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A QUANTUM TREATMENT OF BEAMSTRAHLUNG*

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ABSTRACT

In this paper a straightforward high energy expansion is discussed and applied to the problem of radiation from an extended target. In particular, we discuss bremsstrahlung from an electron-pulse collision. A full quantum treatment of the power spectrum and the average energy loss is given. Scaling laws that smoothly join the quantum regime to the classical limit are derived.

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1. Introduction and Motivation

An important parameter in the design of very high energy electron colliders is the fractional energy loss due to bremsstrahlung as one beam pulse passes through the other pulse.[1] Himel and Siegrist[2] treated this process by adapting a quantum treatment of synchrotron radiation by an electron in a uniform magnetic field given by Sokolov and Ternov.[3–6] This adaptation necessarily involved several assumptions, in particular the approximation of the effects of the pulse by a uniform magnetic field in which the electron was orbiting as it radiated.[7] In fact, the electron sees the rapidly approaching pulse in the collider frame of reference as transverse, mutually orthogonal electric and magnetic fields of equal strengths whose spatial dependence is determined by the distribution of charges in the pulse.

It is the purpose of this note to compute the energy loss more simply and more generally by calculating the radiation of a quantum by a very energetic electron moving through the actual electromagnetic field of a pulse. To this end we find it convenient to work in the rest frame of the pulse which transforms into a very long narrow 'string' of N charges. Since the electric field of the pulse is very strong, it cannot be treated perturbatively. Instead, we employ a high energy scattering approximation which in this case requires retaining corrections beyond the eikonal approximation to one order higher in inverse energy.

The classic use of perturbation theory to describe the bremsstrahlung process from a localized target of low charge Z $(Z\alpha \ll 1)$ was given by Bethe and Heitler.[8] A solution of this problem at high energies for large Z was first given by Bethe and Maximon.[9] A discussion of this high energy phenomenon in a more modern context can be found in Bjorken, Kogut, and Soper.[10]

Here we have developed a method applicable for very extended targets of very large total charge. such as a pulse of finite length in the center of mass frame of the collision which, however, in its own rest frame, has a length proportional to $\gamma\ (= E/m)$. Our approach should be compared with the extensive literature on the eikonal method.[11–14]

2. Summary of Results and Their Physical Interpretation

In this section we will first review the formulas given in Ref. 2 for the average fractional energy loss due to bremsstrahlung by an electron passing through the pulse. We also summarize our new results and discuss their physical interpretation. Using the notation $<>$ to indicate an integration over the cross section *for that part of the incident wave that passes directly through the pulse* (since only those particles are of interest in possible annihilation processes), it becomes

$$\delta = \frac{< k\sigma >}{\pi B^2\, p} \,, \tag{2.1}$$

where B is the radius of the pulse, p is the incident energy and k is the energy of the radiated photon. Note that δ is a Lorentz invariant. The energy of the electron in an annihilation process will be between p and $p(1-\delta)$ depending on whether the interaction occurs near the front or near the rear edge of the pulse. Thus the average energy will be $p(1-\frac{1}{2}\delta)$.

Review:

The result for the fractional energy loss from classical physics for a particle incident at an impact parameter b on a uniformly charged cylinder is

$$\delta_{classical}(b) = \frac{16}{3}\,\frac{\alpha^3 N^2 \gamma}{m^3 \ell_0 B^2} \left(\frac{b^2}{B^2}\right) \,. \tag{2.2}$$

Here N is the total charge of the pulse, ℓ_0 is the length of the pulse in the laboratory (collision center of mass) frame, and the incident energy in this frame is $m\gamma$. It has been assumed that the disruption parameter is small- i.e., the change in impact parameter b during the traversal of the pulse is small and can be ignored. Averaging over the impact parameter then gives

$$\delta_{classical} = \frac{8}{3} \frac{\alpha^3 N^2 \gamma}{m^3 \ell_0 B^2} . \tag{2.3}$$

This classical result excludes all effects of the radiation back on the motion of the radiating electron - i.e., energy loss, recoil, and radiation damping. It is valid only for values of $\delta_{classical} \ll 1$. The parameters pertinent to the SLC lie in this classical regime- i.e.,

$$\begin{aligned} N &\sim 5 \times 10^{10} \qquad & \gamma &= 10^5 \\ B &\sim 1 \times 10^{-4} cm \qquad & \ell_0 &= 10^{-1} cm \end{aligned} \tag{2.4}$$

which gives

$$\delta_{classical}(SLC) \sim 0.014 . \tag{2.5}$$

For higher energies and smaller beam dimensions, as has been suggested for a future extension of the linear collider technique, the energy loss given by (2.3) will grow. Clearly a quantum treatment of the scattering and radiation process is then required since the photon will characteristically carry off a large fraction of the available energy. In Ref. 2 , Himel and Siegrist have given an approximate formula by tying on to the calculation of the power spectrum for synchrotron radiation by an electron moving in a uniform magnetic field. Their result can be written[15]

$$\delta < \delta_{classical} \frac{3}{2} \left[\frac{m^2 \ell_0 B}{6 \gamma N \alpha} \right]^{4/3} , \tag{2.6}$$

or

$$\frac{\delta}{\delta_{classical}} < \left(\frac{2}{3} \right)^{1/3} [C]^{4/3} = 0.87 \ldots [C]^{4/3} \tag{2.7}$$

in terms of the scaling variables y, and C:

$$y = \frac{N\alpha}{mB} \qquad\qquad C = \frac{m\ell_0}{4\gamma y} \ . \tag{2.8}$$

Both of these will play an important role in our development and will also be convenient for later use. Note that y is a purely classical variable, whereas C is inversely proportional to $\hbar$.[16] For the classical regime, $\hbar \to 0$ and $C \gg 1$. This is the regime of the SLC for which the parameters given in (2.4) lead to $C \approx 50$ and $y \approx 140$. Note also that (2.3) is independent of $\hbar$ since $\alpha/m = r_e = 2.8 \times 10^{-13}$cm, the classical electron radius. In terms of these scaling variables (2.3) becomes

$$\delta_{classical} = \frac{2y\alpha}{3C} \ . \tag{2.9}$$

For a *super* linear collider defined with parameters

$$\begin{aligned} N &\sim 3 \times 10^8 & \gamma &= 10^7 \\ B &\sim 5 \times 10^{-8} cm & \ell_0 &= 3 \times 10^{-5} cm \end{aligned} \tag{2.10}$$

the scaling variables become $C \approx 10^{-5}$ and $y \approx 2 \times 10^{+3}$. Thus this 'super' is well into the quantum regime.

<u>New Results:</u>

In this paper we will derive two remarkably simple scaling forms for the fractional energy loss and the power spectrum. The average energy loss obeys the scaling law

$$\frac{\delta}{\delta_{classical}} = F(C) \ , \tag{2.11}$$

where the scaling function or form factor F will be derived in the text. For a uniform cylindrical pulse of constant charge density $N\alpha/\pi LB^2$, it has the limiting behavior

$$F(C) \approx 1 - \frac{11\sqrt{3}}{4C} = 1 - \frac{4.77}{C} \qquad \text{for } C \gg 1$$
$$\approx b_1\,[C]^{4/3}(1 - 2\,C^{2/3}) \qquad \text{for } C \ll 1\ . \tag{2.12}$$

where $b_1 = 0.83\ldots$ for $j = 1/2$ leptons. In this latter limit (2.12) is approximately 5% smaller than (2.7) . Using (2.9) , (2.10) , and (2.12) ,we find for the 'super' that $\delta = 0.55 y\alpha C^{1/3}$ and

$$\delta(Super) \sim 0.17 \tag{2.13}$$

for the fractional loss of beam energy due to radiation. For the SLC, the form factor reduces $\delta_{classical}$ by about 10%. For spin zero electrons, the parameter b_1 is reduced by a factor of 9/16 and the coefficient of the correction term $C^{2/3}$ drops from 2 to 1. The scaling function $F(C)$ is plotted for arbitrary C in Figure 1a and 1b. We also plot $F(C)/C$, which is of interest by (2.9) and (2.11) .

The power spectrum can be written in the scaling form

$$\frac{1}{C\,\delta_{classical}}\,\frac{d\delta}{d\,x^2} = T(x) \times R(u)\ , \tag{2.14}$$

where x is the ratio of the energies of the final and incident electron (the fraction of the incident energy in the photon is therefore $1 - x$). The spectrum $R(u)$ is a function only of the scaling variable u defined by $u^3 = C^2(\frac{1-x}{x})^2$. For spin one-half electrons, $T(x) = (x + 1/x)/2$, while for scalar electrons $T(x) = 1$.[17] For $u < 1/2$, $R(u)$ has the approximate behavior

$$R(u) \approx (u)^{1/2}(A_1 - A_2 u + \ldots)\ , \tag{2.15}$$

where $A_1 = 0.582\ldots$ and $A_2 = 0.50\ldots$ for a uniform cylindrical pulse. The scaling spectrum is shown in Figure 2. The peak near $u \approx 0.4$ indicates that in the classical regime, where $C \gg 1$, predominantly soft photons are radiated with $(1-x) \ll 1$. On the other hand, for the 'super' with $C \ll 1$, the peak is near $x \approx 0$ corresponding to hard photons.

Physical Interpretation:

There is a simple way to understand the general form of the main results quoted above, equations (2.3) , (2.11) , and (2.12) .

