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COVARIANT ONE LOOP FERMION EMISSION AMPLITUDES IN CLOSED STRING THEORIES

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ABSTRACT

We calculate the 2 fermion–2 boson and the 4 fermion scattering amplitudes at the one loop in the NSR formulation of the heterotic and type II superstring theories using the covariant fermion emission vertex. The results are shown to agree with well known amplitudes calculated in the light-cone gauge in the Green-Schwarz formulation. The agreement of the 4 fermion amplitude entails the use of a new ϑ–function identity which we also prove.

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I. Introduction

Calculation of covariant fermionic loop amplitudes in the heterotic or superstring theory have so far been hampered by the apparent complexity of the covariant fermion emission vertex of Friedan, Martinec and Shenker [1] and Knizhnik [2]. In this covariant formalism the vertex describing fermion emission involves spin fields of matter and superconformal ghosts, and the calculation of fermionic scattering amplitudes would necessarily require correlation functions of these fields. Apriori, these are not so straightforward to compute, essentially since the spin fields do not possess a local representation in terms of the fundamental fields on the world-sheet.

At the tree level, one way around this was to use the current algebra of $SO(10)$ [3] to derive a differential equation for the spin field correlator, or more conveniently to proceed through bosonization [1]. Both have successfully been applied in the calculation of tree level fermion emission amplitudes in ref. [4]* and the results seem to agree with those of the light-cone gauge. On the torus or higher genus surfaces on the other hand, current bosonization techniques are plagued by subtle global issues such as the bosonization of the superconformal ghosts. These render their practical utility at this stage doubtful.

In a previous work [6, 7], we have succeeded in deriving integrable first order differential equations for the spin correlators, using complex function theory on arbitrary genus Riemann surfaces and the idea that the stress tensor generates deformations of the moduli [8]. In this paper we shall use these results to explicitly calculate fermion emission amplitudes in the heterotic and superstring theories. In order to avoid subtleties associated with the supermoduli [9] on higher genera, we shall for the time being restrict ourselves to one loop.

One of the motivations behind this work is the need for a better understanding of the covariant formulation of fermionic strings on the torus and higher genus

⋆ Covariant tree level amplitudes for general n-point functions with external bosonic vertices have also been computed in Ref.[5].

Riemann surfaces. As a first step, one would like to at least verify explicitly that this formulation of string theory reproduces the physically sensible 1–loop scattering amplitudes of the light-cone gauge theory of Green and Schwarz [10, 11]. Scattering amplitudes of up to four external bosons have been calculated in ref. [12] and [13]. These results agree with the light-cone gauge answer. However they do not test the validity of the fermion emission vertex or the prescription of Friedan, Martinec and Shenker for calculating scattering amplitudes at the one loop. Calculation of scattering amplitudes involving up to three external lines (fermionic or bosonic) were presented in ref. [7]† . These were all shown to vanish identically in agreement with the light-cone gauge results and the nonrenormalization theorems. In this paper we shall present the calculation of the 2 fermion 2 boson and the 4 fermion scattering amplitudes at the one loop. The results, as we shall see, agree with the light cone gauge scattering amplitudes[11].

The organization of this paper is as follows: In section 2 we review the calculation of spin field correlation functions on the torus. In section 3 we present the details of the 2F2B and the 4F calculations. Section 4 contains our conclusions and some remarks about higher loop scattering amplitudes. Some technical details which include our γ–matrix and ϑ–function conventions, the proof of a new ϑ–function identity and calculation of some relevant bosonic correlators are relegated to three appendices at the end of the paper.

† At genera higher than one, one faces subtleties associated with the supermoduli and the ghost background charge. In ref. [7] we proposed an ansatz for handling these subtleties which made the calculation feasible. Proof of this ansatz awaits a deeper understanding of the supermoduli. Nevertheless, our results in that reference restricted to one loop are independent of the ansatz.

II. Spin Operators

In this section we shall review some of the pertinent facts about spin operators and their correlation functions [6,7]. For the purposes of this paper it is sufficient to restrict ourselves to correlation functions on the torus. The more general case on arbitrary genus Riemann surfaces have been dealt with in ref. [7].

We shall first analyse the $SO(2)$ spin model, which is a system of one complex Weyl fermion $\psi(z)$ and its associated spin fields $S^{\pm}(z)$. Correlation functions of the $SO(10)$ spin fields that appear in the fermion emission vertices can be assembled out of those of the $SO(2)$ model as we indicate at the end of this section.

The fields $\psi(z), S^{\pm}(z)$ obey the following operator product expansions,

$$\begin{aligned}
\bar{\psi}(z)S^{+}(w) &\sim (z-w)^{-\frac{1}{2}}S^{-}(w)+\ldots \\
\bar{\psi}(z)S^{-}(w) &\sim (z-w)^{\frac{1}{2}}\hat{S}^{-}(w)+\ldots \\
\psi(z)S^{+}(w) &\sim (z-w)^{\frac{1}{2}}\hat{S}^{+}(w)+\ldots \\
\psi(z)S^{-}(w) &\sim (z-w)^{-\frac{1}{2}}S^{+}(w)+\ldots \\
\bar{\psi}(z)\bar{\psi}(w) &\sim (z-w)+\ldots \\
\psi(z)\psi(w) &\sim (z-w)+\ldots \\
\bar{\psi}(z)\psi(w) &\sim (z-w)^{-1}+\ldots \\
S^{+}(z)S^{-}(w) &\sim (z-w)^{-\frac{1}{4}}+\ldots
\end{aligned} \tag{2.1}$$

where $\ldots$ denotes less singular terms and $\hat{S}^{\pm}$ are excited spin fields of conformal dimension $\frac{3}{2}$. It is perhaps important to emphasize at this stage that the operator product expansions in (2.1) may be realized explicitly by bosonization [1]. However, in order to avoid the subtle global issues of bosonization that arise on the torus and higher genus surfaces [14,15], we shall at no stage in our calculation use bosonization. Instead we shall calculate the spin field correlation function by first deriving a differential equation for it as we now explain.

We shall be interested in calculating a correlation function of the form,

$$F(y_i, z_i, u_i, v_i) \equiv \left\langle \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i) \right\rangle \tag{2.2}$$

on the torus. $\frac{1}{2}(N_1 - N_2) + (N_4 - N_3)$ here must vanish in order to conserve the total fermionic charge. In order to calculate this we start with another Green function defined by:

$$G(z, w; y_i, z_i, u_i, v_i,) \equiv \frac{\left\langle \bar{\psi}(z)\psi(w) \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i) \right\rangle}{\left\langle \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i) \right\rangle} \tag{2.3}$$

An explicit expression for G can be written down by examining its zeros, singularities and periodicities as a function of z and w. In particular using (2.1) we see that $G(z, w; y_i, z_i, u_i, v_i)$ has to satisfy the following conditions:

$$\begin{aligned}
&\lim_{z \to w} G(z, w; y_i, z_i, u_i, v_i) = \frac{1}{(z-w)} + \ldots \\
&\lim_{z \to y_i} G \sim (z - y_i)^{-\frac{1}{2}}, \qquad \lim_{w \to y_i} G \sim (w - y_i)^{+\frac{1}{2}}, \\
&\lim_{z \to z_i} G \sim (z - z_i)^{+\frac{1}{2}}, \qquad \lim_{w \to z_i} G \sim (w - z_i)^{-\frac{1}{2}}, \\
&\lim_{z \to u_i} G \sim (z - u_i), \qquad \lim_{w \to u_i} G \sim (w - u_i)^{-1}, \\
&\lim_{z \to v_i} G \sim (z - v_i)^{-1}, \qquad \lim_{w \to v_i} G \sim (w - v_i) .
\end{aligned} \tag{2.4}$$

In addition, as a function of z and w, G has to have the periodicity properties dictated by the spin structure of ψ.

