P^-(w)U^{(a)}(\bar z_1)\rangle_{PP}$ may be expressed as

$$\begin{aligned}\langle P^+(z)P^-(w)U^{(a)}(\bar z_1)\rangle_{PP}=&\Big\langle\langle\hat S^+(z)\hat S^-(w)U^{(a)}(\bar z_1)\rangle\Big\rangle_{PP}\\&\times\frac{1}{256\pi^4}\frac{(\eta(\tau)\overline{\eta(\tau)})^{-2}}{(\mathrm{Im}\,\tau)^3}\frac{\vartheta_1(\frac12 z-\frac12 w)}{\eta(\tau)}\frac{(\vartheta_1'(0))^{\frac14}}{(\vartheta_1(z-w))^{\frac14}}e^{-i\frac{3\pi}{4}\tau}e^{i\frac{11\pi}{6}\bar\tau},\end{aligned} \tag{3.51}$$

where $\big\langle\langle\,\rangle\big\rangle$ denotes a trace over all the internal fields φ^j (for σ-model compactifications φ^j consists of X^i,$X^{\bar i}$, ψ^i, $\psi^{\bar i}$ and λ^M), and a sum over spin structures in the left-handed sector; $\eta(\tau)$ is the Dedekind eta function [26]. In writing eq.(3.51) we have extracted all the normal ordering constants, so that the $SL(2,C)$-invariant ground state for the internal superconformal field theory has $L_0^{(int)}=\bar L_0^{(int)}=0$. The explicit factors in (3.51) come from taking the trace over the fields X^μ, ψ^μ and the ghost fields, and dividing by the volume (Im τ) of the group of translations on the torus. The overall normalization has been chosen so that the partition function reproduces (-1) times the sum of the contributions to the cosmological constant from individual states if we extend the integration over τ_2 from 0 to ∞ instead of restricting it to the fundamental region in the moduli space [32]. In fixing the overall normalization, we have also explicitly introduced a factor of $\frac{1}{2}$ in order to take into account the fact that we must average over the spin structures in the τ direction instead of summing over them [24]. The factor of $\frac{1}{2}$ only accounts for the right-handed sector. Similar factors coming from the left-handed sector are still implicit inside the correlator $\langle\langle\ \rangle\rangle$, and will be taken into account during the evaluation of the correlator.

Comparing eqs.(3.49) and (3.51) we get

$$\begin{aligned}\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP} =& \lim_{z\to w}\{(z-w)^{\frac{3}{4}}\langle\langle \hat{S}^+(z)\hat{S}^-(w)U^{(a)}(z_1)\rangle\rangle_{PP}\} \\ =&256\pi^4\tilde{K}(\eta(\tau))^3(\overline{\eta(\tau)})^2(\text{Im }\tau)^3 \\ &\times(\vartheta_1'(0))^{-1}\vartheta_1(-A_1)\vartheta_1(-B_1)\vartheta_1(-C_1)e^{i\frac{3\pi}{4}\tau}e^{-i\frac{11\pi}{6}\bar{\tau}}.\end{aligned} \tag{3.52}$$

Comparing with (3.50) and using the relation $\vartheta_1'(0)=-2\pi(\eta(\tau))^3$ we get

$$\begin{aligned}\langle P^+(z_1)P^-(w)U^{(a)}(\bar{z}_1)\rangle =& -\frac{1}{64\pi^3}(\text{Im }\tau)^{-3}(\overline{\eta(\tau)})^{-2} \\ &\times\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}e^{-i\frac{3\pi}{4}\tau}e^{i\frac{11\pi}{6}\bar{\tau}},\end{aligned} \tag{3.53}$$

and so, from (3.40) and (3.46),

$$\begin{aligned}c^{(a)} =& -\frac{g}{128\pi^4}\int\frac{d^2\tau}{(\text{Im }\tau)^2}\overline{(\eta(\tau)}^{-2}\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}e^{i\frac{11\pi}{6}\bar{\tau}}e^{-i\frac{3\pi}{4}\tau} \\ \equiv& \int\frac{d^2\tau}{(\text{Im }\tau)^2}\mathbf{F}(\tau,\bar{\tau}),\end{aligned} \tag{3.54}$$

where we have explicitly performed the z_1 integral to get a factor of $2i\text{Im }\tau$, using the fact that the integrand is independent of z_1 due to translational invariance on the torus. $\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}$ may now be evaluated using the operator formalism, in which it is given by

$$\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP} = Tr_P\{(-1)^{F^{(int)}} U^{(a)}(\bar{z}_1)\bar{P}_{GSO} e^{2\pi i L_0^{(int)}\tau} e^{-2\pi i \bar{L}_0^{(int)}\bar{\tau}}\}, \qquad (3.55)$$

where the trace is taken over all the states in the Ramond sector of the internal superconformal theory, *i.e.* states whose operators are double valued with respect to T_F^{int}. $F^{(int)}$ counts the number of internal right-handed fermions. $\bar{P}_{GSO}$ denotes the appropriate GSO projection operator [33, 10, 24] in the left-handed sector. If $G_0^{(int)}(= G_0^+ + G_0^-)$ denotes the generator of the (1,0) supersymmetry, then the contribution to the trace from any state $|n\rangle$ and the state $G_0^{(int)}|n\rangle$ cancel each other, since $G_0^{(int)}$ commutes with $L_0^{(int)}$, $\bar{L}_0^{(int)}$, $\bar{P}_{GSO}$ and $U^{(a)}(\bar{z}_1)$ but anticommutes with $(-1)^{F^{(int)}}$. Thus only the states with $G_0^{(int)} = 0$ ($\rightarrow L_0^{(int)} = G_0^2 + \frac{3}{8} = \frac{3}{8}$) contribute to the correlator. (After being combined with the vacuum of the X^μ, ψ^μ and the ghost system, these correspond to massless states of the theory.) Thus we see that the integrand $\mathbf{F}(\tau, \bar{\tau})$ in eq.(3.54) has no dependence on τ. Since $\frac{d^2\tau}{(\text{Im } \tau)^2}$ is a modular invariant measure, this shows that $\mathbf{F}(\tau, \bar{\tau})$ regarded as a function of $\bar{\tau}$ must be a modular form of weight zero, i.e. it must be modular invariant.

Invariance under $\bar{\tau} \rightarrow \bar{\tau} + 1$ shows that $\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}$ can receive contribution from states with integer $\bar{L}_0^{(int)}$ eigenvalues only. The lowest $\bar{L}_0^{(int)}$ eigenvalue ($\bar{L}_0^{(int)} = 0$) is provided by the $\overline{SL(2,C)}$ invariant ground state. But $U^{(a)}(\bar{z}_1)$ is a conformal field of dimension (0,1), so by $\overline{SL(2,C)}$ invariance it has vanishing vacuum expectation value in this state. Thus the lowest states which can contribute to $\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}$ have $\bar{L}_0^{(int)} = 1$. They correspond to massless states of the string theory, after being combined with the ghost vacuum with $\bar{L}_0^{ghost} = -1$. Due to translational invariance on the cylinder, we may replace $U^{(a)}(\bar{z}_1)$ by

$$\int_0^1 d\sigma U^{(a)}(\sigma, \tau) = \frac{1}{i}\oint d\bar{w} U^{(a)}(\bar{w}) \equiv \frac{1}{i}\oint d\bar{y} \hat{U}^{(a)}(\bar{y}), \qquad (3.56)$$

where

$$\bar{w} = \tau + i\sigma, \qquad \bar{y} = e^{2\pi\bar{w}}. \tag{3.57}$$

If $V(w', \bar{w}')$ describes a vertex operator carrying $U^{(a)}$ charge $q^{(a)}$, then on the plane,

$$\hat{U}^{(a)}(\bar{y})V(w', \bar{w}') \sim \frac{q^{(a)}}{\bar{y} - e^{2\pi\bar{w}'}} V(w', \bar{w}'). \tag{3.58}$$

Thus acting on a state created by the vertex operator V at $w' = -\infty$ we get,

$$\frac{1}{i} \oint d\bar{y} \hat{U}^{(a)}(\bar{y})|V\rangle \sim \frac{1}{i} \oint d\bar{y} \frac{q^{(a)}}{\bar{y}} |V\rangle = 2\pi q^{(a)}|V\rangle. \tag{3.59}$$