Three length scales are important in characterizing the electron's path and the radiation pattern. These are:

1. the coherence length of the radiation, l_{coh}, defined as the path length of the electron corresponding to its acquiring a transverse momentum $\sim m$ from the electric field. Since the width of the photon radiation pattern is also $\sim m$, the radiation can be coherent only from a finite length of the curving path, namely

$$l_{coh} \sim \left| \frac{m}{eE_{\perp}} \right| \sim \frac{L}{2y} = \frac{\ell_0}{2y}\gamma \tag{2.16}$$

   for impact parameter B.[18]

2. the radiation length, l_{rad}, related by the uncertainty principle to the reciprocal of the longitudinal momentum transfer,

$$l_{rad} \sim \frac{1}{q_z} \sim \frac{p}{m^2} \tag{2.17}$$

   where the last relation corresponds to giving a transverse momentum $\sim m$ to the target pulse. This is the length of the target that the electron scatters from coherently during the radiation process. An alternative and perhaps more physical notation would be to call l_{rad} the *longitudinal* coherence length, and to call l_{coh} the *transverse* coherence length.

3. the graininess of the bunch- i.e., the average separation of its particles, expressed as

$$l_{grn} \sim \frac{L}{N} \, . \tag{2.18}$$

In all cases of interest, the radiation length l_{rad} is much larger than the graininess; i.e.

$$l_{rad} \sim \frac{p}{m^2} = \frac{2\gamma^2}{m} \gg \frac{L}{N} = l_{grn} \, . \tag{2.19}$$

Note that the incident electron energy is given by $p = 2\gamma^2 m$ in the rest frame of the pulse. As will be confirmed in a later section by explicit calculation, the above justifies our making a smooth approximation for the charge distribution of the pulse. This applies for both the SLC-like parameters corresponding to a dense pulse,

$$\frac{L}{N} \sim 2 \times 10^{-7} cm \ll B \sim 10^{-4} cm \tag{2.20}$$

and to those parameters quoted earlier as envisaged for a 'super' linear collider corresponding to a dilute pulse,

$$\frac{L}{N} \sim 3 \times 10^{-6} cm \ll B \sim 5 \times 10^{-8} cm \, . \tag{2.21}$$

The dimensionless scaling parameter C discussed above is simply the ratio of the coherence length to the radiation length:

$$C = \frac{m\ell_0}{4\gamma y} \sim \frac{l_{coh}}{l_{rad}} \, . \tag{2.22}$$

Hence in the classical region of large $C \gg 1$, as appropriate for the SLC, the deflection of the electron orbit is negligible over the path length l_{rad} and the form factor for radiation along the length l_{rad} is unity.[19]

The result given in (2.3) can then be understood as radiation from L/l_{rad} transverse slices of the pulse, each of thickness l_{rad} and containing $N\,l_{rad}/L$ charges, with each slice radiating incoherently with respect to the others. Using (2.17) , and introducing $d\sigma \propto \alpha^3 N^2 dk/(m^2 k)$ as the cross section for emission of a photon k by charge $N\alpha$, we find

$$\delta_{classical} \sim \frac{L}{l_{rad}} \frac{1}{\pi B^2} \int^{p} \frac{dk}{k} \left(\frac{k}{p}\right) \left(\frac{N\,l_{rad}}{L}\right)^2 \frac{\alpha^3}{m^2} \sim \frac{N^2 \alpha^3 \gamma}{m^3 \ell_0 B^2} \,. \tag{2.23}$$

In the quantum regime of small $C \ll 1$ as appropriate for the 'super', we will calculate the form factor for the overlap of the radiation along the bending path l_{coh} and find that $F(C) \propto C^{4/3}$, clearly showing the diminishing overlap in this situation.

An additional length of importance in this problem is the *disruption parameter* . We shall measure this effect by the fractional change in the impact parameter b of the electron as it traverses the pulse and is deflected by the very strong fields produced by the $N \approx 10^8 - 10^{10}$ leptons (positrons) forming the pulse. In this calculation we treat the pulse as a fixed charge distribution which produces a static (primarily transverse) electric field in its rest system. Therefore we must limit our calculation to small disruption parameters $\delta b \ll b$. Otherwise, as the incident pulse of electrons is squeezed by the attractive field of the positron pulse, the radius of the positron pulse is likewise squeezed by the effects of the electron pulse. A proper treatment of these mutual focussing effects (which if large would set up betatron oscillations) would require a much more extensive and difficult analysis.

According to the classical equation of motion, the condition for small disruption if no photon is emitted can be expressed as

$$\delta b = \frac{1}{2}(eE_\perp)\,\frac{L^2}{p} \ll b \; . \tag{2.24}$$

If a photon is emitted at the point z, then

$$\delta b = \frac{1}{2}\,(eE_\perp)\,\left[\frac{z^2}{p} + \frac{(L-z)^2}{xp}\right] \ll b \tag{2.25}$$

For a uniform cylindrical pulse $e\overrightarrow{E}_\perp(b) = -2N\alpha\,\overrightarrow{b}/(LB^2)$; this restriction then demands that

$$\left.\frac{\delta b}{b}\right|_L \sim \frac{y\ell_0}{2\gamma B} < 1 \; . \tag{2.26}$$

Referring to the nominal 'super' parameters (2.10) , we see that this restriction is satisfied since

$$\left.\frac{\delta b}{b}\right|_{super} \approx 6 \times 10^{-2} \tag{2.27}$$

whereas for the SLC, the parameters (2.4) give a sizable disruption,

$$\left.\frac{\delta b}{b}\right|_{SLC} \approx 4 \times 10^{-1} \; . \tag{2.28}$$

Slicing:

An approximate way of treating beamstrahlung for conditions corresponding to sizable disruption is indicated by the form of (2.11) and (2.14) . As we discussed above in deriving (2.23) , let us slice the pulse into n cylinders, each of length L/n but with unequal radii- the j^{th} slice has radius B_j and the average radius is B .We specify n so that $n < y$, i.e.,

$$\begin{aligned} \frac{L}{n} &> \frac{L}{y} \\ \frac{\delta b}{b} &= \frac{y}{n}\,\frac{\ell_0}{2\gamma B_j} \ll 1 \ , \end{aligned} \tag{2.29}$$

so that the radiation from successive cylinders is incoherent, and so that the disruption during passage through one slice of radius B_j is small. The average fractional energy loss in the j^{th} slice is then

$$\delta(j) = \frac{8}{3}\,\frac{\alpha^3 (N/n)^2 \gamma}{m^3 (\ell_0/n) B_j^2}\, F(C_j) \ , \tag{2.30}$$

where $C_j = (B_j/B)C$. The total loss for the complete pulse is then given by the sum over the n slices and can then be written as

$$\begin{aligned} \delta &= \sum_{j=1}^{n} \delta(j) \\ &= \frac{8}{3}\,\frac{\alpha^3 N^2 \gamma}{m^3 \ell_0 B^2}\, \overline{F}(C) = \delta_{classical}\, \overline{F}(C) \ , \end{aligned} \tag{2.31}$$

where

$$\overline{F}(C) \equiv \frac{1}{n} \sum_j \frac{B^2}{B_j^2}\, F\left(\frac{B_j}{B} C\right) \ . \tag{2.32}$$

So long as we can simultaneously satisfy the two conditions (2.29) , even regimes in which the disruption is sizable can be treated in a straightforward manner. Note that in the classical regime, the effective form factor involves an average of the inverse square of the radii, whereas in the quantum limit, according to (2.8) and (2.11) , the average of the radii to the $(-2/3)$ power enters.

Finally, we remark that the high energy quantum mechanical scattering formalism that we develop in the next two sections is, for the problem of scattering from a long string-like pulse of length $L \propto \gamma$, just an expansion in powers of the disruption parameter. For the 'super' collider regime, the expansion parameter (2.27) is small. For the classical regime of SLC, the disruption (2.28) is large but we can simply slice the pulse into a small number ($say \approx 10$) of pulses and introduce an effective form factor as in (2.32) .

3. Scalar Electrons

In this section we derive an expression for the matrix element for the emission of a photon during the scattering of an spin-zero electron from a pulse of N positrons. This case is treated first to illustrate the physics of our approach. In a later section the case of Dirac electrons will be discussed. The general form of the matrix element of interest is

$$M = \left\langle \phi_f^{(-)} \middle| \vec{A} \cdot \vec{J} \middle| \phi_i^{(+)} \right\rangle , \tag{3.1}$$

where A is the photon field, $\vec{J}$ is the electron current and $\phi_f^{(-)}$ and $\phi_i^{(+)}$ are respectively the final (incoming) and initial (outgoing) scattering eigenstates of the electron in the static external field of the pulse. For simplicity we will assume that the pulse is a cylinder of length L and radius B but any shape can be treated in principle. The calculation will be carried out in the rest frame of the pulse, and in this frame, the length L and incident energy p are

$$L = \ell_0 \gamma \qquad\qquad p = 2m\gamma^2 . \tag{3.2}$$

Let us now turn to a detailed calculation of the relevant wave functions, matrix elements, and cross sections for our problem. It will prove to be necessary to retain corrections of order $(1/p)^2$ to the leading terms, or one order beyond the standard eikonal approximation.