The unique Green's function that satisfies all the above properties can be written as:*

$$G_\nu(z,w;y_i,z_i,u_i,v_i)$$

$$= \left(\frac{\prod_i \vartheta_1(z-z_i)\ \prod_j \vartheta_1(w-y_j)}{\prod_j \vartheta_1(z-y_j)\ \prod_i \vartheta_1(w-z_i)}\right)^{\frac{1}{2}} \left(\frac{\prod_i \vartheta_1(z-u_i)\prod_j \vartheta_1(w-v_j)}{\prod_j \vartheta_1(z-v_j)\prod_i \vartheta_1(w-u_i)}\right)\left(\frac{\vartheta_1'(0)}{\vartheta_1(z-w)}\right)$$

$$\frac{\vartheta_\nu\left(w-z+\frac{1}{2}\sum_{i=1}^{N_1} y_i - \frac{1}{2}\sum_{i=1}^{N_2} z_i + \sum_{i=1}^{N_4} v_i - \sum_{i=1}^{N_3} u_i\right)}{\vartheta_\nu\left(\frac{1}{2}\sum_{i=1}^{N_1} y_i - \frac{1}{2}\sum_{i=1}^{N_2} z_i + \sum_{i=1}^{N_4} v_i - \sum_{i=1}^{N_3} u_i\right)}\ . \tag{2.5}$$

Here $\nu = 1, 2, 3, 4$ correspond to spin structures (P,P), (P,A), (A,A) and (A,P) respectively. Now from (2.5) we may derive a differential equation for F. Let us consider the following object.

$$\langle\langle\ T(z)\ \rangle\rangle_\nu \equiv \frac{\left\langle T(z)\prod_{i=1}^{N_1} S^+(y_i)\prod_{i=1}^{N_2} S^-(z_i)\prod_{i=1}^{N_3}\bar{\psi}(u_i)\prod_{i=1}^{N_4}\psi(v_i)\right\rangle_\nu}{\left\langle \prod_{i=1}^{N_1} S^+(y_i)\prod_{i=1}^{N_2} S^-(z_i)\prod_{i=1}^{N_3}\bar{\psi}(u_i)\prod_{i=1}^{N_4}\psi(v_i)\right\rangle_\nu} \tag{2.6}$$

where $T(z)$ is the stress tensor of the system defined as [1]:

$$T(z) = \lim_{w\to z}\left(\frac{1}{2}(\partial_z\bar{\psi}(z)\psi(w) - \bar{\psi}(z)\partial_w\psi(w)) + \frac{1}{(z-w)^2}\right)\ . \tag{2.7}$$

* See appendix B for definition of ϑ–functions.

From (2.7) we see that $\langle\langle\ T(z)\ \rangle\rangle$ is given by

$$\langle\langle\ T(z)\ \rangle\rangle_\nu = \lim_{w\to z}\left(\frac{1}{2}\left(\partial_z G_\nu(z,w;y_i,z_i,u_i,v_i)\right.\right.$$
$$\left.\left. - \partial_w G_\nu(z,w;y_i,z_i,u_i,v_i)\right) + \frac{1}{(z-w)^2}\right) . \qquad (2.8)$$

The right hand side of the above expression can be calculated explicitly using the Green function in (2.5). Given $\langle\langle\ T(z)\ \rangle\rangle$ one could investigate the limit as z approaches some of the insertion points say y_i. Now since the operator product expansion of T with any primary field ϕ is

$$T(z)\phi(w) \sim \frac{h}{(z-w)^2}\phi(w) + \frac{1}{(z-w)}\partial_w\phi(w), \qquad (2.9)$$

one could identify the singular part of $\langle\langle\ T(z)\ \rangle\rangle$ in the limit $z \to y_i$ with

$$\frac{1}{8}\frac{1}{(z-y_i)^2} + \frac{1}{(z-y_i)}\frac{1}{F(y_i,z_i,u_i,v_i)}\frac{\partial}{\partial y_i}F(y_i,z_i,u_i,v_i) \ . \qquad (2.10)$$

where $\frac{1}{8}$ is the conformal dimension of $S^+(y_i)$. Thus the residue of the simple pole in that limit is equal to the derivative in the variable y_i of the logarithm of the correlation function of interest. Studying other limits (*e.g.*, $z \to u_i, v_i, z_i$) in $\langle\langle\ T(z)\ \rangle\rangle$ furnishes differential equations for F in all other variables. It is straightforward to verify that the resulting set of differential equations is integrable and that furthermore they integrate to give the following answer:

$$F_\nu(y_i,z_i,u_i,v_i)$$
$$= K_\nu \prod_{i<j}\vartheta_1(y_i-y_j))^{\frac{1}{4}}(\prod_{i<j}\vartheta_1(z_i-z_j))^{\frac{1}{4}}(\prod_{i<j}\vartheta_1(u_i-u_j))(\prod_{i<j}\vartheta_1(v_i-v_j))$$
$$(\prod_{i,j}\vartheta_1(z_i-y_j))^{-\frac{1}{4}}(\prod_{i,j}\vartheta_1(u_i-y_j))^{-\frac{1}{2}}(\prod_{i,j}\vartheta_1(v_i-y_j))^{\frac{1}{2}}(\prod_{i,j}\vartheta_1(u_j-z_i))^{\frac{1}{2}}$$
$$(\prod_{i,j}\vartheta_1(v_j-z_i))^{-\frac{1}{2}}(\prod_{i,j}\vartheta_1(v_j-u_i))^{-1}\vartheta_\nu\left(\frac{1}{2}\sum_i y_i - \frac{1}{2}\sum_i z_i + \sum_i v_i - \sum_i u_i\right) \qquad (2.11)$$

Here K_ν is a normalization constant that can be determined by factoring the correlator F_ν on the partition function in the sector with spin structure ν. It is important to notice that since the argument of ϑ_ν contains $\frac{1}{2}z_i$ and $\frac{1}{2}y_i$, it changes to a ϑ–function with a different characteristic as we translate y_i or z_i by 1 or τ on the torus, and hence F_ν is not periodic. As we shall see in the next section, correlation functions of physical vertex operators in string theory involve appropriate powers of the correlation functions of the spin fields given here and of the superconformal ghosts, which make them periodic after summing over the spin structures. The relative phases and normalizations of the contributions from different spin structures are fixed by dragging z_i and y_i around 1 and τ and demanding periodicity as explained in refs. [6,7]. Generalization of eq. (2.11) to arbitrary genus Riemann surfaces has been given in ref. [7].

Next we turn our attention to the superconformal ghost system (β, γ) with the stress tensor

$$T_g(z) = \lim_{z \to w} \left(-\frac{3}{2}\beta(z)\partial_w\gamma(w) - \frac{1}{2}\partial_z\beta(z)\gamma(w) - \frac{1}{(z-w)^2} \right) \quad . \tag{2.12}$$

We need to analyse correlation functions of the spin fields $S_g^\pm$ of this system. In particular we would like to calculate correlation functions of the form

$$F^g(y_i, z_i) = \left\langle \prod_{i=1}^{N} S_g^+(y_i) S_g^-(z_i) \right\rangle \quad . \tag{2.13}$$

These appear in the calculation of fermionic amplitudes as we shall see in the next section.

At one loop order there are no subtleties concerning the supermoduli or the background ghost charge.* So we can proceed as before by first constructing the

* In the presence of fermion emission vertex operators carrying ghost charges that add up to zero, there are no supermoduli, even in the periodic periodic sector.

Green function G^g of the β,γ system in the presence of the ghost spin operators $S_g^{\pm}$, *i.e.*,

$$G_\nu^g(y,z;y_i,z_i) \equiv \frac{\langle \beta(y)\gamma(z)\prod_{i=1}^{N} S_g^+(y_i)S_g^-(z_i)\rangle_\nu}{\langle \prod_{i=1}^{N} S_g^+(y_i)S_g^-(z_i)\rangle_\nu} \tag{2.14}$$

G^g can be written down explicitly by analysing its zeros and poles as dictated by the O.P.E. of β,γ with $S_g^{\pm}$, and its periodicity properties as a function of y and z as dictated by the spin structure ν. The unique answer for G is given by:

$$\begin{aligned} G_\nu^g(y,z;y_i,z_i) = & \left(\frac{\prod_i \vartheta_1(y-y_i)\vartheta_1(z-z_i)}{\prod_i \vartheta_1(z-y_i)\vartheta_1(y-z_i)}\right)^{\frac{1}{2}} \\ & \left(\frac{\vartheta_\nu\left(y-z+\frac{1}{2}\sum(y_i-z_i)\right)}{\vartheta_\nu\left(\frac{1}{2}(y_i-z_i)\right)}\right)\left(\frac{\vartheta_1'(0)}{\vartheta_1(z-y)}\right) \end{aligned} \tag{2.15}$$

From this we derive a set of first order differential equations for F_ν^g as before using the stress tensor (2.12). The answer is:

$$\begin{aligned} F_\nu^g(y_i,z_i) = & K_\nu^g\left(\prod_{i<j}\vartheta_1(z_i-z_j)\right)^{-\frac{1}{4}}\left(\prod_{i<j}\vartheta_1(y_i-y_j)\right)^{-\frac{1}{4}} \\ & \left(\prod_{i<j}\vartheta_1(z_i-y_j)\right)^{\frac{1}{4}}\vartheta_\nu^{-1}\left(\frac{1}{2}\sum_i(y_i-z_i)\right) . \end{aligned} \tag{2.16}$$

We next consider the $SO(10)$ spin operators appearing in the covariant fermion vertex. These operators are most conveniently introduced by first combining the ten right moving Majorana–Weyl fermions $\psi^\mu(z)$ in the NSR formulation of the superstring or the heterotic string into five complex fermions. One may then in the standard fashion introduce five sets of $SO(2)$ spin operators

$(S_i^+,\ S_i^-)$ $i = 1, \ldots 5$. The $SO(10)$ spin operators are then given by

$$S_1^\pm S_2^\pm \ldots S_5^\pm. \tag{2.17}$$

There are 32 such operators. We may divide them into two sets according to their chiralities. We adopt the convention that operators with an even number of S^- are positive chirality and those with an odd number of S^- are negative chirality. In this helicity basis correlation functions of the $SO(10)$ spin operators will simply be products of correlation functions of $SO(2)$ spin fields which we have already calculated. Covariance can readily be restored by first writing down the most general Lorentz structure for a given correlation function and then determining the various Lorentz invariant coefficients from the calculation of correlation functions in the helicity basis with fixed polarizations. This will be illustrated in our calculation of fermion scattering amplitudes in the next section.