Therefore in (3.55) we may replace $U^{(a)}(\bar{z}_1)$ by $U_0^{(a)}$, where $U_0^{(a)}$ measures the total $U^{(a)}(1)$ charge. This gives,

$$\lim_{\bar{\tau} \to -i\infty} \langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP} = \\ 2\pi Tr_P\{(-1)^{F^{(int)}} \bar{P}_{GSO} U_0^{(a)}\}_{L_0^{(int)} = \frac{3}{8}, \bar{L}_0^{(int)} = 1} e^{i\frac{3\pi}{4}\tau} e^{-i2\pi\bar{\tau}}, \tag{3.60}$$

since in this limit $\langle\langle U^{(a)}(\bar{z}_1)\rangle\rangle_{PP}$ receives contribution from the lowest lying states only. Since $\overline{\eta(\tau)}$ goes to $e^{-i\frac{\pi}{12}\bar{\tau}}$ as $\bar{\tau} \to -i\infty$, we see from (3.54) that $\mathbf{F}(\tau, \bar{\tau})$ goes to a constant in this limit. However as is well known a modular form of weight zero which is bounded at infinity must be a constant [34]. Hence $\mathbf{F}(\tau, \bar{\tau})$ may be determined by evaluating it at infinity, *i.e.* by simply evaluating the sum $Tr_P\{(-1)^{F^{(int)}} \bar{P}_{GSO} U_0^{(a)}\}_{L_0^{(int)} = \frac{3}{8}, \bar{L}_0^{(int)} = 1}$. In this expression $(-1)^{F^{(int)}}$ measures the four dimensional chirality of a fermionic state up to a sign. The precise sign is determined as follows. First note that GSO projection requires,*

$$(-1)^{F^{(int)} + F^{(ext)} + F^{(g)}} = -1. \tag{3.61}$$

* The phases δ_ν in eq. (3.49) give a relative minus sign between the partition functions with periodic and antiperiodic boundary conditions in the τ-direction. We interpret this as defining the projection operator to be $(1 - (-1)^{F^{(tot)}})/2$. We may also absorb the minus sign into a redefinition of $F^{(int)}$, $F^{(ext)}$ or $F^{(g)}$. This does not change the final outcome of our analysis.

Let us define $J^{(4)}$ to be the fermion number density for the system $\psi^4, \psi^{\bar{4}}$, so that $S_4^+(S_4^-)$ carries $J_0^{(4)}$ charge $\frac{1}{2}(-\frac{1}{2})$. Then,

$$S_4^+(z)S_4^-(w) \sim \frac{1}{(z-w)^{\frac{1}{4}}}\{1+\frac{1}{2}(z-w)J^{(4)}(w)\}. \tag{3.62}$$

Let us now note that in our normalization convention,

$$Tr_P\{(-1)^{F^{(4)}} S_4^+(z)S_4^-(w)\} \equiv \langle S_4^+(z)S_4^-(w)\rangle_{PP} = \frac{\vartheta_1(\frac{z-w}{2})}{(\vartheta_1(z-w))^{\frac{1}{4}}} \frac{(\vartheta_1'(0))^{\frac{1}{4}}}{\eta(\tau)}, \tag{3.63}$$

as can be seen from eq. (3.51); $F^{(4)}$ denotes the fermion number for the $\psi^4, \psi^{\bar{4}}$ system. Taking the limit $z \to w$ in (3.65), and comparing with (3.64) we get,

$$Tr_P\{(-1)^{F^{(4)}} J^{(4)}(z)\} = -2\pi(\eta(\tau))^2. \tag{3.64}$$

If $J_0^{(4)}$ denotes the total $J^{(4)}$ charge this gives,

$$Tr_P\{(-1)^{F^{(4)}} J_0^{(4)}\} = -(\eta(\tau))^2. \tag{3.65}$$

This shows that at the lowest mass level the $F^{(4)}$ and $J_0^{(4)}$ eigenvalues are negatively correlated, in other words a state with positive $J_0^{(4)}$ eigenvalue ($S_4^+|0\rangle$) has $(-1)^{F^{(4)}} = -1$ and vice versa.

A similar analysis may be carried out for the $(\psi^5, \psi^{\bar{5}})$ and the ghost system.† The net result is that a state of the form $S_4^+S_5^+e^{-\phi/2}|0\rangle$ or $S_4^-S_5^-e^{-\phi/2}|0\rangle$ has $(-1)^{F_{ext}+F_g} = 1$ and states of the form $S_4^+S_5^-e^{-\phi/2}|0\rangle$ and $S_4^-S_5^+e^{-\phi/2}|0\rangle$ have $(-1)^{F_{ext}+F_g} = -1$. From this we see that a state with $(-1)^{F_{int}} = 1$ $((-)^{F_{int}} = -1)$ must combine with a dotted (undotted) four dimensional spin field of chirality $-1(+1)$ to produce a physical vertex operator. As a result

† The easiest way to analyse the ghost system is to consider the uncompactified theory and use the fact that the state $\prod_i^5 S_i^+e^{-\phi/2}|0\rangle$, being an allowed state has $(-1)^{F^{(tot)}} = -1$.

$-Tr_P\{(-1)^{F^{int}} U_0^{(a)} \bar{P}_{GSO}\}_{L_0^{(int)}=\frac{3}{8}, \bar{L}_0^{(int)}=1}$ denotes the number of massless fermions (n_i) in the theory weighted by their chirality (h_i) and the $U^{(a)}(1)$ charge ($q_i^{(a)}$), and we finally get

$$c^{(a)} = \frac{g}{64\pi^3} \int \frac{d^2\tau}{(\text{Im } \tau)^2} \sum_i n_i q_i^{(a)} h_i = \frac{g}{192\pi^2} \sum_i n_i q_i^{(a)} h_i. \tag{3.66}$$

The numerical value of $c^{(a)}$ can be evaluated from (3.66) entirely in terms of properties of the massless spectrum. In the case of (2,2) $Spin(32)/Z_2$ compactifications the parameters in (3.66) can be computed by using index theorems [35,36]. Furthermore we see easily from the above equation that if $U_0^{(a)}$ refers to a $U(1)$ subgroup of a non-Abelian group, then $\sum_i n_i q_i^{(a)} h_i$ vanishes identically. This is due to the fact that the massless fermions belong to some (reducible) representation of this non-Abelian gauge group, and that $TrU_0^{(a)}$ vanishes in any such representation.

The above result may be interpreted as the result of a direct computation of the D-term tadpole in the four dimensional effective field theory with a stringy regularization. Since the massive states always come in pairs carrying opposite charges, only the massless states contribute. Their contribution is given by

$$c^{(a)} = g\frac{1}{2} \sum_i n_i q_i^{(a)} h_i \int \frac{d^4k}{(2\pi)^4 k^2}, \tag{3.67}$$

where the explicit factor of $\frac{1}{2}$ arises due to the fact that $\sum_i n_i q_i h_i$ counts scalar field and its complex conjugate as different states, whereas $\int \frac{d^4k}{(2\pi)^4 k^2}$ is the contribution to the tadpole from the scalar together with its complex conjugate field. We now express the divergent integral as

$$\int_0^\infty \frac{d^4k}{k^2} = \frac{1}{2} \int_0^\infty ds e^{-sk^2/2} d^4k = \frac{4\pi^2}{2} \int_0^\infty \frac{ds}{s^2} = 2\pi^2 \frac{1}{4\pi} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1, \tag{3.68}$$

where following ref.[32] we have identified the proper time variable s with $4\pi\tau_2$ (for $\alpha' = 2$). We now identify $\tau_1 + i\tau_2$ as the Teichmuller parameter τ and regulate the divergence at $s = 0$ by restricting the τ integration to the fundamental region of the Teichmuller space. This gives back the answer obtained in eq. (3.66).

4. EXPLICIT RESULTS ON ORBIFOLDS

A. TWO POINT FUNCTION

In this section we shall illustrate some of the above general results through explicit calculations on orbifolds [17]. In particular we shall explicitly verify that scalars carrying a charge under an anomalous $U(1)$ develop a mass at the one loop level. Although our calculations here can be carried out for a general class of orbifold models, we choose for definiteness the $Spin(32)/Z_2$ heterotic string compactified in the standard fashion [17] on a Z_3 orbifold. In this compactification the Z_3 twist generates the center of both the $SU(3)$ holonomy group and an $SU(3)$ subgroup of the $SO(32)$ gauge group. The resulting four-dimensional gauge group is $SU(3) \times U(1) \times O(26)$ and the $U(1)$ as can easily be verified is anomalous.

As in the previous section, we begin by considering the following two point amplitude at the one loop:

$$A(k) = \frac{g^2}{4\pi^2} \int d^2\tau \int d^2 z_1 d^2 z_2 \langle V_0(k, z_1) \tilde{V}_0(-k, z_2) \rangle. \tag{4.1}$$

Here $V_0(k, z_1)$ is the vertex operator for the emission of a $U(1)$ charged scalar belonging to the untwisted sector of the orbifold Hilbert space. We take this vertex to be:

$$V_0^{(j)}(k, z_1) = \big(i\lambda^P(\bar{z}_1)\lambda^{\bar{j}}(\bar{z}_1)\big)\big(\partial X^j(z_1) + ik_\nu \psi^\nu(z_1)\psi^j(z_1)\big)e^{ik\cdot X(z_1)} \tag{4.2}$$

where $j = 1, 2$ or 3 is an internal index to be distinguished from Minkowski indices $\mu\nu$; no sum over j is implied. P is an $O(26)$ vector index and runs over 7,...,32. $\psi(z)$ are the right-handed fermions on the world-sheet while $\lambda(\bar{z})$ are left-handed fermions representing the gauge degrees of freedom of the heterotic string. $\tilde{V}_0^{(j)}(k_2, z_2)$ is the vertex operator for the CPT conjugate state. It is given by (4.2) with j and $\bar{j}$ interchanged. The overall phase and normalization of the vertex operator have been chosen so as to be consistent with the convention used in sec. 3. The amplitude in (4.1) at zero momentum is equal to $-m^2$, where m is the mass of the $U(1)$ charged scalar at one loop.