<u>Approximate Wave Functions:</u>

The Klein-Gordon equation for a scalar particle of mass m in an external vector potential A_μ is

$$[m^2 - (i\partial - eA)^2\,]\,\phi \,=\, 0 \;. \tag{3.3}$$

In the rest frame of the pulse there is only a static field and the spatial K-G equation can be written as

$$[(E-V)^2 + \vec{\nabla}^2 - m^2]\phi(x) = 0 \;. \tag{3.4}$$

The solution will be written in the form

$$\phi(x) \;=\; \exp(i\Phi(x)) \;, \tag{3.5}$$

where Φ satisfies the equation

$$(E-V)^2 - m^2 \;=\; (\vec{\nabla}\Phi(x))^2 - i\vec{\nabla}^2\Phi \;. \tag{3.6}$$

For the problem of interest, we must solve this equation in the limit of large energies for the requisite boundary conditions, and must exhibit the solutions to a higher accuracy than the familiar eikonal approximation. We will assume that the potential has cylindrical symmetry and write

$$V(x) \;=\; V(z,b) \qquad\qquad b^2 = x^2 + y^2 \;. \tag{3.7}$$

For the incident wave, the leading term in Φ_i must be pz since the incident momentum is along the z-axis. The phase function to order $(1/p_i)$ will be expanded in the form

$$\Phi_i = p_i z - \chi_0(z,b) - \frac{1}{p_i}\left[\chi_1(z,b) + i\chi_2(z,b)\right] . \tag{3.8}$$

Substitution into (3.6) then yields

$$\chi_0(z,b) = \int_{-\infty}^{z} dz'\, V(z',b) , \tag{3.9}$$

which is recognized as the usual eikonal form, and the leading $(1/p)$ corrections are

$$\begin{aligned} \chi_1(z,b) &= \frac{1}{2}\int_{-\infty}^{z} dz' \left[\vec{\nabla}_\perp\, \chi_0(z',b)\right]^2 \\ \chi_2(z,b) &= \frac{1}{2}\int_{-\infty}^{z} dz' \left[\vec{\nabla}^2 \chi_0(z',b)\right] . \end{aligned} \tag{3.10}$$

While the term χ_2 will not be important in this application, the term χ_1 will be crucial in a proper description of the beamstrahlung process.

For the final state with incoming wave boundary conditions, the leading term in Φ_f must contain the final electron momentum which is parametrized in the form $\vec{p}_f = (\hat{z}p_f + \hat{b}p_\perp)$. The phase function to order $(1/p_f)$ will be written as

$$\Phi_f = \vec{p}_f \cdot \vec{r} + \tau_0(z,b) + \frac{1}{p_f}\left[\tau_1(z,b) + i\tau_2(z,b)\right] , \tag{3.11}$$

and then substitution into (3.6) yields the solutions

$$\tau_0(z,b) = \int_{z}^{\infty} dz'\, V(z',b) , \tag{3.12}$$

which is again in the familiar eikonal form, and the leading corrections in this case are

$$\tau_1(z,b) = \frac{1}{2}\int\limits_z^\infty dz' \left[(\overrightarrow{\nabla}_\perp \tau_0)^2 - 2\,p_\perp\hat{b}\cdot\overrightarrow{\nabla}_\perp \tau_0\right]$$

$$\tau_2(z,b) = \frac{1}{2}\int\limits_z^\infty dz' \left[\overrightarrow{\nabla}^2\tau_0\right] . \tag{3.13}$$

In the evaluation of the matrix element, an essential element is the total phase of the wave function product. Including the phase of the photon wave function $A(\overrightarrow{r})$, and defining the momentum transfer to the pulse as $\overrightarrow{q} = \overrightarrow{p}_f + \overrightarrow{k} - \overrightarrow{p}_i$, it can be written in the form

$$\begin{aligned}\Phi_{tot} &= \Phi_i - \Phi_f - \overrightarrow{k}\cdot\overrightarrow{r} \\ &= -\overrightarrow{q}\cdot\overrightarrow{r} - \chi_0^{tot}(b) - \frac{1}{p}\left[\chi_1^{tot}(z,b) + i\chi_2^{tot}(z,b)\right] ,\end{aligned} \tag{3.14}$$

where from now on $p \equiv p_i$ and total phase functions have been introduced as the appropriate sum of a χ and a τ. Therefore the zeroth order term is independent of z

$$\chi_0^{tot}(b) = \int\limits_{-\infty}^{\infty} dz'\, V(z',b) , \tag{3.15}$$

while the first order terms still retain some z-dependence:

$$\begin{aligned}\chi_1^{tot}(z,b) &= \chi_1 + \frac{\tau_1}{x} \\ \chi_2^{tot}(z,b) &= \chi_2 + \frac{\tau_2}{x} .\end{aligned} \tag{3.16}$$

where $x = p_f/p_i$.

Cylindrical Pulse:

Neglecting end effects, the potential due to a uniform cylindrical pulse is of the form

$$V(z,b) = V_0\, b^2 \qquad\qquad V_0 = \frac{N\alpha}{LB^2} \tag{3.17}$$

for $0 < z < L$, and zero otherwise. It is a simple matter to calculate that

$$\chi_0(z,b) = V_0 b^2 z \ , \tag{3.18}$$

and

$$\begin{aligned} \chi_1(z,b) &= \frac{2}{3}\, V_0^2 b^2 z^3 \\ \chi_2(z,b) &= V_0 z^2 \ . \end{aligned} \tag{3.19}$$

The momentum operator applied to the real part of this phase yields the 'local' momentum at a point denoted by $\overrightarrow{p}_i(loc; z, b)$. For the incident wave this gives

$$\begin{aligned} p_i^z(loc) &= p - V_0 \left[1 + \frac{2}{p}\, V_0 z^2\right] b^2 \\ p_i^\perp(loc) &= -2V_0 \left[1 + \frac{2}{3p}\, V_0 z^2\right] z\, \overrightarrow{b} \ . \end{aligned} \tag{3.20}$$

For the final state incoming wave, one finds

$$\tau_0(z,b) = V_0 b^2 (L-z) \ , \tag{3.21}$$

and

$$\begin{aligned} \tau_1(z,b) &= \frac{2}{3}\, V_0^2 b^2 (L-z)^3 - p_\perp \cdot \overrightarrow{b}\, V_0 (L-z)^2 \\ \tau_2(z,b) &= V_0 (L-z)^2 \ . \end{aligned} \tag{3.22}$$

The local momentum, $\overrightarrow{p}_f(loc; z, b)$ is

$$p_f^z(loc) = xp - V_0 \left[1 + \frac{2}{xp} V_0(L-z)^2\right] b^2 + 2\,\frac{p_\perp \cdot \overrightarrow{b}}{xp} V_0(L-z)$$

$$p_f^\perp(loc) = \overrightarrow{p}_\perp + 2V_0 \left[1 + \frac{2}{3xp} V_0(L-z)^2\right] (L-z)\overrightarrow{b} - \frac{V_0}{xp} (L-z)^2 \overrightarrow{p}_\perp \, . \tag{3.23}$$

The elements of the total phase of the matrix element for this type of pulse become[20]

$$\chi_0^{tot}(b) = V_0 b^2 L \tag{3.24}$$

for the zeroth order term, while the first order terms are

$$\begin{aligned} \chi_1^{tot}(z,b) &= \frac{2}{3} V_0^2 \left[z^3 + \frac{1}{x}(L-z)^3\right] b^2 - \frac{p_\perp \cdot \overrightarrow{b}}{x} V_0(L-z)^2 \\ \chi_2^{tot}(z,b) &= \chi_2^{tot}(z) = V_0 \left[z^2 + \frac{1}{x}(L-z)^2\right] . \end{aligned} \tag{3.25}$$

The total phase can be rewritten in more convenient form as

$$\Phi_{tot} = -q_z z - \left[\overrightarrow{q}_\perp \cdot \overrightarrow{b} + V_0 b^2 L\, \eta(z) + \frac{\overrightarrow{p}_\perp \cdot \overrightarrow{b}}{xp} V_0(L-z)^2\right] - i\,\frac{1}{p}\,\chi_2^{tot}(z) \, , \tag{3.26}$$

where

$$\eta(z) = 1 + \frac{2}{3pL} V_0 \left[z^3 + \frac{1}{x}(L-z)^3\right] , \tag{3.27}$$

and

$$\begin{aligned} -q_z &= \frac{m^2 + p_\perp^2}{2p_f} + \frac{k_\perp^2}{2k} - \frac{m^2}{2p} \\ &= \frac{m^2(1-x)}{2xp} + \frac{k_\perp^2}{2(1-x)p} + \frac{p_\perp^2}{2xp} \, . \end{aligned} \tag{3.28}$$

The form of (3.26) shows why it was necessary to keep corrections of order

$(1/p)^2$ in solving the Klein-Gordon equation. In contrast to the transverse momentum transfer $|\vec{q}_\perp|$, which typically is $\sim m$ and finite in the limit of infinite p, the longitudinal momentum transfers are, see (2.17) , $\sim m^2/p$ which is of the same order as the $(1/p)$ terms retained in (3.10) and (3.13) for the phase Φ. Since the very long string-like potential (3.17) has no z-dependence, except for end effects $\sim 1/L$, we must retain these $1/p$ terms to the overall phase to achieve a consistent treatment.

Elsewhere in the calculation, these $1/p$ terms appear as corrections to the leading order. Their magnitude is given approximately by the disruption parameter as is seen by comparison of (3.20) , (3.23) , and (3.27) with (2.25) . Subsequently we shall drop them as small since our present calculation is limited to the study of small disruptions. There is no essential difficulty in retaining them along with higher order terms in the calculation of the relevant phases, (3.8) and (3.11) .