III. Calculation of Fermionic Amplitudes

A. Fermion Emission Vertices

Fermion emission vertices in the covariant formulation of the string theory have been constructed by Friedan, Martinec and Shenker [1], and by Knizhnik [2]. Here we shall briefly review their construction. The basic fermion emission vertex denoted by $V_{-\frac{1}{2}}(u,k,z)$ is given by,

$$V_{-\frac{1}{2}}(u,k,z) = u^{\alpha}(k)\ S_{\alpha}(z)\ S_g^{-}(z)\ e^{ik\cdot X(z)}, \tag{3.1}$$

where u^{α} is a Majorana spinor reflecting the polarization of the external state and k is the momentum of that state. $S_{\alpha}(z)$ and $S_g^{-}(z)$ are the $SO(10)$ and the ghost spin fields introduced in sec. II. Since S_g^{-} carries a ghost charge of $-\frac{1}{2}$ the correlation function involving several $V_{-\frac{1}{2}}$'s on the torus vanishes identically due to the ghost charge conservation. The solution to this problem was given in ref. [1,2] where a new vertex $V_{+\frac{1}{2}}$ was introduced as follows:

$$V_{+\frac{1}{2}}(u,k,z) = u^{\alpha}(k) S_g^{+}(z) \lim_{w\to z}(w-z)^{\frac{1}{2}}\left(\psi_{\mu}(w)\partial X^{\mu}(w) S_{\alpha}(z) e^{ik\cdot X(z)}\right)\ , \tag{3.2}$$

where S_g^{+} is the ghost spin field carrying ghost charge $+\frac{1}{2}$. The appearance of the fermionic component of the stress tensor $\psi_{\mu}(w)\partial X^{\mu}(w)$ is eq. (3.2) may be traced to the integration over the extra supermoduli that appear in the calculation of a correlator if we represent each of the fermion vertices by $V_{-\frac{1}{2}}$ and explicitly remove the integration over the zero modes of $\beta(z)$ [16]. Note that the most singular part of the operator product expansion in eq. (3.2) given by,*

$$u^{\alpha}(k)\ S_g^{+}(z)(w-z)^{-1}\ (\gamma_{\mu})_{\alpha\beta} S^{\beta}(z)(-ik^{\mu}) e^{ik\cdot X(z)}\ , \tag{3.3}$$

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* See appendix A for our γ-matrix conventions.

vanishes by the on-shell condition $\not{k}u = 0$. Hence we can write (3.2) as,

$$\begin{aligned} V_{+\frac{1}{2}}(u,k,z) =& u^\alpha(k)\ S_g^+(z) e^{ik\cdot X(z)} \\ & [(\gamma_\mu)_{\alpha\beta}\ S^\beta(z)\partial X^\mu(z) - ik^\mu(\lim_{w\to z}(w-z)^{-\frac{1}{2}}\psi_\mu(w)S_\alpha(z))] \end{aligned} \tag{3.4}$$

The prescription for calculating an amplitude with $(2n)$ external fermions, as given in ref. [1] is to use the vertex $V_{-\frac{1}{2}}$ for n of the fermions, and the vertex $V_{+\frac{1}{2}}$ for the other n. Since the total ghost charge now adds up to zero, as it should, on the torus we expect a nonzero answer. At the intermediate stages of the calculation we loose manifest (anti)-symmetry under the interchange of the external fermions if one of them is represented by the vertex $V_{-\frac{1}{2}}$ and the other is represented by the vertex $V_{+\frac{1}{2}}$. However, general arguments were presented in ref. [1] showing that the final result is independent of the particular assignment of $V_{+\frac{1}{2}}$ and $V_{-\frac{1}{2}}$ to different vertices, and must be totally antisymmetric in the external lines. We shall verify explicitly that this is indeed the case.

For our calculation we also need the bosonic vertex operator. This is canonically given by,

$$V_0(\varsigma,k,z) = \varsigma_\mu[\partial X^\mu(z) + ik_\nu\psi^\mu(z)\psi^\nu(z)]\ e^{ik\cdot X(z)}\ , \tag{3.5}$$

where ς is the polarization tensor of the external bosonic particle.

Finally we should point out that the various vertex operators listed so far only furnish half of a complete vertex operator, namely they only give the (0, 1) part of a vertex operator (except the $e^{ik\cdot X}$ factor which receives contribution from both the left and the right handed sectors). For example in the $E_8\times E_8$ heterotic string theory, if we are interested in the scattering of gaugino or gauge bosons, we must multiply each of these vertex operators by

$$\lambda^s(\bar{z})(T^a)_{st}\lambda^t(\bar{z})\ , \tag{3.6}$$

or by an appropriate spin operator of $SO(16)\times SO(16)$ [17,6]. Here λ^s are the 32 gauge fermions transforming in the (16, 1) or (1, 16) representation of

$SO(16)\times SO(16)$, and T^a is a generator of the group. If instead we are interested in the scattering of gravitons or gravitinos, the antianalytic part of the vertex operator is given by,

$$\bar{\partial}X^{\mu} \ . \tag{3.7}$$

In the next two subsections we shall calculate the 2F2B and 4F scattering amplitudes. For simplicity we first restrict ourselves to the scattering of gauge bosons and gauginos, since in this case the left and the right sectors decouple from each other, except for the $e^{ik\cdot X}$ factors. We shall discuss the scattering of gravitons, gravitinos and other members of the supergravity multiplet at the end of each subsection.

B. 2 Fermion - 2 Boson Scattering

The relevant correlation function for the calculation of 2 fermion - 2 boson scattering is,

$$\langle V_0(k_1,\varsigma^{(1)},z_1)\ V_0(k_2,\varsigma^{(2)},z_2)\ V_{-\frac{1}{2}}(k_3,u_{(3)},z_3)\ V_{+\frac{1}{2}}(k_4,u_{(4)},z_4)\rangle\ . \tag{3.8}$$

Using eqs. (3.1), (3.4) and (3.5), this correlation function may be written as the sum of eight different correlation functions. For each of them, we shall first use eq. (2.11) and (2.16) of the previous section to evaluate the correlation functions involving the spin operators $S_\alpha, S_g^{\pm}$ and the fermion fields ψ, sum over spin structures, and if the answer is non-zero, multiply it by the relevant correlation function involving the bosonic X fields. We shall illustrate the evaluation of the various terms appearing in (3.8) through a few examples.

i) One of the eight terms in (3.8) looks like

$$\begin{aligned}&\langle(\prod_i e^{ik_i\cdot X(z_i)})\partial X^{\mu_1}(z_1)\partial X^{\mu_2}(z_2)\partial X^{\mu_4}(z_4)\rangle\varsigma^{(1)}_{\mu_1}\varsigma^{(2)}_{\mu_2}u^{\alpha_3}_{(3)}u^{\alpha_4}_{(4)}(\gamma_{\mu_4})_{\alpha_4\beta_4}\\&\qquad\langle S_{\alpha_3}(z_3)S^{\beta_4}(z_4)\rangle\langle S_g^-(z_3)S_g^+(z_4)\rangle\end{aligned} \tag{3.9}$$

The correlation function involving the spin fields can be written as,

$$\langle S_{\alpha_3}(z_3)S^{\beta_4}(z_4)\rangle = A(z_3,z_4)\delta_{\alpha_3}^{\beta_4} \ , \tag{3.10}$$

since this is the only possible Lorentz invariant tensor structure. $A(z_3,z_4)$ may be evaluated by calculating the correlator for specific values of α_3 and β_4, say $\alpha_3 = (+++++)$, $\beta_4 = (-----)$. The answer for this correlator in a given spin structure is next multiplied by the ghost correlator $\langle S_g^-(z_3)S_g^+(z_4)\rangle$ in that spin structure, and a sum over spin structures is performed. The relative normalizations between the contribution from different spin structures are determined by demanding that the final result be periodic in z_3(and z_4) with periods 1 and τ. The combined contribution from the ghost and the $SO(10)$ spin correlators is given by,

$$\vartheta_1(z_3-z_4)^{-1}\sum_\nu \delta_\nu\left(\vartheta_\nu\left(\frac{z_3-z_4}{2}\right)\right)^4 \ , \tag{3.11}$$

where $\delta_1 = 1$, $\delta_2 = -1$, $\delta_3 = 1$ and $\delta_4 = -1$. This expression vanishes identically using the Riemann ϑ–identity (see Appendix B).