In the Z_3 orbifold background, just as in flat space-time, all two-dimensional fields appearing in the correlator (4.1) are free. However, the boundary conditions on the torus for the internal "twisted" fields, X^j, ψ^j, λ^j and their complex conjugates, differ from the usual flat space-time boundary conditions by Z_3 phases $(1, e^{i2\pi/3}, e^{i4\pi/3})$. The complete correlator involves a sum over 9 different boundary conditions, or twist structures; there are 3 possible boundary conditions in the σ-direction and 3 in τ. For the world-sheet fermions ψ^j, λ^j the Z_3 phases multiply the usual ± 1 factors corresponding to different spin structures. Just as the sum over spin sturctures implements a projection onto states with even fermion number and allows both space-time fermions and bosons to propagate around the loop, similarly the sum over twist structures implements a projection onto Z_3-invariant states and allows both untwisted and twisted states to propagate around the loop. Both sums are necessary for modular invariance of the one loop amplitude. We shall denote the 9 twist structures by (c_j, d_j), where $e^{2\pi i c_j}(e^{2\pi i d_j})$ is the Z_3 phase acquired by the internal fields X^j, ψ^j, λ^j when translated around the torus in the $\sigma(\tau)$ direction. The complex conjugate fields of course acquire the conjugate phases $e^{-2\pi i c_j}(e^{-2\pi i d_j})$. The three sets of σ-boundary conditions are described by* $c_1 = c_2 = \frac{1}{3}n$, $c_3 = -\frac{2}{3}n$ with $n = 0, +1, -1$, and similarly for the τ-boundary conditions with c_i replaced by d_i.

Because only free fields appear in (4.1), the correlator can be evaluated quite readily using techniques similar to those implemented in ref. [29] for the calculation of flat space covariant fermionic loop amplitudes. More specifically, one can show that all terms contributing to (4.1) vanish as a consequence of some

* The constraints $c_1 + c_2 + c_3 = 0$, and $d_1 + d_2 + d_3 = 0$ are required for there to be an $N = 1$ supersymmetry preserved after compactification to four dimensions [14].

Riemann ϑ-function identity, except for one term which takes the form:

$$m^2 = \frac{g^2}{4\pi^2} \int d^2\tau d^2 z_1 d^2 z_2 k_\mu k_\nu \langle e^{ik\cdot X(z_1)} e^{-ik\cdot X(z_2)} \rangle \\ \langle \psi^\mu(z_1)\psi^j(z_1)\psi^\nu(z_2)\psi^{\bar{j}}(z_2)\rangle \langle \lambda^M(\bar{z}_1)\lambda^{\bar{j}}(\bar{z}_1)\lambda^N(\bar{z}_2)\lambda^j(\bar{z}_2)\rangle. \tag{4.3}$$

The above correlators are again straightforward to compute in a given twist structure $\{(c_i, d_i)\}$ and given spin structures (a, b) and (a', b') for the right and left handed fermions respectively. After summing over twist and spin structures the answer can be cast in the form:†

$$m^2 = \frac{g^2}{4\pi^2} k^2 \int d^2\tau d^2 z_1 d^2 z_2 \;\; N \;\; (\chi_{12})^{-k^2}$$

$$\sum_{\{(c_i,d_i)\}}{}' \left| \frac{1}{(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_1 - z_2))^2 \vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{smallmatrix}\right](0)\vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_2\\ \frac{1}{2}+d_2\end{smallmatrix}\right](0)\vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_3\\ \frac{1}{2}+d_3\end{smallmatrix}\right](0)} \right|^2$$

$$\left(\sum_{\substack{(a,b)\\ \text{even only}}} (\delta_{(a,b)}) \vartheta\begin{bmatrix} a+c_1 \\ b+d_1 \end{bmatrix}(z_1 - z_2)\vartheta\begin{bmatrix} a+c_2 \\ b+d_2 \end{bmatrix}(0)\vartheta\begin{bmatrix} a+c_3 \\ b+d_3 \end{bmatrix}(0)\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z_1 - z_2) \right)$$

$$\left(\sum_{(a',b')} \vartheta\begin{bmatrix} a'+c_1 \\ b'+d_1 \end{bmatrix}(z_1 - z_2)\vartheta\begin{bmatrix} a'+c_2 \\ b'+d_2 \end{bmatrix}(0)\vartheta\begin{bmatrix} a'+c_3 \\ b'+d_3 \end{bmatrix}(0)\vartheta\begin{bmatrix} a' \\ b' \end{bmatrix}(z_1 - z_2)\vartheta^{12}\begin{bmatrix} a' \\ b' \end{bmatrix}(0) \right)^*. \tag{4.4}$$

where * denotes complex conjugation. The sum in (4.4) excludes the contribution of the twist structure $c_i \equiv d_i \equiv 0, i = 1,2,3$. This contribution can be shown to vanish as a consequence of a standard Riemann ϑ-identity. The spin structure dependent phase $\delta_{(a,b)}$ is just $\delta_{(\frac{1}{2},0)} = \delta_{(0,\frac{1}{2})} = -1$ and $\delta_{(0,0)} = 1$, where these correspond to the spin structures $(P, A), (A, A)$ and (A, P) respectively. As in section 3, the sum here runs only over even spin structures for the right-handed fermions; the contribution of the (P, P) sector is zero due to the presence of fermion zero modes in that sector. The correlator of the free bosonic fields X^μ

† see ref. [29] for our ϑ-function notation and conventions.

yields a factor of $(\chi_{12})^{-k^2}$ in (4.4), where χ_{12} is given by:

$$\chi_{12} = \left|\frac{\vartheta_1(z_1 - z_2)}{\vartheta_1'(0)}\right|^2 exp\{ - \frac{2\pi}{\mathrm{Im}\ \tau}(\mathrm{Im}(z_1 - z_2))^2\}. \tag{4.5}$$

The factor $\left(\prod_i \vartheta\left[{\frac{1}{2}+c_i \atop \frac{1}{2}+d_i}\right](0)\right)^{-1}$ in (4.4) is the twist-structure-dependent contribution of the twisted bosons $X^i, X^{\bar{i}}$ to the partition function. All the remaining τ-dependent contribution of the various fields to the partition function has been lumped into an overall normalization constant N. To determine N we factor the above correlator on the partition function of this orbifold theory by taking the limit $z_1 \to z_2$, $\bar{z}_1 \to \bar{z}_2$. The latter in a given twist structure $\{(c_i, d_i)\}$ and spin structure (a, b) and (a', b') for the right and left handed fermions, is given by:

$$\begin{aligned}\langle\ I\ \rangle =&\left(\frac{1}{4}\right)\left(\frac{27}{3}\right)\left(\frac{1}{128\pi^4}\right)\frac{1}{(\mathrm{Im}\ \tau)^3}(\eta(\tau))^{-3}(\overline{\eta(\tau)})^{-15}\\ &\left|\vartheta\begin{bmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_2\\ \frac{1}{2}+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_3\\ \frac{1}{2}+d_3\end{bmatrix}(0)\right|^{-2}\\ &\left(\delta_{(a,b)}\right)\left[\vartheta\begin{bmatrix}a+c_1\\ b+d_1\end{bmatrix}(0)\vartheta\begin{bmatrix}a+c_2\\ b+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}a+c_3\\ b+d_3\end{bmatrix}(0)\right]\\ &\left[\vartheta\begin{bmatrix}a'+c_1\\ b'+d_1\end{bmatrix}(0)\vartheta\begin{bmatrix}a'+c_2\\ b'+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}a'+c_3\\ b'+d_3\end{bmatrix}(0)\vartheta^{13}\begin{bmatrix}a'\\ b'\end{bmatrix}(0)\right]^*.\end{aligned} \tag{4.6}$$