Matrix Element- Stationary Phase:

Neglecting certain normalization factors for the moment, the matrix element now achieves the form

$$M = \frac{e}{\pi} \int_0^L dz \int^B d^2b \; \vec{\epsilon} \cdot \vec{P}(z,b) \quad \exp[i\Phi_{tot}(z,b)] \quad , \tag{3.29}$$

where the factor $\vec{P}(z,b)$ is the gauge invariant (to the order of this calculation in $1/p$) average of the initial and final momentum

$$\vec{P}(z,b) = \frac{1}{2}\left[\vec{p}_i(loc;z,b) + \vec{p}_f(loc;z,b)\right] . \tag{3.30}$$

In component form this is

$$\begin{aligned} P_z(z,b) &= \frac{(1+x)}{2}p - V_0 b^2 \\ \vec{P}_\perp(z,b) &= \frac{1}{2}\,\vec{p}_\perp + V_0\,\vec{b}_0(L-2z) \ , \end{aligned} \tag{3.31}$$

where terms of order $1/p$ were neglected.

The phase Φ_{tot} is quadratic in the impact parameter for a long uniform cylindrical pulse. Since the coefficient of b^2 is very large in units of the radius of the pulse-i.e., $V_0 b^2 L = N\alpha(b/B)^2$ - we will carry out the d^2b integral via the method of stationary phase. To do this it is necessary to solve for the stationary impact parameter $\vec{b}_0$, where

$$\vec{\nabla}_\perp \Phi_{tot}(z,b_0) = 0 . \tag{3.32}$$

This gives

$$2V_0 L\eta(z)\,\vec{b}_0 = -\left[\vec{q}_\perp + p_\perp \frac{V_0}{xp}\,(L-z)^2\right] , \tag{3.33}$$

which fixes the 'classical' impact parameter in terms of the final momentum transfer and the coordinate and energy of the radiated photon. The factor $\eta(z)$ induces a z-dependence in b_0 which is a reflection of the curved classical trajectory and also is the quantum source of the disruption parameter. Note that since $b_0 \leq B$, the momentum transfer $\vec{q}_\perp$ has a maximum allowed magnitude (otherwise the stationary point does not exist). Expanding the impact parameter around the value $\vec{b} = \vec{b}_0 + \vec{b}_1$ yields

$$\Phi_{tot}(z,b) = \Phi_{tot}(z,b_0) - b_1^2 V_0 L\eta(z) \ . \tag{3.34}$$

To leading order in $1/p$, the phase $\Phi_{tot}(z,b_0)$ is

$$\begin{aligned}\Phi_{tot}(z) &\equiv \Phi_{tot}(z,b_0(z)) \\ &= -q_z z + \frac{q_\perp^2}{4V_0L\eta(z)} + \frac{\vec{q}_\perp \cdot \vec{p}_\perp}{2xpL\eta(z)}\,(L-z)^2 - i\,\frac{1}{p}\,\chi_2^{tot}(z) \ .\end{aligned} \tag{3.35}$$

Also, since $b_0 = b_0(z)$, we introduce the notation for (3.30)

$$\vec{P}(z) \equiv \vec{P}(z, b_0(z)) \,. \tag{3.36}$$

The integral over the relative impact parameter b_1 can now be performed with the result

$$M = -i\,\frac{e}{LV_0}\int_0^L \frac{dz}{\eta(z)}\ \vec{\epsilon}\cdot\vec{P}(z)\ \ \exp[i\Phi_{tot}(z)] \ . \tag{3.37}$$

The integral over b_1 effectively extends only over the range $0 < b_1^2 < 1/(LV_0) = B^2/(N\alpha)$. This is a measure of the localization of the incident packet as it enters the pulse.

The square of the matrix element, summed over photon polarizations, is of the form

$$\sum_{pol} M^* M = \frac{\alpha}{(LV_0)^2} \int_0^L \frac{dz_1\, dz_2}{\eta(z_1)\eta(z_2)} S \ \exp[i(\Phi_{tot}(z_1) - \Phi_{tot}(z_2))] \ , \tag{3.38}$$

where the polarization sum that we require is

$$S(Boson) = \sum \vec{\epsilon} \cdot \vec{P}(z_1) \times \vec{\epsilon} \cdot P(z_2) \ . \tag{3.39}$$

<u>Polarization Trace:</u>

In the coulomb gauge, this sum becomes

$$S(Boson) = \vec{P}(z_1) \cdot \vec{P}(z_2) - \frac{1}{k^2} \vec{P}(z_1) \cdot \vec{k}\, \vec{k} \cdot \vec{P}(z_2) \ . \tag{3.40}$$

Using (3.31) , and retaining again only the leading term in $1/p$, this can be written

$$S(Boson) = \frac{1}{(1-x)^2} \left[(k'_\perp)^2 - w^2 \frac{1}{4}(1-x)^2 q_\perp^2 \right] \ , \tag{3.41}$$

where for convenience, we introduce the quantities

$$\begin{aligned} w &= \frac{(z_1 - z_2)}{L} \\ \vec{k}'_\perp &\equiv \vec{k}_\perp - \frac{(z_1 + z_2)}{2L} (1-x) \vec{q}_\perp \ . \end{aligned} \tag{3.42}$$

For notational simplicity we define

$$S(Boson) = \frac{1}{(1-x)^2} D[w^2] \ , \tag{3.43}$$

where $D[w^2]$ is a polynomial given by (3.41) .

Properties of the Phase:

The properties of the matrix element are largely determined by the detail properties and behavior of the real part of the phase $\Phi_{tot}(z, b_0)$ as a function of z. Since the imaginary term $\tau_2^{tot}(z)$ induces a small change in amplitude of the integrand of M, it will be neglected here. Its effects can be easily included.

First, recall from (3.28) that

$$-q_z = \frac{m^2(1-x)}{2xp} + \frac{k_\perp^2}{2k} + \frac{p_\perp^2}{2xp}\,, \tag{3.44}$$

where $\vec{p}_\perp = \vec{q}_\perp - \vec{k}_\perp$ and $k = (1-x)p$. To simplify the discussion, consider the derivative of the phase with respect to z:

$$\frac{d\Phi_{tot}(z)}{dz} = -q_z - \frac{q^2}{4V_0 L \eta(z)^2}\frac{d\eta(z)}{dz} - \frac{(L-z)\,\vec{q}_\perp \cdot \vec{p}_\perp}{xpL\eta(z)}\,, \tag{3.45}$$

where

$$\frac{d\eta(z)}{dz} = \frac{2V_0}{pL}\left[z^2 - \frac{1}{x}(L-z)^2\right]. \tag{3.46}$$

Utilizing all of the above, one finds after some reduction and after neglecting terms of order p^{-2},

$$\frac{d\Phi_{tot}(z)}{dz} = \frac{1}{2x(1-x)p}\left[m^2(1-x)^2 + \left[\vec{k}_\perp - \frac{z(1-x)}{L}\vec{q}_\perp\right]^2\right]\,, \tag{3.47}$$

and the phase itself is restored by integration over z.

Finally, in the evaluation of the *absolute square* of the matrix element, the relevant total phase will be difference of the above phase evaluated at different z-values. This phase difference has the remarkable property that it depends only on the difference of z-coordinates and a 'natural' photon transverse momentum variable that rotates as the particle traverses the pulse following the classical (curved) path:[21]

$$[\Phi_{tot}(z_1) - \Phi_{tot}(z_2)] = sw + \frac{1}{3}\, r^3 w^3 \;, \tag{3.48}$$

where

$$r^3 \equiv \frac{L(1-x)}{8xp}\, q_\perp^2 \qquad\qquad s \equiv \frac{L}{2x(1-x)p}\left[m^2(1-x)^2 + (\overrightarrow{k}'_\perp)^2\right] \tag{3.49}$$

with the photon transverse momentum dependence characterized by the same variable as was found in the polarization trace, namely

$$\overrightarrow{k}_\perp{}' = \overrightarrow{k}_\perp - \frac{(z_1+z_2)}{2L}\,(1-x)\,\overrightarrow{q}_\perp \;. \tag{3.50}$$

In order to estimate the magnitude of these variables, note that they can be written to leading order in the form

$$\begin{aligned} s &= 2y\left[\left(C\frac{1-x}{x}\right)\frac{m^2(1-x)^2 + (\overrightarrow{k}'_\perp)^2}{2m^2(1-x)^2}\right] \\ r^3 &= \frac{1}{4}\, y^3\left[\left(C\frac{1-x}{x}\right)\frac{q_\perp^2}{q_\perp^2(max)}\right] \;. \end{aligned} \tag{3.51}$$

where by (3.33) , $q_\perp^2(max) = L^2 e^2 E_\perp^2(B)$ to leading order, corresponding to the classical path. As we shall see, the square brackets above are of order unity; therefore the important values of w are $\sim 1/y$, which can be interpreted as due to the fact that emission differing in position such that $|z_1 - z_2| \sim L/y$ can be coherent.