ii) A less trivial term in the correlator (3.8) is the term

$$\begin{aligned}&\varsigma^{(1)}_{\mu_1}\ \varsigma^{(2)}_{\mu_2}u^{\alpha_3}_{(3)}u^{\alpha_4}_{(4)}(\gamma_{\mu_4})_{\alpha_4\beta_4}i(k_1)_{\nu_1}\big\langle\big(\prod_i e^{ik_i\cdot X(z_i)}\big)\ \partial X^{\mu_2}(z_2)\partial X^{\mu_4}(z_4)\big\rangle \\ &\quad\big\langle\psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1)S_{\alpha_3}(z_3)S^{\beta_4}(z_4)\big\rangle\langle S_g^-(z_3)S_g^+(z_4)\rangle \ .\end{aligned} \tag{3.12}$$

The most general tensor structure for the correlator involving the spin fields is

$$\begin{aligned}&\langle\psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1)S_{\alpha_3}(z_3)S^{\beta_4}(z_4)\rangle\langle S_g^-(z_3)S_g^+(z_4)\rangle \\ &\qquad = \ A\ \delta^{\mu_1\nu_1}\delta_{\alpha_3}^{\beta_4} + B\ (\gamma^{\mu_1}\gamma^{\nu_1})_{\alpha_3}^{\beta_4} \ ,\end{aligned} \tag{3.13}$$

of this the first term does not contribute to (3.12) due to the mass shell condition $\varsigma^{(1)}\cdot k_1 = 0$. The second term may be evaluated by setting $\mu_1 = 1$, $\nu_1 = 2$

$\alpha_3 = (--+++)$, $\beta_4 = (-----)$. The answer after multiplying by the ghost correlator and summing over spin structures turns out to be proportional to:

$$\sum_\nu \delta_\nu \left(\vartheta_\nu\left(z_1 - \frac{z_3+z_4}{2}\right)\right)^2 \left(\vartheta_\nu\left(\frac{z_3-z_4}{2}\right)\right)^2, \tag{3.14}$$

which again vanishes as a consequence of the Riemann ϑ–identity.

In this fashion we may analyse the rest of the eight terms appearing in (3.8). All of these can shown to vanish either by on shell constraints or through a Riemann ϑ–identity, except for one term which takes the form:

$$\begin{aligned} \Lambda = \; & \varsigma^{(1)}_{\mu_1}\varsigma^{(2)}_{\mu_2} u^{\alpha_3}_{(3)} u^{\alpha_4}_{(4)} (-ik_1)_{\nu_1}(-ik_2)_{\nu_2}(-ik_4)_{\mu_4} \\ & \lim_{w_4\to z_4} (w_4 - z_4)^{-\frac{1}{2}} \langle \psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1)\psi^{\mu_2}(z_2)\psi^{\nu_2}(z_2)\psi^{\mu_4}(w_4) S_{\alpha_3}(z_3) S_{\alpha_4}(z_4)\rangle \\ & \qquad \langle S^-_g(z_3) S^+_g(z_4)\rangle \langle \prod_i e^{ik_i\cdot X(z_i)} \rangle \end{aligned} \tag{3.15}$$

A simple group theoretic analysis shows that Λ has 26 independent tensor structures. (This is the number of independent singlets in the $10\otimes 10\otimes 10\otimes 10\otimes 10\otimes 16\otimes 16$ representation of $SO(10)$). We write down the general expansion for the correlator as

$$\langle\psi^{\mu_1}(z_1)\psi^{\nu_1}(z_1)\psi^{\mu_2}(z_2)\psi^{\nu_2}(z_2)\psi^{\mu_4}(w_4)S_{\alpha_3}(z_3)S_{\alpha_4}(z_4)\rangle\langle S_g^-(z_3)S_g^+(z_4)\rangle$$

$$=A\delta^{\mu_1\nu_1}\delta^{\mu_2\nu_2}(\gamma^{\mu_4})_{\alpha_3\alpha_4}+B\delta^{\mu_1\nu_1}\delta^{\mu_2\mu_4}(\gamma^{\nu_2})_{\alpha_3\alpha_4}+C\delta^{\mu_1\nu_1}\delta^{\nu_2\mu_4}(\gamma^{\mu_2})_{\alpha_3\alpha_4}$$

$$+D\delta^{\mu_2\nu_2}\delta^{\mu_1\mu_4}(\gamma^{\nu_1})_{\alpha_3\alpha_4}+E\delta^{\mu_2\nu_2}\delta^{\nu_1\mu_4}(\gamma^{\mu_1})_{\alpha_3\alpha_4}+F\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}(\gamma^{\mu_4})_{\alpha_3\alpha_4}$$

$$+G\delta^{\mu_1\nu_2}\delta^{\mu_2\nu_1}(\gamma^{\mu_4})_{\alpha_3\alpha_4}+H(\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_2\nu_2}+I(\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\mu_2})_{\alpha_4\alpha_3}\delta^{\nu_1\nu_2}$$

$$+J(\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\nu_1\mu_2}+K(\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_1\nu_2}+L(\gamma^{\mu_4}\gamma^{\mu_2}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\nu_1\mu_1}$$

$$+M(\gamma^{\mu_4}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_1\mu_2}+O(\gamma^{\mu_4}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}$$

$$+N(\gamma^{\mu_1})_{\alpha_3\alpha_4}\delta^{\mu_2\nu_1}\delta^{\mu_4\nu_2}+P(\gamma^{\mu_1})_{\alpha_3\alpha_4}\delta^{\mu_2\mu_4}\delta^{\nu_2\nu_1}+Q(\gamma^{\mu_2})_{\alpha_3\alpha_4}\delta^{\mu_1\mu_4}\delta^{\nu_1\nu_2}$$

$$+R(\gamma^{\mu_2})_{\alpha_3\alpha_4}\delta^{\mu_1\nu_2}\delta^{\mu_4\nu_1}+S(\gamma^{\nu_1})_{\alpha_3\alpha_4}\delta^{\mu_1\nu_2}\delta^{\mu_4\mu_2}+T(\gamma^{\nu_1})_{\alpha_3\alpha_4}\delta^{\nu_2\mu_4}\delta^{\mu_2\mu_1}$$

$$+U(\gamma^{\nu_2})_{\alpha_3\alpha_4}\delta^{\nu_1\mu_4}\delta^{\mu_1\mu_2}+V(\gamma^{\nu_2})_{\alpha_3\alpha_4}\delta^{\mu_1\mu_4}\delta^{\nu_1\mu_2}+W(\gamma^{\mu_1}\gamma^{\nu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_2\mu_4}$$

$$+X(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\nu_2\mu_4}+Y(\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_1\mu_4}+Z(\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\nu_1\mu_4}$$

(3.16)

The tensor structures have been chosen in such a way that the contribution from the tensors multiplying A through O vanish by on-shell constraints when substituted in eq. (3.15). Hence we only need to evaluate the coefficients N through Z. The calculation simplifies by noting that the result must be symmetric under the simultaneous exchange $\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$, $z_1 \leftrightarrow z_2$. Symmetry under this exchange gives

$$\begin{aligned} W(z_1,z_2) &= Y(z_2,z_1) \\ R(z_1,z_2) &= N(z_2,z_1) \\ P(z_1,z_2) &= Q(z_2,z_1) \\ X(z_1,z_2) &= Z(z_2,z_1) \\ S(z_1,z_2) &= V(z_2,z_1) \\ T(z_1,z_2) &= U(z_2,z_1) \ . \end{aligned} \qquad (3.17)$$

This has the advantage of cutting down the number of terms that we need to evaluate by half.