In the above expression the factor of 27 for $c_i \not\equiv 0$, *i.e.* in the twisted sector, comes from the fact that there are 27 fixed points in this orbifold model, each contributing equally to the partition function of the twisted sector. On the other hand, in the sector where $c_i \equiv 0$, *i.e.* in the untwisted sector the same factor of 27 appears. The reason for this is that the partition function for a pair of bosons with $c = 0$ and twisted in the τ direction by d and $1 - d$ is given by $(1 - e^{2\pi i d})\eta(\tau)/\vartheta\left[{\frac{1}{2} \atop \frac{1}{2}+d}\right]$ instead of $\eta(\tau)/\vartheta\left[{\frac{1}{2} \atop \frac{1}{2}+d}\right]$ as is the case for $c_i \not\equiv 0$. This accounts for the multiplicative factor $|1 - e^{2\pi i/3}|^2|1 - e^{2\pi i/3}|^2|1 - e^{-4\pi i/3}|^2 = 27$ in the partition function for that sector. The factors of $\frac{1}{3}$ and $\frac{1}{4}$ appear since we must average over twist and spin structures in the τ-direction instead of summing over them. Using eq. (4.4) and (4.6) we find that:

$$N = (\frac{9}{512\pi^4})(\vartheta_1'(0))^2(\bar{\vartheta}_1'(0))^2(\eta(\tau))^{-3}(\overline{\eta(\tau)})^{-15}\frac{1}{(\text{Im }\tau)^3}. \tag{4.7}$$

At this stage it may seem that m^2 in (4.4) vanishes trivially because of the on-shell condition $k^2 = 0$. However as in the calculation in sec. 3 the integral over z_1 in (4.1) in the region $z_1 \to z_2$ generates a factor of $1/k^2$. Therefore $A(k)$ in (4.1) contains a term of the form k^2/k^2 which is somewhat ambiguous. This is of course the same problem that occurred previously. As we did earlier, we will refrain from implementing the on-shell condition $k^2 = 0$ until the term proportional to k^2/k^2 has been extracted. To justify this procedure we also calculate below the four point function involving appropriate fermions on the Z_3 orbifold. From that we isolate the two point function for the $U(1)$ charged scalars by factorization. As we shall see below, both results actually agree. One should view this agreement as a check on the validity of the corresponding factorization in the previous arbitrary background calculation.

To extract the (k^2/k^2) term in (4.1), we first carry out the sum over even spin structures for the right-handed fermions in (4.4) using the following ϑ-identity:

$$\begin{aligned}&\sum_{\substack{(a,b)\\ even}} \delta_{(a,b)}\vartheta\begin{bmatrix}a+c_1\\ b+d_1\end{bmatrix}(z_1-z_2)\vartheta\begin{bmatrix}a+c_2\\ b+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}a+c_3\\ b+d_3\end{bmatrix}(0)\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z_1-z_2)\\ &= \vartheta\begin{bmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{bmatrix}(z_1-z_2)\vartheta\begin{bmatrix}\frac{1}{2}+c_2\\ \frac{1}{2}+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_3\\ \frac{1}{2}+d_3\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z_1-z_2).\end{aligned} \tag{4.8}$$

This gives:

$$m^2 = \frac{(g^2)}{4\pi^2}k^2 \int d^2\tau \quad N \quad \int d^2 z_1 \int d^2 z_2 (\chi_{12})^{-k^2}$$

$$\sum_{\{(c_i,d_i)\}} \Big[\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{smallmatrix}\right](z_1-z_2)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_1-z_2)\vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{smallmatrix}\right](0)}\Big]\Big[\frac{1}{(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_1-z_2))^2\prod_i \vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_i\\ \frac{1}{2}+d_i\end{smallmatrix}\right](0)}\Big]^*$$

$$\Big[\sum_{(a',b')}\vartheta\begin{bmatrix}a'+c_1\\ b'+d_1\end{bmatrix}(z_1-z_2)\vartheta\begin{bmatrix}a'+c_2\\ b'+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}a'+c_3\\ b'+d_3\end{bmatrix}(0)\vartheta\begin{bmatrix}a'\\ b'\end{bmatrix}(z_1-z_2)\vartheta^{12}\begin{bmatrix}a'\\ b'\end{bmatrix}(0)\Big]^*. \tag{4.9}$$

We next Taylor expand in powers of $(z_1 - z_2)$ and $(\bar{z}_1 - \bar{z}_2)$ and isolate the singularity. The most singular term that goes as $(z_1 - z_2)^{-1-k^2}(\bar{z}_1 - \bar{z}_2)^{-2-k^2}$ vanishes by phase integration. The subleading singularity $(z_1 - z_2)^{-1-k^2}(\bar{z}_1 - \bar{z}_2)^{-1-k^2}$ survives to generate the desired $1/k^2$ factor after integration over z_1. The final result takes the form:

$$m^2 = -\Big(\frac{9}{256\pi^4}\Big)g^2 \int \frac{d^2\tau}{(\text{Im }\tau)^2}$$

$$\left[(\eta(\tau))^{-15}\sum_{(c_i,d_i)}\sum_{(a',b')}\frac{\vartheta^{13}\left[\begin{smallmatrix}a'\\ b'\end{smallmatrix}\right](0)\vartheta'\left[\begin{smallmatrix}a'+c_1\\ b'+d_1\end{smallmatrix}\right](0)\vartheta\left[\begin{smallmatrix}a'+c_2\\ b'+d_2\end{smallmatrix}\right](0)\vartheta\left[\begin{smallmatrix}a'+c_3\\ b'+d_3\end{smallmatrix}\right](0)}{\prod_i \vartheta\left[\begin{smallmatrix}\frac{1}{2}+c_i\\ \frac{1}{2}+d_i\end{smallmatrix}\right](0)}\right]^*, \tag{4.10}$$

where we have substituted eq. (4.7) for N and used the fact that $\vartheta_1'(0) = -2\pi\eta^3(\tau)$.

One can check by examining the modular transformation properties of the expression inside the square brackets in (4.10) that it is a modular form of weight zero. Furthermore one can check that it is bounded as $\tau \to i\infty$. However the only bounded modular form of weight zero is a constant [34]. Evaluating the expression at $\tau = i\infty$ gives the value of that constant. We find a value of -256π.

Putting this in (4.10) we finally get:

$$m^2 = (\frac{9}{\pi^3})g^2 \int \frac{d^2\tau}{(\text{Im } \tau)^2} \\ = \frac{3g^2}{\pi^2}, \tag{4.11}$$

where the integration over τ covers the fundamental domain of the modular group.

Let us now compare (4.11) with what we would expect from the general expression in (3.66). The massless spectrum of the Z_3 orbifold theory contains thirty six generations, each generation containing a **26** of $SO(26)$ carrying $U(1)$ charge $\frac{1}{\sqrt{3}}$, and an $SO(26)$ singlet, carrying $U(1)$ charge $-\frac{2}{\sqrt{3}}$. The scalar whose mass is being calculated is of the former type. Substituting these numbers into eq.(3.66) and using the scalar mass formula $m^2 = gh\sum_a q^{(a)}c^{(a)}$, we can verify that eqs.(4.11) and (3.66) agree with each other.

B. FOUR POINT FUNCTION

To justify the above analysis we now turn to an orbifold amplitude involving four space-time fermions. In particular we shall calculate the amplitude

$$A(k_1,k_2,k_3,k_4) = \frac{g^4}{(2\pi)^4}\int d^2\tau \int \prod_i d^2 z_i \\ \langle V_{-\frac{1}{2}}(u_1,k_1,z_1)V_{\frac{1}{2}}(u_2,k_2,z_2)V_{-\frac{1}{2}}(u_3,k_3,z_3)V_{\frac{1}{2}}(u_4,k_4,z_4)\rangle, \tag{4.12}$$

where the fermionic vertex operators are chosen such that this amplitude factors on the two point function for the $U(1)$ charged scalars in the s-channel. To isolate this two point function we shall look for the coefficient of the $1/(k_1 \cdot k_2)^2$ pole in (4.12) as has been discussed in section three.