4. Spectrum and Cross Section

Final State Sum:

The square of the matrix element summed over the polarization and integrated over the transverse momentum of the photon is defined as

$$\int M*M \equiv \int \frac{d^2k_\perp}{(2\pi)^2} \sum_{pol} M^*M \; , \tag{4.1}$$

so that our next task is to evaluate

$$\int M{*}M \;=\; \frac{\alpha}{((1-x)V_0)^2} \int \frac{d^2k_\perp}{(2\pi)^2} \int_0^L \frac{dz_1 dz_2}{L^2\eta(z_1)\eta(z_2)}\, D[w^2] \exp\left[i\left(sw + \frac{1}{3}\, r^3w^3\right)\right] \; . \tag{4.2}$$

The inverse factors of $\eta(z)$ in the integrand can be set equal to one to the accuracy required. Introducing the difference variable w, we can interchange orders of integration, perform one z-integral and obtain

$$\int M*M \;=\; 2\,\frac{\alpha}{((1-x)V_0)^2} \int \frac{d^2k_\perp}{(2\pi)^2} \int_0^1 dw\,(1-w)\,D[w^2]\,\cos\left(sw + \frac{1}{3}\, r^3w^3\right) \; . \tag{4.3}$$

Since the parameters r and s are large, both of order y by (3.51) , the integral can be well approximated by

$$\int M*M \;=\; 2\,\frac{\alpha}{((1-x)V_0)^2} \int \frac{d^2k'_\perp}{(2\pi)^2} \int_0^\infty dw\;\, D[w^2]\,\cos\left(sw + \frac{1}{3}\, r^3w^3\right) \; . \tag{4.4}$$

Using the standard definition of the Airy function,[22] this becomes

$$\int M*M \;=\; 2\pi\frac{\alpha}{((1-x)V_0)^2} \int \frac{d^2k'_\perp}{(2\pi)^2}\, D\left[-\frac{d^2}{ds^2}\right]\, \frac{1}{r} Ai\left[\frac{s}{r}\right] \; . \tag{4.5}$$

Finally, using the differential equation satisfied by the Airy function $Ai(w)$, namely $Ai(w)'' = wAi(w)$, one achieves

$$\int M * M = 2\pi \frac{\alpha}{((1-x)V_0)^2} \int \frac{d^2k'_\perp}{(2\pi)^2} D\left[-\frac{s}{r^3}\right] \frac{1}{r} Ai\left[\frac{s}{r}\right] , \tag{4.6}$$

where, using the definition of r and s, D simplifies to

$$D\left[-\frac{s}{r^3}\right] = m^2(1-x)^2 + 2(k'_\perp)^2 . \tag{4.7}$$

Now what follows is a succession of variable changes to make this integral tractable. First define

$$s = r\,v , \tag{4.8}$$

and using (3.49) , the integration over $d^2k'_\perp$ can be replaced by an integral over v . Paying attention to the limits of integration, and introducing the value of v at $(k'_\perp)^2 = 0$,

$$v_0 \equiv \frac{m^2L(1-x)}{2xpr} , \tag{4.9}$$

one finds

$$\int M * M = 2\alpha \left(\frac{xp}{LV_0}\right)^2 r \int_{v_0}^{\infty} dv \; [2v - v_0]\, Ai(v) . \tag{4.10}$$

Now the variables left to integrate are $d^2q_\perp$, and the photon energy $p(1-x)$. Since the variable r^3 is linear in $q_\perp^2$, and since $q_\perp^2$ has a maximum value of $(2V_0LB)^2$, r^3 will also have an upper limit of

$$(r_{max})^3 \equiv \frac{V_0^2L^3B^2(1-x)}{2xp} . \tag{4.11}$$

If we write $r = r_{max} t$, where $0 < t < 1$, then $v_0 = u/t$, where u is the scaling variable defined earlier, and

$$d^2 q_\perp = 24\pi x p (V_0 L B)^2 \, 3\, t^2 dt \ , \tag{4.12}$$

and the partial cross section for fixed photon energy fraction $(1-x)$ becomes

$$\int \frac{d^2 q_\perp}{(2\pi)^2} \int M * M = J \int_0^1 dt \; 3\, t^3 \int_{u/t}^{\infty} dv \left[2v - \frac{u}{t} \right] Ai(v) \ , \tag{4.13}$$

with

$$J = 12\alpha (xpB)^2 \left[\frac{L^3 V_0^2 B^2}{2p} \, \frac{(1-x)}{x} \right]^{1/3} \ . \tag{4.14}$$

The integrations can be interchanged, the dt integration performed, and the result is

$$\int \frac{d^2 q_\perp}{(2\pi)^2} \int M * M = J \int_u^{\infty} dv \, \left[\frac{3}{2} v - u - \frac{u^4}{2v^3} \right] Ai(v) \ , \tag{4.15}$$

where the important scaling variable u is given by

$$u^3 = C^2 \left(\frac{1-x}{x} \right)^2 \ . \tag{4.16}$$

Scaling Laws:

The differential cross section for beamstrahlung is achieved by dividing by the normalization factors for the initial and final electron and photon wave functions, $[p^3 x(1-x)]$; the fractional power spectrum by then multiplying by an extra factor of $(1-x)$ together with trivial numerical factors. The final result can be written

$$\frac{d\delta}{dx} = [\ 2\ x\ C\ R(u)\ ]\ \delta_{classical}\ , \tag{4.17}$$

where the scaling spectrum function $R(u)$ is defined as

$$R(u)\ =\ \frac{3}{2}\, u^{1/2} \int_u^\infty dv \left[\frac{3}{2}v - u - \frac{u^4}{2v^3}\right] Ai(v)\ , \tag{4.18}$$

The form factor described earlier is easily computed from the above results. Now

$$F(C)\ \equiv\ \frac{\delta}{\delta_{classical}}\ , \tag{4.19}$$

and since from the definition of u one has $x = [1 + u^{3/2}/C]^{-1}$, it follows that

$$F(C)\ =\ 2\int_0^1 dx\, xC\, R(u)\ =\ 3\int_0^\infty du\ \frac{u^{1/2}}{[1+\frac{u^{3/2}}{C}]^3}\, R(u)\ . \tag{4.20}$$

Explicitly, the form factor for the spin-zero case is

$$F(C)\ =\ \frac{9}{2}\int_0^\infty du\ \frac{u}{[1+\frac{u^{3/2}}{C}]^3} \int_u^\infty dv\ \left[\frac{3}{2}v - u - \frac{u^4}{2v^3}\right] Ai(v)\ . \tag{4.21}$$

The normalization can be checked by taking the limit of large C and interchanging the order of integration:

$$
\begin{aligned}
F(C=\infty) &= \frac{9}{2}\int\limits_0^\infty du\, u \int\limits_u^\infty dv \left[\frac{3}{2}v - u - \frac{u^4}{2v^3}\right] Ai(v) \\
&= \frac{3}{2}\int\limits_0^\infty dv Ai(v) v^3 \; = 1 \, .
\end{aligned}
\tag{4.22}
$$

5. Dirac Electrons

The extension of our analysis to Dirac electrons is straightforward. The general form of the matrix element for this case is

$$M = e \left\langle \phi_f^{(-)} \middle| \vec{A} \cdot \vec{\alpha} \middle| \phi_i^{(+)} \right\rangle , \tag{5.1}$$

where A is the photon field. It is convenient to use a chiral basis for the Dirac spinors. The Hamiltonian and the Dirac equation take the form

$$H \, \Psi(r) = E \, \Psi(r) , \tag{5.2}$$

where in terms of the ordinary two-component Pauli matrices,

$$H = \begin{pmatrix} -i \vec{\sigma} \cdot \vec{\nabla} + V & m \\ m & +i \vec{\sigma} \cdot \vec{\nabla} + V \end{pmatrix} . \tag{5.3}$$

The wave function will be written as

$$\Psi = \begin{pmatrix} \psi_u \\ \psi_l \end{pmatrix} , \tag{5.4}$$

and the equation satisfied by the upper and lower components are

$$\begin{aligned} m\psi_u &= (E - V - i \vec{\sigma} \cdot \vec{\nabla})\psi_l \\ m\psi_l &= (E - V + i \vec{\sigma} \cdot \vec{\nabla})\psi_u . \end{aligned} \tag{5.5}$$

The second order equation satisfied by these components is

$$\begin{aligned} m^2\psi_u &= \left[(E - V)^2 + \vec{\nabla}^2 - i \vec{\sigma} \cdot e \vec{E}(r)\right] \psi_u \\ m^2\psi_l &= \left[(E - V)^2 + \vec{\nabla}^2 + i \vec{\sigma} \cdot e \vec{E}(r)\right] \psi_u , \end{aligned} \tag{5.6}$$

where $e\vec{E}(r) = -\vec{\nabla} V(r)$.

In order to solve this equation as an expansion in inverse powers of the energy, we will write it in terms of the phase $\Phi(z,b)$ which solves Klein-Gordon equation:

$$\begin{aligned} \psi_u &= \exp[i\Phi(z,b)]\, w_u \\ \psi_l &= \exp[i\Phi(z,b)]\, w_l \quad . \end{aligned} \tag{5.7}$$

Using the equation satisfied by the phase function, we find

$$\begin{aligned} \left[2i(\vec{\nabla}\Phi)\cdot\vec{\nabla} + \vec{\nabla}^2 - i\vec{\sigma}\cdot e\vec{E}(r)\right] w_u &= 0 \\ \left[2i(\vec{\nabla}\Phi)\cdot\vec{\nabla} + \vec{\nabla}^2 + i\vec{\sigma}\cdot e\vec{E}(r)\right] w_l &= 0\ . \end{aligned} \tag{5.8}$$

For the incident wave one sets $\Phi = \Phi_i$ and requires continuity at $z = 0$ with the initial plane wave solution. For the final state, choose Φ_f and demand continuity at $z = L$ with the outgoing plane wave. Thus the matrix element (5.1) splits into a spacetime factor that is the same as that found for the spin zero case, and a spinor factor which will be computed below.