Let us now set $\mu_2 = 1, \nu_1 = \bar{1}, \mu_4 = 2, \nu_2 = \bar{2}, \mu_1 = 3, \alpha_3 = (- - - - +)$ and $\alpha_4 = (+ + - + -)$. The only term in (3.16) that contributes for this configuration is N. On the other hand the correlator of the spin fields on the left hand side of (3.16) contributes by

$$
\begin{aligned}
N =& c\Big\{\vartheta_1(z_3 - z_4)^{-\frac{1}{2}}\vartheta_1(z_3 - w_4)^{-\frac{1}{2}}\vartheta_1(z_1 - z_4)^{-1} \\
&\vartheta_1(w_4 - z_4)^{\frac{1}{2}}\vartheta_1(z_1 - z_2)^{-1}\vartheta_1(z_2 - w_4)^{-1}\Big\} \\
&\sum_\nu \delta_\nu \vartheta_\nu\big(\frac{z_3 - z_4}{2} + z_1 - z_2\big)\vartheta_\nu\big(\frac{z_3 - z_4}{2} + z_2 - w_4\big) \\
&\qquad \vartheta_\nu\big(\frac{z_3 + z_4}{2} - z_1\big)\vartheta_\nu\big(\frac{z_3 - z_4}{2}\big) ,
\end{aligned}
\tag{3.18}
$$

where c is an overall normalization factor independent of z_i. c can be fixed as follows: consider the following limit of the correlation function (3.15) for the particular polarizations that we have chosen

$$
\begin{aligned}
\lim_{z_2\to z_1}\lim_{w_4\to z_2}\lim_{z_3\to z_1}\lim_{z_4\to z_3}\Big(&\langle\psi^3(z_1)\psi^{\bar{1}}(z_1)\psi^1(z_2)\psi^{\bar{2}}(z_2)\psi^2(w_4)S_{\alpha_3}(z_3)S_{\alpha_4}(z_4)\rangle \\
&\langle S_g^-(z_3)S_g^+(z_4)\rangle\Big) .
\end{aligned}
\tag{3.19}
$$

From the operator product expansion we see that in this limit the correlation function should diverge as

$$
(z_4 - z_3)^{-\frac{1}{2}}(z_1 - z_3)^{-1}(w_4 - z_2)^{-1}(z_2 - z_1)^{-1}\langle\ I\ \rangle_\nu \tag{3.20}
$$

where $\langle\ I\ \rangle_\nu$ is the expectation value of the identity operator in a spin structure characterized by ν. In this limit the contibution to (3.18) from a given spin structure diverges as:

$$c(\vartheta_\nu(0))^4(\vartheta_1'(0))^{-\frac{7}{2}}(z_4-z_3)^{-\frac{1}{2}}(z_1-z_3)^{-1}(w_4-z_2)^{-1}(z_1-z_2)^{-1} \quad . \tag{3.21}$$

But $\langle\ I\ \rangle_\nu$ is the partition function of a system of ten Majorana-Weyl fermions together with ten bosons and the corresponding ghosts. As is well known*

$$\langle\ I\ \rangle_\nu = \left(\frac{\vartheta_\nu(0)}{\vartheta_1'(0)}\right)^4 , \tag{3.22}$$

Comparing (3.20) and (3.21) we get that

$$c = (\vartheta_1'(0))^{-\frac{1}{2}} , \tag{3.23}$$

where we have ignored an overall total phase in c.

We may now simplify (3.18) by using the Riemann ϑ–identity (B.4). We further need to multiply by $(w_4 - z_4)^{-\frac{1}{2}}$ and take the limit $w_4 \to z_4$. In this limit, we see that all the z_i dependence of N cancels except for a prefactor of $2(w_4 - z_4)^{+\frac{1}{2}}\ (\vartheta_1'(0))^{\frac{1}{2}}$. So we finally get,

$$\lim_{w_4\to z_4}(w_4-z_4)^{-\frac{1}{2}}N = \lim_{w_4\to z_4}(w_4-z_4)^{-\frac{1}{2}}2c(\vartheta_1'(0))^{\frac{1}{2}}(w_4-z_4)^{\frac{1}{2}} = 2. \tag{3.24}$$

Next consider the configuration $\mu_2 = 1, \nu_1 = \bar{1}, \mu_4 = 2,\ \nu_2 = \bar{2},\ \mu_1 = 3, \alpha_3 = (+----)$, $\alpha_4 = (-+-++)$. From (3.16) we see that this contribution is given by $-N - X$. On the other hand application of the Riemann ϑ–identity shows that this correlator vanishes in the limit $w_4 \to z_4$. Thus we conclude that

$$X \simeq -N \simeq -2(w_4 - z_4)^{\frac{1}{2}} , \tag{3.25}$$

where $\simeq$ denotes equality up to the desired accuracy. Evaluation of the correlator with $\mu_1 = 1, \nu_2 = \bar{1},\ \mu_4 = 2, \mu_2 = \bar{2},\ \nu_1 = 3,\ \alpha_3 = (---+-)$, $\alpha_4 = (++--+)$ gives

---

* We have used only the analytic part of the partition function for normalization. The other parts containing the antianalytic dependence, $Im\ \tau$ and an overall numerical factor will be put in later.

$$S \simeq 0 \quad , \tag{3.26}$$

and with $\nu_2 = 1, \mu_4 = \bar{1}, \mu_2 = 2, \mu_1 = \bar{2}, \nu_1 = 3, \alpha_3 = (++-+-), \alpha_4 = (----+)$ gives

$$T \simeq 0 \quad . \tag{3.27}$$

Finally setting $\nu_2 = 1, \nu_1 = \bar{1}, \mu_4 = 2, \mu_2 = \bar{2}, \mu_1 = 3, \alpha_3 = (----+), \alpha_4 = (++-+-)$ gives

$$P \simeq 2\eta(w_4 - z_4)^{\frac{1}{2}}. \tag{3.28}$$

and $\nu_2 = 1, \nu_1 = \bar{1}, \mu_4 = 2, \mu_2 = \bar{2}, \mu_1 = 3, \alpha_3 = (+----), \alpha_4 = (-+-++)$ gives,

$$P + W \simeq 0 \quad . \tag{3.29}$$

Here η is a phase factor which is undetermined at this stage. (η could be determined by several ways. One way would be to find polarizations that would contribute to say P and N at the same time. This will determine the relative phase between P and N just as in eqs. (3.25) and (3.29). Another way which is typically easier is to use symmetry considerations as we do below).

Using eq. (3.17) we may now determine all the coefficients N through Z up to the phase η. We obtain

$$\begin{aligned} R \simeq N \simeq -Z \simeq -X \simeq\ & 2(w_4 - z_4)^{\frac{1}{2}} \ , \\ P \simeq Q \simeq -W \simeq -Y \simeq\ & 2\eta(w_4 - z_4)^{\frac{1}{2}} \ . \end{aligned} \tag{3.30}$$

with this (3.16) reduces to

$$\begin{aligned} 2(w_4 - z_4)^{\frac{1}{2}} [& (\gamma^{\mu_1}\gamma^{\nu_1}\gamma^{\mu_2})_{\alpha_4\alpha_3} \ \delta^{\mu_4\nu_2} + (\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_1})_{\alpha_4\alpha_3}\delta^{\mu_4\nu_1} \\ & + \eta((\gamma^{\mu_1}\gamma^{\nu_1}\gamma^{\nu_2})_{\alpha_4\alpha_3}\delta^{\mu_2\mu_4} + (\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\nu_1})_{\alpha_4\alpha_3}\delta^{\mu_1\mu_4})] \ , \end{aligned} \tag{3.31}$$

up to terms which do not contribute on-shell.† η may now be determined by demanding that (3.31) is antisymmetric under the interchange of μ_1 and ν_1 up to terms which vanish on-shell. This gives

† We have used the γ–matrix commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = \delta^{\mu\nu}$ in deriving eq.(3.31).

$$\eta = -1 \ . \tag{3.32}$$

Substitution into eq. (3.15) and some trivial γ–matrix manipulations yield

$$\Lambda = \left\langle \prod_i e^{ik_i \cdot X(z_i)} \right\rangle K(\varsigma^{(1)}, \varsigma^{(2)}, u_{(3)}, u_{(4)}, k_i) \ , \tag{3.33}$$

where

$$\begin{aligned} K \ &(\varsigma^{(1)}, \varsigma^{(2)}, u_{(3)}, u_{(4)}, k_i) \\ =& u_{(4)} \varsigma^{(1)} \cdot \gamma (k_1 + k_4) \cdot \gamma \ \varsigma^{(2)} \cdot \gamma u_{(3)} (2k_2 \cdot k_4) \\ &+ u_{(4)} \varsigma^{(2)} \cdot \gamma (k_2 + k_4) \cdot \gamma \ \varsigma^{(1)} \cdot \gamma u_{(3)} (2k_1 \cdot k_4) \ . \end{aligned} \tag{3.34}$$

Thus the final amplitude has the form

$$\begin{aligned} A \ (1, 2, 3, 4) = \int \frac{d^2 \tau}{(Im \ \tau)^5} \\ \int \prod_{i=2}^{4} d^2 z_i \ \left\{ \left\langle \prod_i \ e^{ik_i \cdot X(z_i)} \right\rangle \ \bar{F}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \ \right\} \\ K(\varsigma^{(1)}, \ \varsigma^{(2)}, \ u_{(3)}, \ u_{(4)}, \ k_i) \ . \end{aligned} \tag{3.35}$$

Here $\bar{F}$ is the properly normalized contribution to the correlator involving the antianalytic part of the vertex operators. $\langle \prod_i e^{ik_i \cdot X(z_i)} \rangle$ is the standard bosonic correlator normalized to unity. More explicitly it is given by

$$\left\langle \prod_i \ e^{ik_i \cdot X(z_i)} \right\rangle = \prod_{i<j} (\chi_{ij})^{k_i \cdot k_j} \ , \tag{3.36}$$

where $\chi_{ij} = | \vartheta_1 (z_i - z_j) / \vartheta_1' (0) |^2 \ exp \ \{ - \frac{2\pi}{Im \ \tau} \ (Im(z_i - z_j))^2 \}$ in the convention that the slope parameter α' has been set to 1. The results in (3.35) and (3.34) agree with the light-cone gauge answer (see for example ref. [11]).