More explicitly we shall take the vertices appearing in (4.11) as follows:

$$\begin{aligned} V_{-\frac{1}{2}}(u_1,k_1,z_1) &= (\lambda^{M_1}(\bar z_1)\lambda^{N_1}(\bar z_1))u_1^{\alpha_1}S_{\alpha_1}(z_1)S_g^-(z_1)e^{ik_1\cdot X(z_1)} \\ V_{\frac{1}{2}}(u_2,k_2,z_2) &= (\lambda^{\bar N_2}(\bar z_2)\lambda^{i_2}(\bar z_2))u_2^{\alpha_2}S_g^+(z_2)\Big[\lim_{w_2\to z_2}(w_2-z_2)^{\frac{1}{2}} \\ &\times[\psi^{\mu_2}(w_2)\partial X^{\mu_2}(w_2)+\psi^{\bar j}(w_2)\partial X^{j}(w_2)+\psi^{j}(w_2)\partial X^{\bar j}(w_2)]S_{\alpha_2}(z_2)e^{ik_2\cdot X(z_2)}\Big] \\ V_{-\frac{1}{2}}(u_3,k_3,z_3) &= (\lambda^{\bar M_3}(\bar z_3)\lambda^{\bar N_3}(\bar z_3))u_3^{\alpha_3}S_{\alpha_3}(z_3)S_g^-(z_3)e^{ik_3\cdot X(z_3)} \\ V_{\frac{1}{2}}(u_4,k_4,z_4) &= (\lambda^{N_4}(\bar z_4)\lambda^{\bar i_4}(\bar z_4))u_4^{\alpha_4}S_g^+(z_4)\Big[\lim_{w_4\to z_4}(w_4-z_4)^{\frac{1}{2}} \\ &\times[\psi^{\mu_4}(w_4)\partial X^{\mu_4}(w_4)+\psi^{\bar j}(w_4)\partial X^{j}(w_4)+\psi^{j}(w_4)\partial X^{\bar j}(w_4)]S_{\alpha_4}(z_4)e^{ik_4\cdot X(z_4)}\Big]. \end{aligned} \tag{4.13}$$

Here M_i, N_i run over $7,...,32$ and we also choose $N_1 \neq M_1$ and $N_3 \neq M_3$; μ_i are four dimensional uncompactified vector indices. To ensure that $V_{-\frac{1}{2}}V_{\frac{1}{2}}$ factors on the correct vertex for the $U(1)$ charged scalar we furthermore need to choose:

$$\begin{aligned} \alpha_1 &= +++(--\ \ or\ \ ++), \qquad \alpha_3 ---(+-\ \ or\ \ -+), \\ \alpha_2 &= --+(--\ \ or\ \ ++), \qquad \alpha_4 ++-(+-\ \ or\ \ -+), \end{aligned} \tag{4.14}$$

for the spinor polarizations in the helicity basis. The terms inside the parantheses stand for the four-dimensional polarizations.

We first analyse the right-handed sector in a given twist structure $\{(c_i, d_i)\}$. Among all the terms that could have contributed to (4.12), the only term that does not vanish as a consequence of some Riemann ϑ-identity is the following:

$$
\begin{aligned}
\Lambda =& (u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\dot\alpha_3} u_4^{\dot\alpha_4}) \langle S_g^-(z_1) S_g^+(z_2) S_g^-(z_3) S_g^+(z_4)\rangle \\
& \lim_{w_4 \to z_4} (w_4 - z_4)^{\frac{1}{2}} \lim_{w_2 - z_2} (w_2 - z_2)^{\frac{1}{2}} \langle S_{+++}(z_1) S_{--+}(z_2) S_{---}(z_3) S_{++-}(z_4)\rangle \\
& \times \langle S_{\alpha_1}(z_1) S_{\alpha_2}(z_2) S_{\dot\alpha_3}(z_3) S_{\dot\alpha_4}(z_4) \psi^{\mu_2}(w_2) \psi^{\mu_4}(w_4)\rangle \\
& \times \langle \partial X^{\mu_2}(w_2) \partial X^{\mu_4}(w_4) \prod_i e^{ik_i \cdot X(z_i)}\rangle .
\end{aligned}
\tag{4.15}
$$

In the above expression, all polarizations are now four-dimensional. In evaluating (4.15) we use many of the results and techniques given in ref. [29] for the evaluation of fermionic amplitudes in flat ten-dimensional space, to which we refer the reader for further details. Here we suppress many of the details and instead highlight some of the main features pertaining to this calculation.

A simple group theoretic analysis shows that Λ has 4 independent tensor structures under the Lorentz group. (This is the number of independent singlets in $(2,1) \otimes (2,1) \otimes (1,2) \otimes (1,2) \otimes (2,2) \otimes (2,2)$ representation of $SU(2) \times SU(2)$). We write down the general expansion for the correlator as:*

---

* We are using a normalization scheme where $\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$.

$$\langle S_{\alpha_1}(z_1)S_{\alpha_2}(z_2)S_{\dot\alpha_3}(z_3)S_{\dot\alpha_4}(z_4)\psi^{\mu_2}(w_2)\psi^{\mu_4}(w_4)\rangle\langle S_g^-(z_1)S_g^+(z_2)S_g^-(z_3)S_g^+(z_4)\rangle$$

$$\langle S_{+++}(z_1)S_{--+}(z_2)S_{---}(z_3)S_{++-}(z_4)\rangle$$

$$=A\delta_{\alpha_1\alpha_2}\delta_{\dot\alpha_3\dot\alpha_4}\delta^{\mu_2\mu_4}+B(\gamma^{\mu_2}\gamma^{\mu_4})_{\alpha_2\alpha_1}\delta_{\dot\alpha_3\dot\alpha_4}$$

$$+C(\gamma^{\mu_4}\gamma^{\mu_2})_{\dot\alpha_4\dot\alpha_3}\delta_{\alpha_1\alpha_2}+D(\gamma^{\mu_4}\gamma^{\rho})_{\dot\alpha_4\dot\alpha_3}(\gamma^{\mu_2}\gamma^{\rho})_{\alpha_2\alpha_1}. \tag{4.16}$$

The individual Lorentz invariant coefficients (A, B, C, D) can be evaluated by calculating the correlator on the left-hand side for particular polarizations that contribute to that Lorentz structure only. For example A in (4.16) is given by the correlator on the left hand side with polarizations $\alpha_1 = ++, \alpha_2 = --, \dot\alpha_3 = -+, \dot\alpha_4 = +-, \mu_2 = \bar{4}$ and $\mu_4 = 4$ and so on. The answer for Λ can be written in the following form:

$$\Lambda = \prod_{i<j}\chi_{ij}^{k_i\cdot k_j}$$

$$\Big\{(\bar u_2u_1)(\bar u_4u_3)\big[(k_1\cdot k_4)(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)+(k_1\cdot k_2)(\Lambda_1+\Lambda_4)\big] \tag{4.17}$$

$$+(\bar u_2\ \not k_3\gamma^\rho u_1)(\bar u_4\gamma_\rho\ \not k_1u_3)\Lambda_5\Big\},$$

where

$$\Lambda_1 = +K\vartheta_1'(0)\big[(\vartheta_1(z_2-z_4))^{-3}(\vartheta_1(z_1-z_2)\vartheta_1(z_1-z_3)\vartheta_1(z_3-z_4))^{-1}$$

$$\vartheta_1(z_1-z_4)\vartheta_1(z_2-z_3)\big]$$

$$\Big[\sum_{(a,b)}(\delta_{(a,b)})\vartheta\begin{bmatrix}a+c_1\\b+d_1\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})\vartheta\begin{bmatrix}a+c_2\\b+d_2\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a+c_3\\b+d_3\end{bmatrix}(\frac{z_1+z_2-z_3-z_4}{2})\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-3z_2-z_3+3z_4}{2})\Big], \tag{4.18}$$

$$\Lambda_2 = +K\big[(\vartheta_1(z_2-z_4))^{-2}(\vartheta_1(z_1-z_3)\vartheta_1(z_3-z_4))^{-1}\vartheta_1(z_2-z_3)\big]$$

$$[\frac{\partial}{\partial z_2}\ln\vartheta_1(z_2-z_1) - \frac{\partial}{\partial z_2}\ln\vartheta_1(z_2-z_3) + \frac{2i\pi}{\operatorname{Im}\tau}\operatorname{Im}(z_3-z_1)]$$

$$\Big[\sum_{(a,b)}(\delta_{(a,b)})\vartheta\begin{bmatrix}a+c_1\\b+d_1\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})\vartheta\begin{bmatrix}a+c_2\\b+d_2\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a+c_3\\b+d_3\end{bmatrix}(\frac{z_1+z_2-z_3-z_4}{2})\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1+z_2+z_3-3z_4}{2})\vartheta^{-1}\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-z_2+z_3-z_4}{2})\Big],$$

(4.19)

$$\Lambda_3 = +K\big[(\vartheta_1(z_2-z_4))^{-2}(\vartheta_1(z_1-z_3)\vartheta_1(z_1-z_2))^{-1}\vartheta_1(z_1-z_4)\big]$$

$$[\frac{\partial}{\partial z_4}\ln\vartheta_1(z_4-z_3) - \frac{\partial}{\partial z_4}\ln\vartheta_1(z_4-z_1) + \frac{2i\pi}{\operatorname{Im}\tau}\operatorname{Im}(z_1-z_3)]$$

$$\Big[\sum_{(a,b)}(\delta_{(a,b)})\vartheta\begin{bmatrix}a+c_1\\b+d_1\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})\vartheta\begin{bmatrix}a+c_2\\b+d_2\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a+c_3\\b+d_3\end{bmatrix}(\frac{z_1+z_2-z_3-z_4}{2})\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-3z_2+z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})\vartheta^{-1}\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1-z_2+z_3-z_4}{2})\Big],$$