The solutions to order $(1/p)$ for the incident wave are required since $\frac{d\Phi}{dz} = p$ to leading order. We find:

positive helicity-

$$w_u = \left[1 + \frac{z}{2p}\vec{\sigma}\cdot e\vec{E}\right] w_+(i) \qquad\qquad w_l = 0\ . \tag{5.9}$$

negative helicity-

$$w_u = 0 \qquad\qquad w_l = \left[1 - \frac{z}{2p}\vec{\sigma}\cdot e\vec{E}\right] w_-(i)\ . \tag{5.10}$$

The spinors $w_\pm(i)$ are defined by

$$w_+(i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad w_-(i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ . \tag{5.11}$$

For the final state, the solutions to order $(1/p_f)$ are:
positive helicity-

$$w_u = \left[1 - \frac{(L-z)}{2p_f}\vec{\sigma}\cdot e\vec{E}\right] w_+(f) \qquad w_l = 0 \; . \tag{5.12}$$

negative helicity-

$$w_u = 0 \qquad w_l = \left[1 + \frac{(L-z)}{2p_f}\vec{\sigma}\cdot e\vec{E}\right] w_-(f) \; . \tag{5.13}$$

The spinors $w_\pm(f)$ are helicity states along the final momentum, and to first order they are

$$w_+(f) = \begin{pmatrix} 1 \\ \frac{p_\perp}{2p_f} \end{pmatrix} \qquad w_-(f) = \begin{pmatrix} -\frac{p_\perp}{2p_f} \\ 1 \end{pmatrix} \; . \tag{5.14}$$

The matrix element of the current is straightforward to evaluate from these explicit solutions. Defining the transverse vectors $v_\pm = v_x \pm iv_y$, one finds that there is no helicity flip amplitude, of course, and for:
positive helicity-

$$\vec{M}_+ \cdot \vec{\epsilon} = \epsilon_z - \frac{1}{2p_f}\,\epsilon_+ k_- + e\frac{z}{2}\left[\frac{\epsilon_- E_+}{p} + \frac{\epsilon_+ E_-}{p_f}\right] \; . \tag{5.15}$$

negative helicity-

$$\vec{M}_- \cdot \vec{\epsilon} = \epsilon_z - \frac{1}{2p_f}\,\epsilon_- k_+ + e\frac{z}{2}\left[\frac{\epsilon_+ E_-}{p} + \frac{\epsilon_- E_+}{p_f}\right] \; . \tag{5.16}$$

We now perform the sum over polarizations and the average over the initial helicity states,

$$S(fermion) = \frac{1}{2} \sum_{pol} (|\overrightarrow{M}_{+} \cdot \overrightarrow{\epsilon}|^2 + (|\overrightarrow{M}_{-} \cdot \overrightarrow{\epsilon}|^2) \ . \tag{5.17}$$

Taking into account the differing normalization conventions between fermions and bosons, the fermion sum turns out to be a simple multiple of the boson sum:

$$S(fermion) = S(boson) \frac{1 + x^2}{2x} \ . \tag{5.18}$$

6. Pulse Granularity and the Smoothness Assumption

In our previous discussion, we have assumed the distribution of charges in the pulse to be smooth and uniform. In reality the pulse is a very long string of length $L = \gamma \ell_0$ as viewed in its rest frame. For the SLC, the parameters, (2.4) , indicate the pulse to be a dense string with radius $B \approx 10^{-4}$ cm and interparticle spacing $L/N \approx 2 \times 10^{-7}$ cm $\ll B$. In this case it is natural to make a smooth averaging of the charge distribution for the purpose of determining the trajectory of the incident electron.

On the other hand, for the 'super' regime with parameters (2.10) , the pulse is dilute with $B \approx 5 \times 10^{-8}$ cm and $L/N \approx 10^{-6}$ cm $\gg B$, and it is thus necessary to verify by direct calculation the validity of smoothing which was suggested by the argument below (2.19) .

Our demonstration of the validity of this approximation follows from the form of the electron wave function. The phase Φ in (3.8) contains integrals (3.9) and (3.10) over the length of the pulse and hence effects of longitudinal variations are suppressed. Consider a long thin pulse with charges at the points z_i and $\vec{b}_i$. Neglecting end effects and retaining only leading order in $1/N\alpha$, the phase becomes

$$\begin{aligned}\chi_0(z,b) &= \int_{-\infty}^{z} dz'\, V(z',b) \\ &= -\alpha \sum_i \int_{-\infty}^{z} dz' [(\vec{b} - \vec{b}_i)^2 + (z'-z_i)^2]^{-1/2} \,. \end{aligned} \tag{6.1}$$

The integral is straightforward:

$$\chi_0(z,b) = constant - \alpha \sum_i \ell n(v_i) \;, \tag{6.2}$$

where

$$v_i = \left((z-z_i) + [(z-z_i)^2 + |\vec{b} - \vec{b}_i|^2]^{1/2} \right) \times \left(\frac{2L}{|\vec{b} - \vec{b}_i|^2} \right) \tag{6.3}$$

Thus χ_0 has only a logarithmic dependence on the position of the individual charges. The sum over i is then well approximated by an integral when introduced into the matrix element (3.29) . Note that the length over which z is averaged in the integrand is controlled by the small variable q_z, and this averaging extends over long intervals. If the charges are uniformly distributed, then the important behavior of the phase is given for $z < L$ by

$$\chi_0(z,b) = constant' + N\alpha \left[\frac{z}{L}\left(1 + \frac{b^2}{B^2}\right) - \frac{z}{L}\, \ell n\, \frac{4zL}{B^2} - \frac{L-z}{L}\, \ell n\, \frac{L}{L-z} \right] , \tag{6.4}$$

while for $z > L$,

$$\chi_0(z,b) \;=\; constant' + N\alpha \left[\left(1 + \frac{b^2}{B^2}\right) - \;\ell n\; \frac{4zL}{B^2} - \frac{z-L}{L} \;\ell n\; \frac{z}{z-L} \right] \;, \tag{6.5}$$

where the logarithmic coulomb phase is extant. In particular, the relevant total phase is of the form (see ref 20)

$$\chi_0^{tot}(b) \;=\; \int\limits_{-\infty}^{\infty} dz' \, V(z',b) \;=\; \chi_0^{tot}(0) + N\alpha \frac{b^2}{B^2} \;. \tag{6.6}$$

Similarly,

$$\int\limits_{-\infty}^{z} dz' \, e \vec{E}_\perp \;=\; -\frac{2N\alpha}{LB^2} \, z \, \vec{b} \;. \tag{6.7}$$

In addition, the current operators $P(z,b)$ in (3.30) are unchanged. This completes our demonstration that smoothing is a good approximation, as was argued qualitatively below (2.19) .

7. Review of Scaling Laws

At this point it may be useful to emphasize again results that are explicitly contained in the above formulas. The fractional energy loss δ for an electron (of spin one-half) is a function of a (quantum) scaling variable that smoothly extrapolates between the classical and the quantum results:

$$\begin{aligned} F(C) \;&=\; \frac{\delta}{\delta_{classical}} \\ &=\; \frac{3}{2} \int\limits_0^{\infty} du \, \left\{ \frac{1}{[1 + \frac{u^{3/2}}{C}]^2} + \frac{1}{[1 + \frac{u^{3/2}}{C}]^4} \right\} u^{1/2} R(u) \;, \end{aligned} \tag{7.1}$$

where $R(u)$ is given by (4.18) . The power spectrum can also be written in a scaling form:

$$\frac{1}{C\delta_{classical}} \frac{d\delta}{dx^2} = T(x)\ R(u)\ , \tag{7.2}$$

where $T(x) = (x + 1/x)/2$ for electrons.

Both of these are expressed in terms of the scaling variable u defined by

$$u^3 = C^2 \left(\frac{1-x}{x}\right)^2\ , \tag{7.3}$$

and

$$C = \frac{m^3}{2pV_0B} = \frac{m\ell_0}{4\gamma y}\ . \tag{7.4}$$

Finally, the form factor and spectrum can be approximated by quite simple analytic forms. By examining the limiting behavior of $F(C)$ for large and small C, and by evaluating analytically the integrals involved, one is lead to the form

$$F(C) = \left(1 + \frac{1}{b_1}\left[C^{-4/3} + 2\,C^{-2/3}\,(1+0.20C)^{-1/3}\right]\right)^{-1}\ , \tag{7.5}$$

where $b_1 = 0.83...$ The asymptotic behavior of the Airy function at large argument, and the behavior of $R(u)$ for small argument, suggest the approximate form

$$R(u) = A_1\sqrt{u}\left(1 + a_2 u\ \exp\left(\frac{2}{3}u^{2/3}\right)\right)^{-1}\ , \tag{7.6}$$

where $A_1 = 0.58...$ and $a_2 = 1.05...$ These approximate forms fit the exact curves to within a few per cent and are graphed along with an exact numerical integration in Figures 1a and 2. The fits can easily be improved.

8. Summary

We have studied the beamstrahlung process and derived formulas for the photon spectrum and the average fractional energy loss. These results were expressed in terms of scaling laws which should be convenient when comparing different collider parameters.

Intermediate Collider:

Finally we apply these results to the parameters under study for a near term collider at a center of mass energy of 650 Gev. One set of proposed parameters is[1]

$$N \sim 1 \times 10^{10} \qquad \gamma = 6.5 \times 10^5$$
$$B \sim 7 \times 10^{-6} cm \qquad \ell_0 = 6 \times 10^{-2} cm \ . \tag{8.1}$$

With these choices we find $C \approx 1.5$ and $y \approx 4 \times 10^{+2}$. Note that this implies $\delta_{classical} = 1.3$. Our measure of beam disruption is sizable, $\delta b/b \sim 2$, requiring a more careful treatment of the slicing technique than given in (2.24) . Neglecting that important effect, we find that the fractional energy loss is also large, $\delta = 0.38$. Note that by doubling the beam length, C doubles, but the situation is not changed much since the form factor increases by about 50% , leading to $\delta = 0.29$, a net 25% decrease. To decrease the fractional energy loss to $\delta \sim 0.1$, it is necessary to increase the pulse length ℓ_0 by a factor of 10 over that given in (8.1) . In Figure 3 we have plotted the photon spectra (normalized to one) for the SLC, the intermediate collider, and the super.