Note that in order to obtain the final form (3.35) we never had to use correlators involving the X fields . As a result, our analysis goes through even if the contribution to the anti-analytic part of the vertex operator involves the field X. For example in the heterotic string theory, the two graviton two gravitino scattering amplitude is given by,

$$
\begin{aligned}
A(1,2,3,4) =& \{u_{(4)\mu_4}\, \varsigma^{(1)}_{\mu_1\nu_1}\, \gamma^{\nu_1}(k_1+k_4)\cdot\gamma\, \varsigma^{(2)}_{\mu_2\nu_2}\, \gamma^{\mu_2} u_{(3)\mu_3}(2k_2\cdot k_4) \\
& + u_{(4)\mu_4}\, \varsigma^{(2)}_{\mu_2\nu_2}\, \gamma^{\nu_2}(k_2+k_4)\cdot\gamma\, \varsigma^{(1)}_{\mu_1\nu_1}\gamma^{\nu_1} u_{(3)\mu_3}(2k_1\cdot k_4)\} \\
& \int \frac{d^2\tau}{(Im\ \tau)^5}\left\{\int\prod_{i=1}^{3} d^2 z_i \left\langle \prod_i e^{ik_i\cdot X(z_i)}\bar\partial X^{\mu_1}(z_1)\bar\partial X^{\mu_2}(z_2)\right.\right. \\
& \left.\left. \bar\partial X^{\mu_3}(z_3)\bar\partial X^{\mu_4}(z_4)\right\rangle\right\},
\end{aligned}
\tag{3.37}
$$

where $\varsigma^{(1)}$, $\varsigma^{(2)}$ are symmetric and traceless and $u_{(4)}, u_{(3)}$ are γ–traceless. Scattering amplitudes involving antisymmetric tensor fields are also given by (3.37) if we take ς to be antisymmetric.

This concludes our analysis of the 2F2B scattering amplitudes.

C. 4 Fermion Scattering

Next we turn our attention to the four fermion scattering amplitude. In this case the relevant correlator is,

$$
\langle V_{-\frac{1}{2}}(k_1, u_{(1)}, z_1) V_{-\frac{1}{2}}(k_2, u_{(2)}, z_2) V_{\frac{1}{2}}(k_3, u_{(3)}, z_3) V_{\frac{1}{2}}(k_4, u_{(4)}, z_4)\rangle \tag{3.38}
$$

Using Eqs.(3.1) and (3.4) we may express this correlator as a sum of four terms. Of these the term,

$$
\begin{aligned}
& u^{\alpha_1}_{(1)} u^{\alpha_2}_{(2)} u^{\alpha_3}_{(3)} u^{\alpha_4}_{(4)} (\gamma_{\mu_3})_{\alpha_3\beta_3} (\gamma_{\mu_4})_{\alpha_4\beta_4} \langle \prod_i e^{ik_i\cdot X(z_i)} \partial X^{\mu_3}(z_3) \partial X^{\mu_4}(z_4)\rangle \\
& \langle S_g^-(z_1) S_g^-(z_2) S_g^+(z_3) S_g^+(z_4)\rangle \langle S_{\alpha_1}(z_1) S_{\alpha_2}(z_2) S^{\beta_3}(z_3) S^{\beta_4}(z_4)\rangle
\end{aligned}
\tag{3.39}
$$

may be shown to vanish as a consequence of Riemann ϑ–identity.

The next term to be considered is,

$$u^{\alpha_1}_{(1)}u^{\alpha_2}_{(2)}u^{\alpha_3}_{(3)}u^{\alpha_4}_{(4)}(\gamma_{\mu_3})_{\alpha_3\beta_3}(-ik_4)_{\mu_4}\langle\prod_i e^{ik_i.X(z_i)}\partial X^{\mu_3}(z_3)\rangle$$
$$\lim_{w_4\to z_4}(w_4-z_4)^{-\frac{1}{2}}\langle S_{\alpha_1}(z_1)S_{\alpha_2}(z_2)S^{\beta_3}(z_3)S_{\alpha_4}(z_4)\psi^{\mu_4}(w_4)\rangle \tag{3.40}$$
$$\langle S_g^-(z_1)S_g^-(z_2)S_g^+(z_3)S_g^+(z_4)\rangle.$$

Although this correlator may be evaluated directly, in order to fix the relative phases of various terms, it is more convenient to express (3.40) as,

$$\Lambda_1 = u^{\alpha_1}_{(1)}u^{\alpha_2}_{(2)}u^{\alpha_3}_{(3)}u^{\alpha_4}_{(4)}(-ik_4)_{\mu_4}\langle\prod_i e^{ik_i.X(z_i)}\partial X^{\mu_3}(z_3)\rangle\langle S_g^-(z_1)S_g^-(z_2)S_g^+(z_3)S_g^+(z_4)\rangle$$
$$\lim_{w_4\to z_4}\lim_{w_3\to z_3}\Big[(w_4-z_4)^{-\frac{1}{2}}(w_3-z_3)^{\frac{1}{2}}\langle S_{\alpha_1}(z_1)S_{\alpha_2}(z_2)S_{\alpha_3}(z_3)S_{\alpha_4}(z_4)$$
$$\psi^{\mu_3}(w_3)\psi^{\mu_4}(w_4)\rangle\Big]. \tag{3.41}$$

The spin field correlator appearing in (3.41) has 11 independent tensor structures (number of independent singlets in $16\otimes16\otimes16\otimes16\otimes10\otimes10$). We may express this as,

$$\langle S_g^-(z_1)S_g^-(z_2)S_g^+(z_3)S_g^+(z_4)\rangle\langle S_{\alpha_1}(z_1)S_{\alpha_2}(z_2)S_{\alpha_3}(z_3)S_{\alpha_4}(z_4)\psi^{\mu_3}(w_3)\psi^{\mu_4}(w_4)\rangle$$
$$= A(\gamma^{\mu_3})_{\alpha_3\alpha_1}(\gamma^{\mu_4})_{\alpha_4\alpha_2} + B(\gamma^{\mu_3})_{\alpha_3\alpha_2}(\gamma^{\mu_4})_{\alpha_4\alpha_1} + C(\gamma^{\mu_3})_{\alpha_3\alpha_4}(\gamma^{\mu_4})_{\alpha_1\alpha_2}$$
$$+ D\delta^{\mu_3\mu_4}(\gamma^{\rho})_{\alpha_3\alpha_1}(\gamma^{\rho})_{\alpha_4\alpha_2} + E\delta^{\mu_3\mu_4}(\gamma^{\rho})_{\alpha_3\alpha_2}(\gamma^{\rho})_{\alpha_4\alpha_1} + F(\gamma^{\mu_4})_{\alpha_3\alpha_1}(\gamma^{\mu_3})_{\alpha_4\alpha_2}$$
$$+ G(\gamma^{\mu_4})_{\alpha_3\alpha_2}(\gamma^{\mu_3})_{\alpha_4\alpha_1} + H(\gamma^{\mu_4})_{\alpha_3\alpha_4}(\gamma^{\mu_3})_{\alpha_1\alpha_2} + I(\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\rho})_{\alpha_3\alpha_1}(\gamma^{\rho})_{\alpha_4\alpha_2}$$
$$+ J(\gamma^{\mu_4}\gamma^{\mu_3}\gamma^{\rho})_{\alpha_4\alpha_1}(\gamma^{\rho})_{\alpha_3\alpha_2} + K(\gamma^{\mu_4}\gamma^{\rho}\gamma^{\mu_3})_{\alpha_4\alpha_3}(\gamma^{\rho})_{\alpha_1\alpha_2} \tag{3.42}$$

The contribution to (3.41) from the tensors multiplying A, B, H, J and K vanish due to the on-shell constraint $\not{k}_4 u_{(4)} = 0$. Also, as can be seen from eq.(3.41), the only terms in the correlator which may contribute to (3.41) are the ones which blow up at least as fast as $(w_3 - z_3)^{-\frac{1}{2}}$ as $w_3 \to z_3$. Hence in the evaluation of (3.41) we may ignore terms which are less singular than $(w_3 - z_3)^{-\frac{1}{2}}$ in this limit.