(4.20)

while

$$\Lambda_4 = +K(\vartheta_1'(0))^{-1}\big[(\vartheta_1(z_1-z_3)\vartheta_1(z_2-z_4))^{-1}\big]\big\{M_{31}-M_{34}-M_{21}+M_{24}\big\}$$

$$\Big[\sum_{(a,b)}(\delta_{(a,b)})\vartheta\begin{bmatrix}a+c_1\\b+d_1\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})\vartheta\begin{bmatrix}a+c_2\\b+d_2\end{bmatrix}(\frac{z_1-z_2-z_3+z_4}{2})$$

$$\vartheta\begin{bmatrix}a+c_3\\b+d_3\end{bmatrix}(\frac{z_1+z_2-z_3-z_4}{2})\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\frac{z_1+z_2-z_3-z_4}{2})\Big].$$

(4.21)

Finally,

$\Lambda_5 = \Lambda_4$ with the replacement of the factor in the curly brackets by

$$\{M_{23} - M_{24} - M_{13} + M_{14}\}, \tag{4.22}$$

where in (4.18),(4.19) and (4.21) M_{ij} is the matrix given by,

$$M_{ij} = (\frac{\partial}{\partial z_4} \ln \vartheta_1(z_4 - z_i) - \frac{2\pi i}{\text{Im } \tau}\text{Im } z_i)(\frac{\partial}{\partial z_2} \ln \vartheta_1(z_2 - z_j) - \frac{2\pi i}{\text{Im } \tau}\text{Im } z_j) \tag{4.23}$$

Furthermore K is an overall normalization that is determined by factoring the correlator on the partition function. We shall fix K below after we include the contribution of the left-handed sector (antianalytic sector).

One could attempt to simplify (4.17), however here we are only interested in isolating the contribution of the two point function for the $U(1)$ charged scalars in the s-channel. As mentioned earlier this entails isolating a pole of the form $(k_1 \cdot k_2)^{-2}$. Therefore it is only necessary to isolate the singularities in (4.17) in the limit $z_1 \to z_2$ and/or $z_3 \to z_4$, and then $z_2 \to z_4$. This is what we shall do now.

Consider the term in (4.17) proportional to $(k_1 \cdot k_2)$. For that term to contribute to the $\frac{1}{(k_1 \cdot k_2)^2}$ pole $\Lambda_1 + \Lambda_4$ has to be singular enough in the above limit to yield a $\frac{1}{(k_1 \cdot k_2)^3}$ pole. After summing over spin structures using a Riemann ϑ-identity, one can see that Λ_1 has an explicit factor of $\vartheta_1(z_1 - z_2 - z_3 + z_4)$ in the numerator and hence is not sufficiently singular to give the $1/(k_1 \cdot k_2)^3$ pole. On the other hand Λ_4 in the limit $z_1 \to z_2$, $z_3 \to z_4$ has the required singularity so as to yield a term proportional to $\frac{1}{(k_1 \cdot k_2)^3}$ after we carry out the integration, as we shall explicitly see below.

We next examine the contribution of the terms in (4.17) proportional to $k_1 \cdot k_4$. In this case the sum $\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ needs to develop only a $\frac{1}{(k_1 \cdot k_2)^2}$ pole. This is equivalent to the sum developing a singularity as $z_1 \to z_2$ *or* as $z_3 \to z_4$. However it is not difficult to check that both $\Lambda_1 + \Lambda_2$ and $\Lambda_3 + \Lambda_4$ are singularity free in the limit $z_1 \to z_2$, while in the limit $z_3 \to z_4$, $\Lambda_1 + \Lambda_3$ and $\Lambda_2 + \Lambda_4$ are singularity free. Therefore the sum $\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ does not develop any singularity in either limit and hence cannot give rise to any $\frac{1}{(k_1 \cdot k_2)^2}$ pole upon integration. The same is actually true for the term proportional to Λ_5. The above analysis shows that Λ_4 is the only contribution from the right-handed fermions to the charged scalar two point function in the s-channel.

To complete our analysis we need to calculate the contribution of the anti-analytic sector in the same limit; namely as $\bar{z}_1 \to \bar{z}_2$, $\bar{z}_3 \to \bar{z}_4$ and subsequently $\bar{z}_2 \to \bar{z}_4$. The relevant correlator for the left-handed gauge fermions is given by,

$$I = \langle \lambda^{M_1}(\bar{z}_1)\lambda^{N_1}(\bar{z}_1)\lambda^{\bar{N}_2}(\bar{z}_2)\lambda^{i_2}(\bar{z}_2)\lambda^{\bar{M}_3}(\bar{z}_3)\lambda^{\bar{N}_3}(\bar{z}_3)\lambda^{N_4}(\bar{z}_4)\lambda^{\bar{i}_4}(\bar{z}_4)\rangle. \qquad (4.24)$$

We shall take $M_1 \neq N_1$, $M_3 \neq N_3$, $N_2 = N_1, N_3 = N_4, M_1 = M_3, i_2 = 1$ and $\bar{i}_4 = \bar{1}$. With these polarizations (4.24) will factorize on the antianalytic part of the $U(1)$ charged scalar vertex. The answer for the correlator in (4.24) can be written as,

$$\begin{aligned} I =& \bar{K}\Big(\big(\vartheta_1(z_1 - z_2)\vartheta_1(z_3 - z_4)\vartheta_1(z_1 - z_3)\vartheta_1(z_1 - z_4)\big)^*\Big)^{-1} \\ & \Bigg[\sum_{(a',b')} \vartheta^{10}\begin{bmatrix} a' \\ b' \end{bmatrix}(0)\vartheta\begin{bmatrix} a' \\ b' \end{bmatrix}(z_1 - z_2)\vartheta\begin{bmatrix} a' \\ b' \end{bmatrix}(z_3 - z_4)\vartheta\begin{bmatrix} a' \\ b' \end{bmatrix}(z_1 - z_3) \\ & \qquad \vartheta\begin{bmatrix} a' + c_1 \\ b' + d_1 \end{bmatrix}(z_2 - z_4)\vartheta\begin{bmatrix} a' + c_2 \\ b' + d_2 \end{bmatrix}(0)\vartheta\begin{bmatrix} a' + c_3 \\ b' + d_3 \end{bmatrix}(0)\Bigg]^*. \end{aligned} \qquad (4.25)$$

We are now ready to combine our results given in Eqs.(4.21) and (4.25) and extract the $(k_1 \cdot k_2)/(k_1 \cdot k_2)^3$ term. For that we only need to isolate the singularities in (4.17) in the limit $z_1 \to z_2$, $z_3 \to z_4$ and subsequently $z_2 \to z_4$.

In a given twist structure we get

$$
\begin{aligned}
A_{(c_i,d_i)}(k_1\cdot k_2) = & -2(k_1\cdot k_2)\left(\frac{g^4}{(2\pi)^4}\right) \\
& \int d^2z_4 \int \frac{d^2z_2}{|z_2-z_4|^{2+4k_1\cdot k_2}} \int \frac{d^2z_1}{|z_1-z_2|^{2-2k_1\cdot k_2}} \int \frac{d^2z_3}{|z_3-z_4|^{2-2k_1\cdot k_2}} \\
& (K\bar{K})(\bar{u}_2u_1)(\bar{u}_4u_3)(\vartheta_1'(0))^{-2}(\bar{\vartheta}_1'(0))^{-4} \\
& \left[\vartheta\begin{bmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_2\\ \frac{1}{2}+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_3\\ \frac{1}{2}+d_3\end{bmatrix}(0)\right] \\
& \left[\sum_{(a',b')}\vartheta^{13}\begin{bmatrix}a'\\ b'\end{bmatrix}(0)\vartheta'\begin{bmatrix}a'+c_1\\ b'+d_1\end{bmatrix}(0)\bar{\vartheta}\begin{bmatrix}a'+c_2\\ b'+d_2\end{bmatrix}(0)\bar{\vartheta}\begin{bmatrix}a'+c_3\\ b'+d_3\end{bmatrix}(0)\right]^*.
\end{aligned}
\tag{4.26}
$$

Again we have retained only terms that do not vanish by phase integration. The normalization $K\bar{K}$ can be determined by factoring the four point function on the identity operator and comparing with (4.6). We find that

$$
\begin{aligned}
(K\bar{K}) = & \left(\frac{9}{512\pi^4}\right)\frac{1}{(\mathrm{Im}\ \tau)^2}(\vartheta_1'(0))^3(\bar{\vartheta}_1'(0))^4(\eta(\tau))^{-3}(\overline{\eta(\tau)})^{-15} \\
& \left|\vartheta\begin{bmatrix}\frac{1}{2}+c_1\\ \frac{1}{2}+d_1\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_2\\ \frac{1}{2}+d_2\end{bmatrix}(0)\vartheta\begin{bmatrix}\frac{1}{2}+c_3\\ \frac{1}{2}+d_3\end{bmatrix}(0)\right|^{-2}.
\end{aligned}
\tag{4.27}
$$

It is not difficult to verify that after carrying out the integrals in (4.26), using (4.27), and summing over all twist structures the answer reduces to the expected form,

$$
-(2g)^2\bar{u}_2u_1\bar{u}_4u_3\frac{m^2}{(2k_1\cdot k_2)^2}, \tag{4.28}
$$

where m is the scalar mass calculated in eq.(4.11).