Regimes to Avoid:

The full luminosity is proportional to

$$Lum \equiv Lump \times f = \frac{N^2}{\pi B^2} f \,, \tag{8.2}$$

where f is the number of pulses per second and $Lump$ is equal to the luminosity per pulse. Now consider the behavior of the fractional energy loss δ as the parameters are varied but with the luminosity per pulse, $Lump$, held fixed. Other constraints will be ignored in this brief and incomplete discussion.

This means that the scaling variable y is fixed, since $y = \frac{\alpha}{m}\sqrt{\pi Lump}$. Thus we can write

$$\delta = \frac{2\alpha^2}{3m}\sqrt{\pi Lump}\left[\frac{F(C)}{C}\right] . \tag{8.3}$$

In order to choose a value of C that minimizes the fractional energy loss, note that the ratio of $F(C)$ to C vanishes for small C as $C^{1/3}$, and vanishes for large C as $1/C$. The worst choice for C is at the peak of the ratio which occurs at $C \sim 0.20$ with a value ~ 0.275. This maximum, as shown in Figure 1a, is quite wide; the ratio falls by a factor of 2 when C changes to 0.004 and 3.2. This is the range of C to avoid in order to minimize δ at a fixed luminosity. Thus in order to minimize the fractional energy loss, one is forced into the classical regime of large C or into the quantum regime of small C. Since $C = m\ell_0/(4\gamma y)$, this means either very long or very short pulses.

General Remarks:

Our results quantitatively confirm the arguments of Himel and Siegrist[2] and their adaptation of synchrotron radiation formulas to the collider situation. Their final formula is remarkably accurate in the full quantum regime, $C \ll 1$.

Possible further extensions of this work include studies of more general pulse charge distributions, studies of the effects of the curved trajectory on angular distributions in annihilation processes, especially for polarized beams, and a more accurate discussion of the neglected end effects. It will be particularly important to better understand the regime of large beam disruption. The case of electron-proton collisions would be interesting to study since betatron-like oscillations can be induced in the electron's trajectory with minimal response by the proton pulse. There are other applications of our technique which may prove interesting and useful; among these are bremsstrahlung processes involving relativistic interactions with plasmas, astronomical phenomena, etc.

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15. This has been reduced by a factor of two from the formula given in 2 to allow for the fact that the crossing speed of the collider pulses is $2c$.

16. The scaling variable C is related to the ratio of the photon energy E to its critical energy E_c as defined in 2 , $C = \frac{3}{2}\frac{E}{E_c}$.

17. In the case of a localized scatterer, relevant for the Bethe-Heitler case, one finds that $T(x) = \frac{3}{4}(x + 1/x - 2/3)$.

18. This length plays the same role as the ρ/γ, a parameter which is used to describe an electron orbiting with radius ρ in the magnetic field of a synchrotron.

19. Note that by (2.16) that the transverse momentum transfer will be $\sim m$ for path lengths $\lesssim L/y$.

20. This neglect of the coulomb phase is equivalent to the neglect of the radiation that occurs before and after passage through the pulse.

21. This property is crucial since the phase itself is very large, of order $N\alpha$, which is $\sim 10^8$ for our case, and few approximations can be tolerated.

22. See p. 447 of M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, U.S. Government Printing Office (1964), Washington, D.C.

FIGURE CAPTIONS

Figure 1a. The form factor plotted as a function of the scaling variable C, where $C = m^2 \ell_0 B / 4\gamma N \alpha$. The approximate form given in section 7 is plotted as the dashed line. The ratio of the form factor to C is also plotted as the dot-dash curve.

Figure 1b. The same as Figure 1 except that the form factors are plotted on a logarithmic scale.

Figure 2. The scaling form of the photon power spectrum is plotted as a function of the variable u. The scaling spectrum is defined as $R(u) = \frac{d\delta}{dx^2} \left[T(x) C \delta_{classical} \right]^{-1}$. The approximate form given in section 7 is also plotted as the dashed curve. Recall that the fractional photon energy is $(1 - x)$.

Figure 3. Three sample power spectra are plotted to illustrate the classical, transition, and fully quantum regimes. They are normalized to unit area. The SLC curve is divided by 10 for plotting purposes.

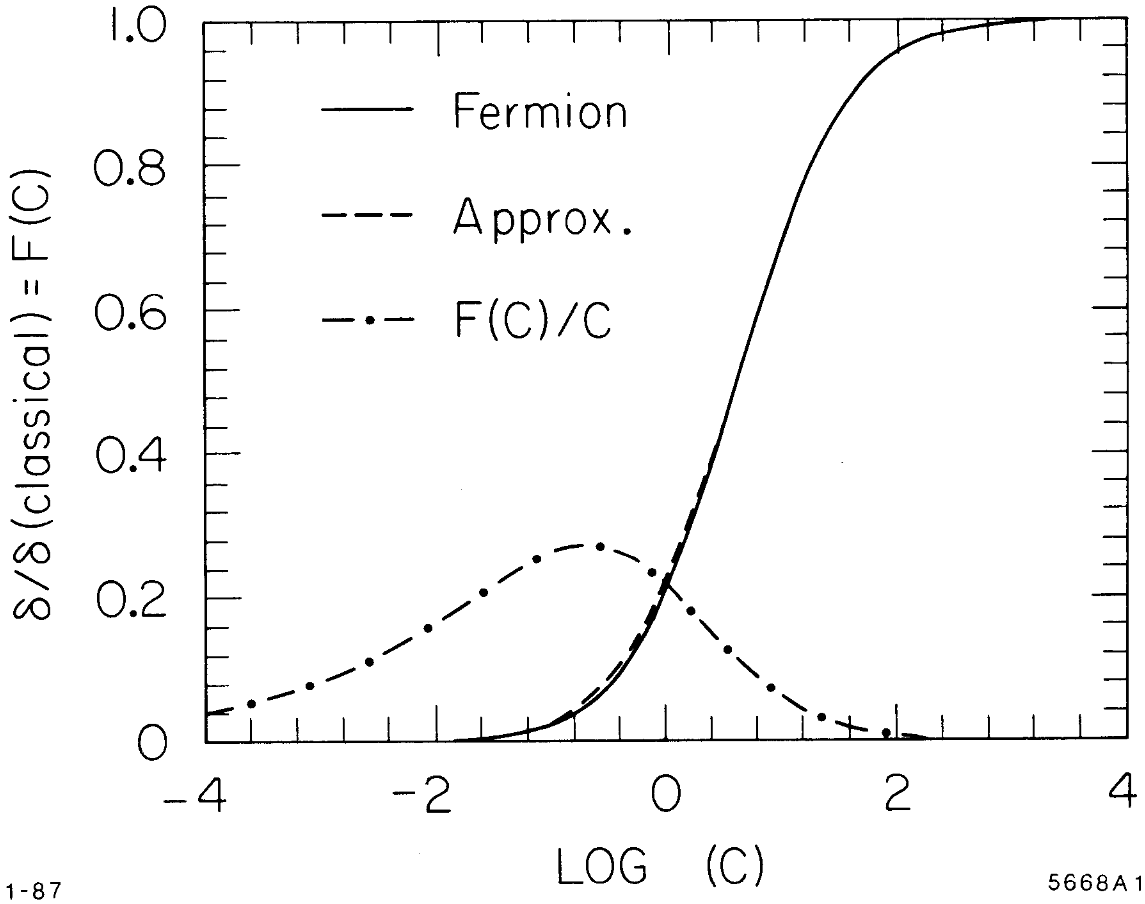

1.0
0.8
0.6
0.4
0.2
0
Fermion
Approx.
F(C)/C
δ/δ(classical) = F(C)
-4
-2
0
2
4
LOG (C)
1-87
5668A1