Before we start evaluating the various coefficients, let us investigate the two other terms that appear in the evaluation of the correlator (3.38). One of them (Λ_2) is related to (3.40) by an interchange $3 \leftrightarrow 4$, and an extra $-$ sign due to the antisymmetry of the fermionic vertex operators. The last term to be evaluated is,

$$\begin{aligned}\Lambda_3 =& u^{\alpha_1}_{(1)} u^{\alpha_2}_{(2)} u^{\alpha_3}_{(3)} u^{\alpha_4}_{(4)} (-ik_3)_{\mu_3} (-ik_4)_{\mu_4} \langle \prod_i e^{ik_i.X(z_i)} \rangle \langle S_g^-(z_1) S_g^-(z_2) S_g^+(z_3) S_g^+(z_4) \rangle \\ & \lim_{w_4 \to z_4} \lim_{w_3 \to z_3} \Big[(w_4 - z_4)^{-\frac{1}{2}} (w_3 - z_3)^{-\frac{1}{2}} \langle S_{\alpha_1}(z_1) S_{\alpha_2}(z_2) S_{\alpha_3}(z_3) S_{\alpha_4}(z_4) \\ & \qquad \psi^{\mu_3}(w_3) \psi^{\mu_4}(w_4) \rangle \Big].\end{aligned} \tag{3.43}$$

We see that we need to evaluate the same correlator that appears in the evaluation of (3.41). But now all terms except the ones proportional to D, E, F and G vanish by the on-shell condition $\not k_3 u_{(3)} = \not k_4 u_{(4)} = 0$. Furthermore, we need to evaluate D, E, F and G to an accuracy of order $(w_3 - z_3)^{\frac{1}{2}} (w_4 - z_4)^{\frac{1}{2}}$, as can be seen from the prefactor appearing in (3.43).

Thus in order to evaluate the complete amplitude given in (3.38), we need to know the coefficients C and I to an accuracy of order $(w_3 - z_3)^{-\frac{1}{2}} (w_4 - z_4)^{\frac{1}{2}}$, and D, E, F and G to order $(w_3 - z_3)^{\frac{1}{2}} (w_4 - z_4)^{\frac{1}{2}}$. We proceed as before, by evaluating the correlator given in (3.41) for various polarizations using eq.(2.11) and (2.16), and comparing with the results expected from (3.42). Taking $\mu_3 = 1$, $\mu_4 = \bar{1}$, $\alpha_1 = (- + - - -)$, $\alpha_2 = (+ - - + +)$, $\alpha_3 = (+ + + + +)$, and $\alpha_4 = (- - + - -)$, we obtain,

$$D \simeq 0, \tag{3.44}$$

$\mu_4 = 1$, $\mu_3 = \bar{1}$, $\alpha_1 = (- + - - -)$, $\alpha_2 = (+ - - + +)$, $\alpha_4 = (+ + + + +)$, and $\alpha_3 = (- - + - -)$ gives,

$$E \simeq 0, \tag{3.45}$$

$\mu_3 = 1$, $\mu_4 = 2$, $\alpha_1 = (--+++)$, $\alpha_2 = (----+)$, $\alpha_3 = (+-++-)$, and $\alpha_4 = (-+---)$ gives,

$$G \simeq 2\epsilon(w_3 - z_3)^{\frac{1}{2}}(w_4 - z_4)^{\frac{1}{2}} \tag{3.46}$$

Normalization of G has been determined as before. ϵ is an arbitrary phase.

Since F is the only other term that contributes to (3.43) on shell, and since the tensor structure multiplying F is related to the one multiplying G by the interchange $3 \leftrightarrow 4$, we get,

$$F \simeq -G \simeq -2\epsilon(w_3 - z_3)^{\frac{1}{2}}(w_4 - z_4)^{\frac{1}{2}} \tag{3.47}$$

The rest of the terms need to be evaluated only to an accuracy $(w_3 - z_3)^{-\frac{1}{2}}(w_4 - z_4)^{\frac{1}{2}}$. Setting $\mu_3 = 1$, $\mu_4 = 2$, $\alpha_1 = (----+)$, $\alpha_2 = (+-+-+)$, $\alpha_3 = (---+-)$, and $\alpha_4 = (-+++-)$, and using the fact that G is of order $(w_3 - z_3)^{\frac{1}{2}}(w_4 - z_4)^{\frac{1}{2}}$, we get,

$$I \simeq 0 \tag{3.48}$$

We are now left with the coefficient C. As we shall see it will turn out to be non-zero. In order to evaluate it completely, we must also determine its phase relative to G. This requires a careful analysis which we present now. First, setting $\mu_3 = 1$, $\mu_4 = 2$, $\alpha_1 = (+-++-)$, $\alpha_2 = (----+)$, $\alpha_3 = (--+++)$, and $\alpha_4 = (-+---)$, we get

$$\begin{aligned}
C =&\eta(\vartheta_1'(0))^{-1}\big(\vartheta_1(z_1 - z_2)\big)^{-1}\big(\vartheta_1(z_1 - z_3)\big)^{\frac{1}{2}}\big(\vartheta_1(z_1 - w_3)\big)^{\frac{1}{2}}\big(\vartheta_1(z_1 - z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(z_1 - w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2 - z_3)\big)^{\frac{1}{2}}\big(\vartheta_1(z_2 - w_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2 - z_4)\big)^{\frac{1}{2}}\\
&\big(\vartheta_1(z_2 - w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_3 - z_4)\big)^{-1}\big(\vartheta_1(z_3 - w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_3 - z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(w_3 - z_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_4 - z_4)\big)^{\frac{1}{2}}\\
&\sum_\nu \Big[\delta_\nu\vartheta_\nu\Big(\frac{z_1 + 2w_3 - z_2 - z_3 - z_4}{2}\Big)\vartheta_\nu\Big(\frac{z_1 + z_2 + z_3 - z_4 - 2w_4}{2}\Big)\\
&\vartheta_\nu^2\Big(\frac{z_1 - z_2 + z_3 - z_4}{2}\Big)\vartheta_\nu\Big(\frac{z_1 - z_2 - z_3 + z_4}{2}\Big)\vartheta_\nu^{-1}\Big(\frac{z_1 + z_2 - z_3 - z_4}{2}\Big)\Big]
\end{aligned} \tag{3.49}$$

where the normalization of C has been determined as before. Since the prefactor diverges as $(w_3 - z_3)^{-\frac{1}{2}}(w_4 - z_4)^{\frac{1}{2}}$, we may set $w_3 = z_3$, $w_4 = z_4$ inside the sum over spin structures. This gives,

$$\sum_\nu \delta_\nu \frac{\vartheta_\nu\left(\frac{z_1-z_2-z_3+z_4}{2}\right)\vartheta_\nu\left(\frac{z_1+z_2+z_3-3z_4}{2}\right)\left(\vartheta_\nu\left(\frac{z_1-z_2+z_3-z_4}{2}\right)\right)^3}{\vartheta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)} \tag{3.50}$$

We may evaluate this sum with the help of the identity,

$$\begin{aligned}&\sum_\nu \delta_\nu \vartheta_\nu(x_1)\vartheta_\nu(x_2)\vartheta_\nu(x_3)\vartheta_\nu(x_4)\vartheta_\nu(x_5)\vartheta_\nu^{-1}(x_1+x_2+x_3+x_4+x_5)\\ &= -2\vartheta_1(x_1+x_2+x_3+x_4)\vartheta_1(x_2+x_3+x_4+x_5)\vartheta_1(x_1+x_5+x_3+x_4)\\ &\vartheta_1(x_1+x_2+x_4+x_5)\vartheta_1(x_1+x_2+x_3+x_5)\vartheta_1^{-1}(2(x_1+x_2+x_3+x_4+x_5)).\end{aligned} \tag{3.51}$$

This identity has been proved in appendix B. Setting, $x_1 = \frac{1}{2}(z_1+z_3-z_2-z_4)$, $x_2 = x_3 = -\frac{1}{2}(z_1+z_3-z_2-z_4)$, $x_4 = \frac{1}{2}(z_1+z_4-z_2-z_3)$ and $x_5 = \frac{1}{2}(z_1+z_2+z_3-3z_4)$ in (3.51) we see that (3.50) is equal to

$$-2\vartheta_1(z_4-z_3)\left(\vartheta_1(z_1-z_4)\right)^2\vartheta_1(z_2-z_4)\vartheta_1(z_2-z_3)\left(\vartheta_1(z_1+z_2-z_3-z_4)\right)^{-1}. \tag{3.52}$$