5. CONCLUSION

In this paper we have calculated the one-loop Fayet-Iliopoulos D-terms generated in string theories compactified on an arbitrary background preserving space-time supersymmetry at string tree level. In particular we have calculated the one loop charged scalar masses and shown that they are in general non-zero if the four dimensional gauge group of the theory contains one or more $U(1)$ factors. From this we may extract the coefficient of the linear term in D, generated at the one loop order. The actual value of these terms is shown to depend on the background only through the massless spectrum of the theory and hence is in principle calculable through an index theorem. Furthermore, in the process we have also shown that the calculation of the scalar mass term at the one loop ultimately reduces to the calculation of the expectation value of a dimension $(1,1)$ operator. This indicates that we may interpret this operator as the vertex for the D-term. Additional evidence for this interpretation was obtained by analysing the space-time supersymmetry transformation properties of this operator. As is shown in appendix A this operator indeed transforms in the right way to be the vertex for the D-term.

We would also like to mention that in the calculation of the scalar masses, had we apriori set the momentum to zero (or set $k^2 = 0$) in the corresponding vertex operators, we would have found no contribution to the mass term. As we saw in section 3 and 4, integration over the location of the vertices on the string world-sheet generates a factor of $1/k^2$, which cancels a factor of k^2 in the numerator and gives a non-zero answer. This indicates that in general, the decoupling of a zero momentum vertex operator does not necessarily imply the decoupling of the corresponding state at zero momentum. Arguments relying on decoupling of zero momentum vertex operators should therefore be made with some degree of caution.

Finally we discuss the significance of our calculation for the fate of a tree-level string vacuum which is destabilized [8] by one loop D-terms. In particular, the sign of the D-term coefficient $c^{(a)}$ in (3.66)—in conjunction with data about the massless spectrum and string tree-level couplings—enables one to determine whether or not nearby stable vacua exist in which certain massless fields are given expectation values of order g [11].

For example, the sign in (3.66) leads to the conclusion that all standard Calabi-Yau compactifications of the $Spin(32)/Z_2$ heterotic string can be stabilized in this way [11]. To see this, first consider the massless spectrum of a compactification on a Calabi-Yau manifold with Betti numbers $b_{1,1}, b_{2,1}$, etc., and Euler characteristic $\chi = 2(b_{1,1} - b_{2,1})$. The four dimensional gauge group is $SO(26) \times U(1)$. At the first order of σ-model perturbation theory, there are $b_{1,1}$ positive chirality scalar multiplets transforming as $(\mathbf{26}, \frac{1}{\sqrt{3}}) \oplus (\mathbf{1}, -\frac{2}{\sqrt{3}})$ and $b_{2,1}$ multiplets transforming as $(\mathbf{26}, -\frac{1}{\sqrt{3}}) \oplus (\mathbf{1}, +\frac{2}{\sqrt{3}})$. So for $\chi \neq 0$ the $U(1)$ is 'anomalous'.

Suppose for the sake of the argument that $\chi > 0$. Then at least $b_{1,1} - b_{2,1}$ of the $(\mathbf{1}, -\frac{2}{\sqrt{3}})$ multiplets –call them S_I– must remain exactly massless at string tree-level. (In fact there are general arguments that none of the $O(26)$ singlets can acquire a mass from either perturbative or non-perturbative [37] σ-model effects, but we do not need this strong a result here.) Furthermore by $U(1)$ gauge invariance any term in the superpotential containing an S_I field contains at least two other types of fields (since the mass terms involving S_I are absent by assumption). Thus the string tree-level F-terms do not prevent expectation values for any combination of the S_I. Furthermore, the charge of the S_I fields has the correct sign to cancel the one-loop D-term if they are given appropriate vacuum expectation values. This is because the quantity $c = \frac{g}{192\pi^2} \sum_i n_i q_i h_i = \frac{g}{192\pi^2} \cdot 2 \cdot \frac{24}{\sqrt{3}}(b_{1,1} - b_{2,1})$ is positive for $\chi > 0$, whereas the tree-level couplings of $\langle S_I \rangle$ to D is negative: $q_S |\langle S_I \rangle|^2 < 0$. If $\chi < 0$, the same argument shows that vacuum expectation values for the $b_{2,1} - b_{1,1}$ massless fields $\tilde{S}_I$ transforming as $(\mathbf{1}, +\frac{2}{\sqrt{3}})$ can be chosen to cancel the one-loop D-term. Finally if $\chi = 0$, no D-term is generated.

Other examples of supersymmetric string vacua with nonzero one-loop D-terms are provided by $(2,0)$ orbifolds. The question of whether stable shifted vacua exist in these models is difficult to answer except on a case-by-case basis. The models have a variety of four dimensional gauge groups and massless field quantum numbers. One must now ensure that any vacuum expectation values for massless scalars do not generate D-terms for non-anomalous factors in the gauge group, as well as not generating F-terms. For the specific models we have analysed, however, there is always some combination of scalar expectation values which cancels the one-loop D-term. We conjecture that such 're-stabilization' is possible in any supersymmetric compactification, but we have no explanation for why this should be so. Work along these lines is in progress.

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APPENDIX A

Vertex Operators For Auxiliary Fields

It was remarked in section 3 that the operator $H(z)U^{(a)}(\bar{z})$ may be regarded as the vertex operator for the auxiliary field $D^{(a)}$ associated with the $U^{(a)}(1)$ gauge current. More generally, if $U^{(a)}(\bar{z})$ denotes the current associated with a non-Abelian symmetry, the vertex operator for the corresponding auxiliary $D^{(a)}$ field will be given by $H(z)U^{(a)}(\bar{z})$. In this appendix we show that these operators do satisfy the correct commutation relations with the space-time supersymmetry charge so as to be interpretable as the vertex operator for auxiliary D fields. We also construct the vertex operator for the auxiliary F fields for any chiral supermultiplet.

If $(A_\mu^{(a)}, \lambda_\alpha^{(a)}, \bar{\lambda}_{\dot{\alpha}}^{(a)}, D^{(a)})$ denote the vector supermultiplet,* then in the Wess-Zumino gauge the supersymmetry transformation laws of various fields are given by,

$$
\begin{aligned}
[Q_\alpha, A_\mu^{(a)}] &= -\frac{i}{\sqrt{2}}(\gamma_\mu)_{\alpha\dot{\beta}}\bar{\lambda}^{(a)\dot{\beta}} \\
\{Q_\alpha, \lambda^{(a)\beta}\} &= -\frac{i}{\sqrt{2}}\delta_\alpha^\beta D^{(a)} - \frac{i}{\sqrt{2}}(\gamma^\mu\gamma^\nu)_\alpha{}^\beta(k_\nu A_\mu^{(a)} - k_\mu A_\nu^{(a)}) \\
\{Q_\alpha, \bar{\lambda}^{(a)\dot{\beta}}\} &= 0 \\
[Q_\alpha, D^{(a)}] &= i\frac{1}{\sqrt{2}}(\not{k})_{\alpha\dot{\beta}}\bar{\lambda}^{(a)\dot{\beta}}
\end{aligned}
\tag{A1}
$$

Commutation relations of various fields with $Q_{\dot{\alpha}}$ are given by similar relations with dotted and undotted indices interchanged, and replacing $D^{(a)}$ by $-D^{(a)}$ everywhere. The vertex operators are related to the fields by truncation of the external propagator, hence the vertex operators $(V_\mu^{(a)}, V_\alpha^{(a)}, V_{\dot{\alpha}}^{(a)}, V_D^{(a)})$ for the

---

* Here we follow the notation of *Superspace* [2]

supermultiplet are given by $(k^2 A_\mu^{(a)}, \sqrt{2}(\not k)_{\alpha\dot\beta}\bar\lambda^{(a)\dot\beta}, \sqrt{2}(\not k)_{\dot\alpha\beta}\lambda^{(a)\beta}, D^{(a)})$ respectively. Transformation laws of these vertex operators are given by

$$
\begin{aligned}
[Q_\alpha, \varsigma^\mu V_\mu^{(a)}] &= -i(\not\varsigma\ \not k)_{\alpha\beta} V^{(a)\beta} \\
\{Q_\alpha, u^\beta V_\beta^{(a)}\} &= 0 \\
\{Q_\alpha, u^{\dot\beta} V_{\dot\beta}^{(a)}\} &= -i(\not k)_{\alpha\dot\beta} u^{\dot\beta} V_D^{(a)} - i(\gamma^\mu)_{\alpha\dot\beta} u^{\dot\beta} V_\mu^{(a)} + \frac{i}{k^2}(\not k)_{\alpha\dot\beta} u^{\dot\beta} k^\mu V_\mu^{(a)} \\
[Q_\alpha, V_D^{(a)}] &= i\frac{1}{2} V_\alpha^{(a)}.
\end{aligned}
\tag{A2}
$$