FIG. 1A

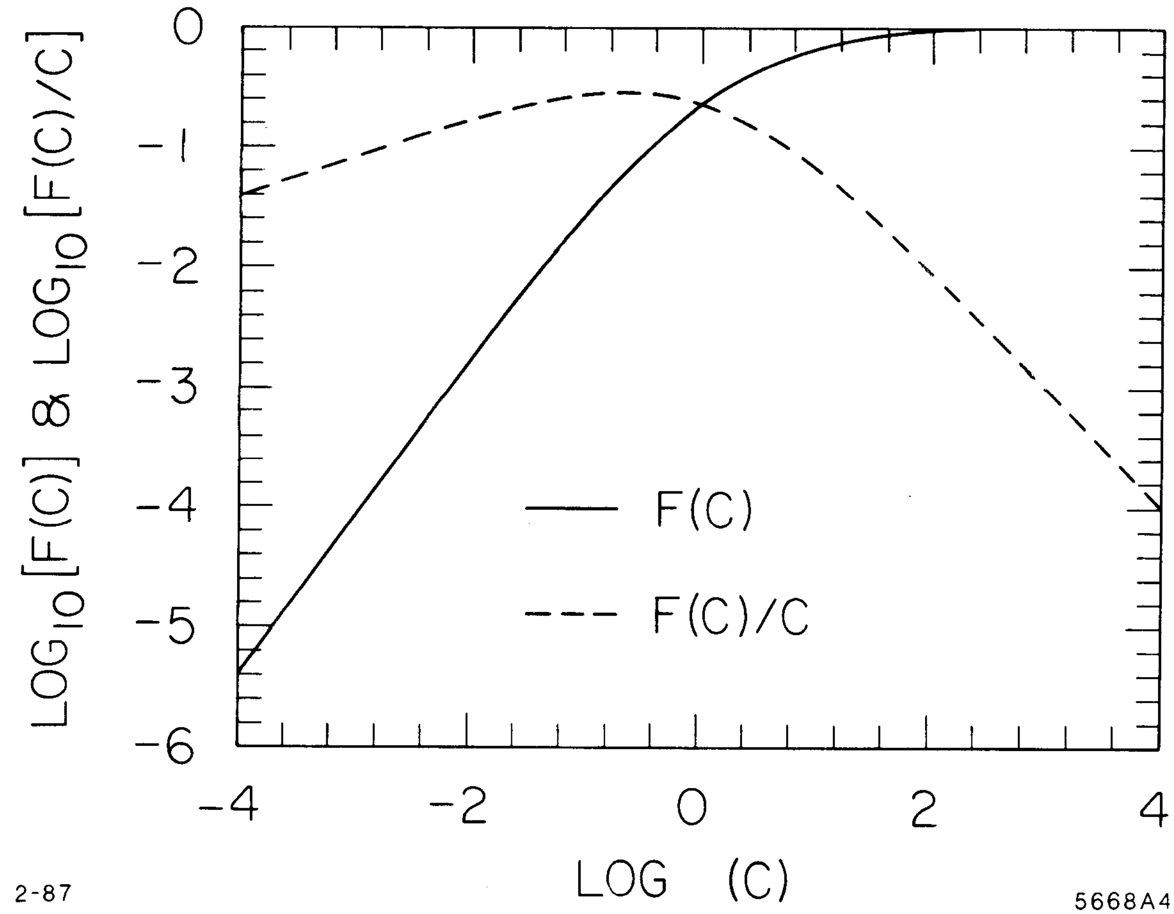


FIG. 1B

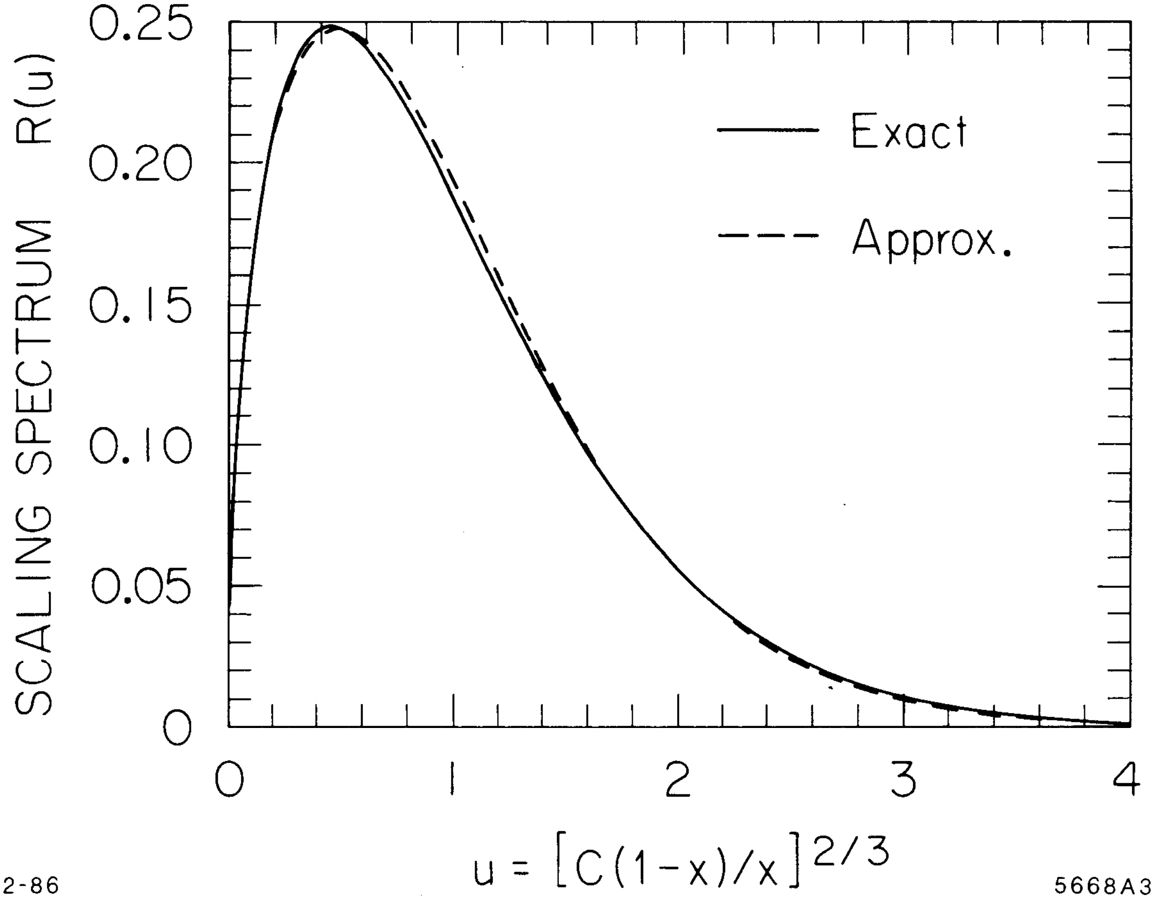
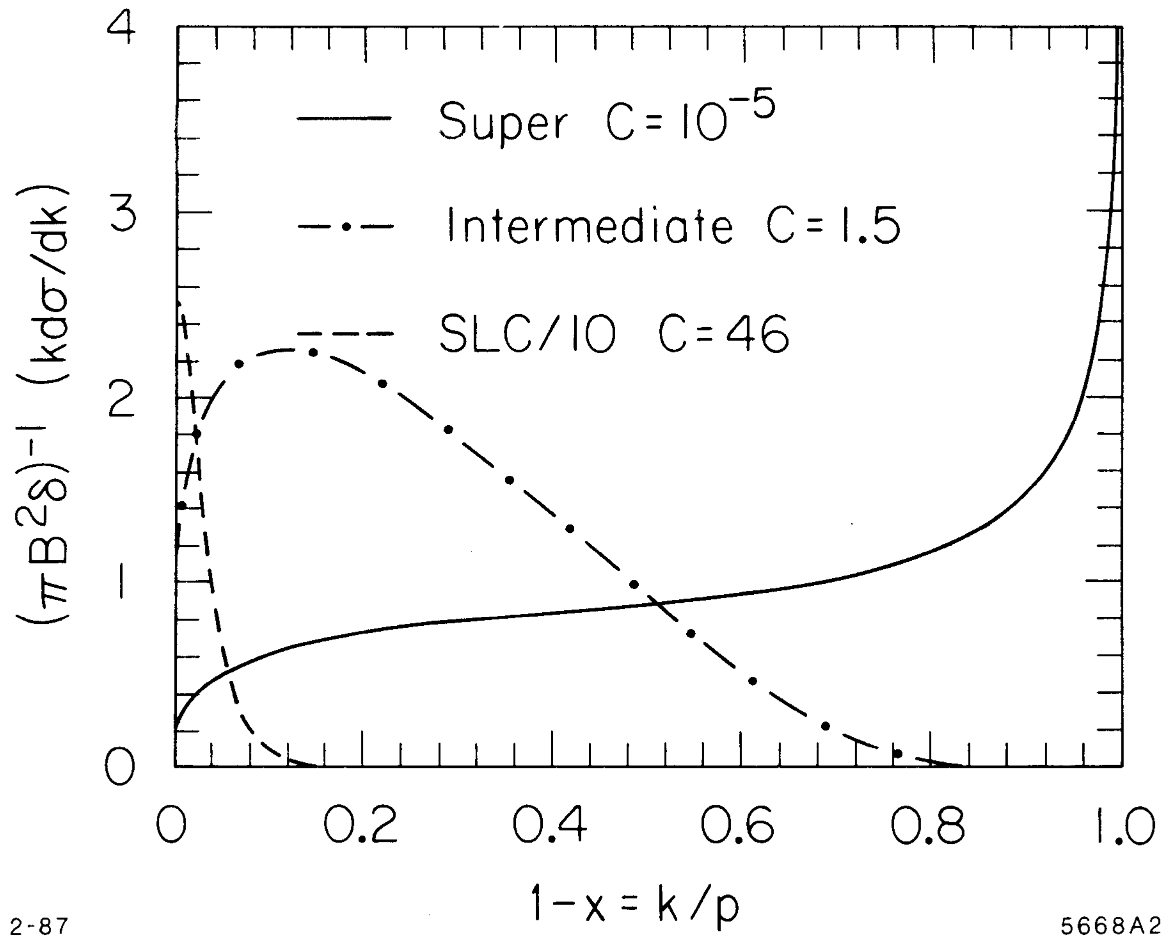

SCALING SPECTRUM R(u)
0.25
0.20
0.15
0.10
0.05
0
Exact
Approx.
0
1
2
3
4
u = [C(1−x)/x]^{2/3}
2-86
5668A3

FIG. 2



Super C = 10^-5
Intermediate C = 1.5
SLC/10 C = 46
0
1
2
3
4
0
0.2
0.4
0.6
0.8
1.0
(πB²δ)⁻¹ (kdσ/dk)
1 − x = k/p
2-87
5668A2

FIG. 3