From this we see that on shell, where $(\not k)_{\alpha\dot\beta} u^{\dot\beta} = 0$, the D term disappears from the transformation laws of $V_\mu^{(a)}, V_\beta^{(a)}$ and $V_{\dot\beta}^{(a)}$. However $[Q_\alpha, V_D^{(a)}]$ is given by the fermion emission vertex which does not vanish on-shell, hence the last Eq. (A.2) provides a nontrivial check to see if $-\frac{i}{3}J(z)U^{(a)}(\bar z)$ can be regarded as the vertex operator for the D field. Since

$$
\hat S^+(z) J(w) U^{(a)}(\bar w) \sim -\frac{3}{2}\frac{1}{z-w}\hat S^+(z) U^{(a)}(\bar z),
\tag{A3}
$$

we see that the supersymmetry charge indeed converts the operator $-\frac{i}{3}J(z)U^{(a)}(\bar z)$ to the vertex operator for the emission of a gaugino in the $-\frac{1}{2}$ picture. Hence $J(z)U^{(a)}(\bar z)$ may be interpreted as the vertex operator for the D term.

Next let us turn to the construction of the vertex operators for the auxiliary field F. If (A, ψ_α, F) denote a negative chirality scalar supermultiplet, then the supersymmetry transformation laws take the following form,

$$
\begin{aligned}
[Q_\alpha, A] &= \frac{1}{\sqrt{2}}\psi_\alpha \\
[Q_{\dot\alpha}, A] &= 0 \\
\{Q_\alpha, \psi^\beta\} &= \frac{1}{\sqrt{2}}\delta_\alpha^\beta F \\
\{Q_{\dot\alpha}, \psi_\beta\} &= \sqrt{2}(\not{k})_{\dot\alpha\beta} A \\
[Q_\alpha, F] &= 0 \\
[Q_{\dot\alpha}, F] &= \sqrt{2}(\not{k})_{\dot\alpha\beta}\psi^\beta
\end{aligned}
\tag{A4}
$$

Again introducing the vertex operators $(V_A, V_\psi^{\dot\alpha}, V_F) \equiv (k^2 A, \sqrt{2}(\not{k})^{\dot\alpha}{}_\beta\psi^\beta, F)$ we get,

$$
\begin{aligned}
[Q_{\dot\alpha}, V_A] &= 0 \\
[Q_\alpha, V_A] &= (\not{k})_{\alpha\dot\beta} V_\psi^{\dot\beta} \\
[Q_\alpha, u_{\dot\beta} V_\psi^{\dot\beta}] &= (\not{k})^{\dot\beta}{}_\alpha u_{\dot\beta} V_F \\
[Q_{\dot\alpha}, u_{\dot\beta} V_\psi{}^{\dot\beta}] &= u_{\dot\alpha} V_A \\
[Q_\alpha, V_F] &= 0 \\
[Q_{\dot\alpha}, V_F] &= (V_\psi)_{\dot\alpha}.
\end{aligned}
\tag{A5}
$$

As was the case for V_D, similarly the vertex operator V_F for the auxiliary F field disappears from the transformation laws for the physical fields on-shell; however, the transformation law of V_F does not vanish on-shell and provides a constraint on V_F.

Given any massless scalar superfield, the vertex operator for the scalar component in the -1 picture may be written as,

$$e^{-\phi(z)}\hat{V}_A(z)e^{ik\cdot X} \tag{A6}$$

where $\hat{V}_A(z)$ is a function of the internal fields φ^j. We propose the following form for the auxiliary field vertex operator V_F in the zero picture:

$$V_F(z) \equiv \hat{V}_F(z) \equiv \lim_{w\to z}\left\{(w-z)\epsilon^+(w)\hat{V}_A(z)\right\}, \tag{A7}$$

where ϵ^+ is the holomorphic tensor field introduced in section 2.

In order to show that V_F defined in this fashion does satisfy the commutation relation given in (A.5), we need to show that

$$\hat{S}^-(z)V_F(w) \sim \frac{1}{z-w}\hat{V}_\psi(w) \tag{A8}$$

where $\hat{V}_\psi(w)$ is the internal part of the vertex operator V_ψ, and is related to $\hat{V}_A(z)$ by

$$\hat{S}^+(z)\hat{V}_A(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\hat{V}_\psi(w). \tag{A9}$$

We also have, from the first equation in (A.5)

$$\hat{S}^-(z)\hat{V}_A(w) \sim (z-w)^{\frac{1}{2}}. \tag{A10}$$

In order to prove eq. (A.8) using (A.9) and (A.10) it is most convenient to introduce a pair of free right-moving bosonic fields χ, ξ which commute with

$\hat{S}^{\pm}, \epsilon^{\pm}, \hat{V}_A, \hat{V}_\psi$ and $\hat{V}_F$, and to define the operators:

$$\begin{aligned}
\tilde{S}^-(z) &= \hat{S}^-(z)e^{i(\xi(z)+\chi(z))/2} \\
\tilde{S}^+(z) &= \hat{S}^+(z)e^{i(3\xi(z)+\chi(z))/2} \\
\tilde{\epsilon}^+(z) &= \epsilon^+(z)e^{i\xi(z)} \\
\tilde{V}_\psi(z) &= \hat{V}_\psi(z)e^{i(3\xi(z)-\chi(z))/2} \\
\tilde{V}_A(z) &= \hat{V}_A(z)e^{-i\chi(z)} \\
\tilde{V}_F(z) &= \hat{V}_F(z)e^{i(\xi(z)-\chi(z))}.
\end{aligned} \tag{A11}$$

Using (A.7), (A.9), (A.10) and the operator product

$$\hat{S}^-(z)\epsilon^+(w) \sim (z-w)^{-\frac{3}{2}}\hat{S}^+(w), \tag{A12}$$

we see that the operators defined above are mutually local:

$$\begin{aligned}
\tilde{S}^+(z)\tilde{V}_A(w) &\sim (z-w)^{-1}\tilde{V}_\psi(w) \\
\tilde{S}^-(z)\tilde{V}_A(w) &\sim O(1) \\
\tilde{\epsilon}^+(z)\tilde{V}_A(w) &\sim (z-w)^{-1}\tilde{V}_F(w) \\
\tilde{S}^-(z)\tilde{\epsilon}^+(w) &\sim (z-w)^{-1}\tilde{S}^+(w).
\end{aligned} \tag{A13}$$

Therefore we may mode expand $\tilde{S}^{\pm}(z)$ and $\tilde{\epsilon}^+(z)$ in the plane,

$$\begin{aligned}
\tilde{S}^+(z) &= \sum_n \tilde{S}^+_n z^{-n-1} \\
\tilde{S}^-(z) &= \sum_n \tilde{S}^-_n z^{-n-1} \\
\tilde{\epsilon}^+(z) &= \sum_n \tilde{\epsilon}^+_n z^{-n-1}.
\end{aligned} \tag{A14}$$

where the sums over n do not necessarily run over integers. Using Eqs. (A.13) we get,

$$\begin{aligned} [\tilde{S}_n^+, \tilde{V}_A(w)] &= w^n \tilde{V}_\psi(w) \\ [\tilde{S}_n^-, \tilde{V}_A(w)] &= 0 \\ [\tilde{\epsilon}_n^+, \tilde{V}_A(w)] &= w^n \tilde{V}_F(w) \\ [\tilde{S}_n^-, \tilde{\epsilon}_m^+] &= \tilde{S}_{n+m}^+ \end{aligned} \tag{A15}$$

Then

$$\begin{aligned} w^m [\tilde{S}_n^-, \tilde{V}_F(w)] &= [\tilde{S}_n^-, [\tilde{\epsilon}_m^+, \tilde{V}_A(w)]] \\ &= [[\tilde{S}_n^-, \tilde{\epsilon}_m^+], \tilde{V}_A(w)] - [[\tilde{S}_n^-, \tilde{V}_A(w)], \tilde{\epsilon}_m^+] \\ &= w^{n+m} \tilde{V}_\psi(w), \end{aligned} \tag{A16}$$

i.e.

$$[\tilde{S}_n^-, \tilde{V}_F(w)] = w^n \tilde{V}_\psi(w). \tag{A17}$$

This gives back the operator product,

$$\tilde{S}^-(z) \tilde{V}_F(w) \sim \frac{1}{z-w} \tilde{V}_\psi(w). \tag{A18}$$

Using Eqs. (A.11) we get

$$\hat{S}^-(z) V_F(w) \sim \frac{1}{(z-w)} \hat{V}_\psi(w), \tag{A19}$$

which is the relation required to show that V_F is indeed the vertex operator for the auxiliary field F.

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