Substituting (3.52) in eq.(3.49) we get in the $w_3 \to z_3$, $w_4 \to z_4$ limit,

$$C = 2\eta\left(\vartheta_1'(0)\right)^{-1}(w_3-z_3)^{-\frac{1}{2}}(w_4-z_4)^{\frac{1}{2}}\frac{\vartheta_1(z_1-z_4)\vartheta_1(z_2-z_4)\vartheta_1(z_2-z_3)\vartheta_1(z_1-z_3)}{\vartheta_1(z_1-z_2)\vartheta_1(z_3-z_4)\vartheta_1(z_1+z_2-z_3-z_4)}. \tag{3.53}$$

This determines C up to the phase factor η. In order to determine η we consider another set of polarizations: $\mu_3 = 1$, $\mu_4 = 2$, $\alpha_1 = (----+)$, $\alpha_2 = (+-++-)$,

$\alpha_3 = (- - - - +)$, and $\alpha_4 = (- + + + -)$. This gives,

$$
\begin{aligned}
C+G =&\eta'(\vartheta_1'(0))^{-1}\big(\vartheta_1(z_1-z_2)\big)^{-1}\big(\vartheta_1(z_1-z_3)\big)^{\frac{3}{2}}\big(\vartheta_1(z_1-w_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_1-z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(z_1-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2-z_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2-w_3)\big)^{\frac{1}{2}}\big(\vartheta_1(z_2-z_4)\big)^{\frac{1}{2}}\\
&\big(\vartheta_1(z_2-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_3-z_4)\big)^{-1}\big(\vartheta_1(z_3-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_3-z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(w_3-z_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_4-z_4)\big)^{\frac{1}{2}}\\
&\sum_\nu\Big[\delta_\nu\vartheta_\nu(\frac{z_1-2w_3-z_2+z_3+z_4}{2})\vartheta_\nu(\frac{z_1+z_2+z_3-z_4-2w_4}{2})\\
&\Big(\vartheta_\nu(\frac{z_1-z_2+z_3-z_4}{2})\Big)^3\vartheta_\nu^{-1}(\frac{z_1+z_2-z_3-z_4}{2})\Big]
\end{aligned}
\tag{3.54}
$$

where η' is another as yet undetermined phase. First we determine the relative phase of η and η' by demanding that in the $w_3 \to z_3$, $w_4 \to z_4$ limit the leading term in (3.54) must be equal to (3.53), since G is nonleading in this limit. The sum over spin structures in (3.54) may again be performed using eq.(3.51) and we get,

$$\eta' = \eta \tag{3.55}$$

We may now evaluate G by subtracting (3.49) from (3.54). This gives,

$$
\begin{aligned}
G =&\eta'(\vartheta_1'(0))^{-1}\big(\vartheta_1(z_1-z_2)\big)^{-1}\big(\vartheta_1(z_1-z_3)\big)^{\frac{1}{2}}\big(\vartheta_1(z_1-w_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_1-z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(z_1-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2-z_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2-w_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_2-z_4)\big)^{\frac{1}{2}}\\
&\big(\vartheta_1(z_2-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(z_3-z_4)\big)^{-1}\big(\vartheta_1(z_3-w_4)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_3-z_4)\big)^{-\frac{1}{2}}\\
&\big(\vartheta_1(w_3-z_3)\big)^{-\frac{1}{2}}\big(\vartheta_1(w_4-z_4)\big)^{\frac{1}{2}}\\
&\sum_\nu\Big[\delta_\nu\vartheta_\nu^{-1}(\frac{z_1+z_2-z_3-z_4}{2})\vartheta_\nu(\frac{z_1+z_2+z_3-z_4-2w_4}{2})\vartheta_\nu^2(\frac{z_1-z_2+z_3-z_4}{2})\\
&\Big(\vartheta_\nu(\frac{z_1-2w_3-z_2+z_3+z_4}{2})\vartheta_\nu(\frac{z_1+z_3-z_2-z_4}{2})\vartheta_1(z_1-z_3)\vartheta_1(z_2-w_3)\\
&-\vartheta_\nu(\frac{z_1+2w_3-z_2-z_3-z_4}{2})\vartheta_\nu(\frac{z_1+z_4-z_2-z_3}{2})\vartheta_1(z_1-w_3)\vartheta_1(z_2-z_3)\Big)\Big]
\end{aligned}
\tag{3.56}
$$

We now use the following identity,

$$\Big(\vartheta_\nu(\frac{z_1-2w_3-z_2+z_3+z_4}{2})\vartheta_\nu(\frac{z_1+z_3-z_2-z_4}{2})\vartheta_1(z_1-z_3)\vartheta_1(z_2-w_3)$$
$$-\vartheta_\nu(\frac{z_1+2w_3-z_2-z_3-z_4}{2})\vartheta_\nu(\frac{z_1+z_4-z_2-z_3}{2})\vartheta_1(z_1-w_3)\vartheta_1(z_2-z_3)\Big)$$
$$=-\vartheta_1(w_3-z_3)\vartheta_1(z_1-z_2)\vartheta_\nu(\frac{z_1+z_2-z_3-z_4}{2})\vartheta_\nu(-w_3+\frac{z_1+z_2-z_3+z_4}{2}) \tag{3.57}$$

This identity may be proved by noting that:

i) Both sides of eq.(3.57) have exactly the same periodicity properties as a function of the variables z_1, z_2, z_3, z_4 and w_3.

ii) The left hand side vanishes whenever the right hand side does.

Thus the ratio of the laft hand side and the right hand side must be a constant. This constant may be determined to be unity by evaluating both sides at $z_1 = w_3$.

After substituting (3.57) into (3.56) we may evaluate the sum over spin structures using ordinary Riemann ϑ–identity. The result is,

$$G \simeq -2\eta(w_3-z_3)^{\frac{1}{2}}(w_4-z_4)^{\frac{1}{2}} \tag{3.58}$$

Comparing this with eq.(3.46) we get,

$$\eta = -\epsilon \tag{3.59}$$

From now on we shall set $\epsilon = 1$ for convenience. This determines all the relevant constants that appear in eq.(3.42). We may now proceed to evaluate (3.41) and (3.43). The evaluation of (3.41) requires the relation (see appendix C),

$$\langle\partial X^{\mu_3}(z_3)\prod_i e^{ik_i.X(z_i)}\rangle$$
$$=-i\langle\prod_i e^{ik_i.X(z_i)}\rangle\sum_{i\neq 3}k_i^{\mu_3}\Big(\frac{\partial}{\partial z_3}\ln\left(\vartheta_1(z_3-z_i)\right)+\frac{2i\pi}{Im\ \tau}Im(z_3-z_i)\Big) \tag{3.60}$$

The $i = 4$ term and the terms proportional to $Im\ z_3$ in (3.60) do not contribute to (3.41) on-shell, as can be easily seen using eq.(3.40). Substituting (3.60) in (3.41) and subtracting from it the term with 3 and 4 interchanged, we get,

$$\Lambda_1 + \Lambda_2 = (-2)\sum_{i=1}^{2}\Big[(\vartheta_1'(0))^{-1}\big(u_{(1)}(-i\ \not{k}_4)u_{(2)}\big)\big(u_{(3)}(-i\ \not{k}_i)u_{(4)}\big)\langle\prod_j e^{ik_j.X(z_j)}\rangle$$
$$\left(\frac{\partial}{\partial z_3}\ln\left(\vartheta_1(z_3 - z_i)\right) - \frac{2i\pi}{Im\tau}Imz_i\right)\frac{\vartheta_1(z_1 - z_4)\vartheta_1(z_2 - z_4)\vartheta_1(z_2 - z_3)\vartheta_1(z_1 - z_3)}{\vartheta_1(z_1 - z_2)\vartheta_1(z_3 - z_4)\vartheta_1(z_1 + z_2 - z_3 - z_4)}\Big]$$
$$- (3 \leftrightarrow 4). \tag{3.61}$$

Here we have used the expression (3.53) for C. Using on-shell constraints and momentum conservation (3.61) may be reduced to,

$$\Lambda_1 + \Lambda_2$$
$$= 2(\vartheta_1'(0))^{-1}\big(u_{(1)}\ \not{k}_4 u_{(2)}\big)\big(u_{(3)}\ \not{k}_1 u_{(4)}\big)\langle\prod_i e^{ik_i.X(z_i)}\rangle$$
$$\Big[\frac{\partial}{\partial z_3}\Big\{\ln\left(\vartheta_1(z_3 - z_1)\right) - \ln\left(\vartheta_1(z_3 - z_2)\right)\Big\} - \frac{\partial}{\partial z_4}\Big\{\ln\left(\vartheta_1(z_4 - z_1)\right) - \ln\left(\vartheta_1(z_4 - z_2)\right)\Big\}\Big]$$
$$\frac{\vartheta_1(z_1 - z_4)\vartheta_1(z_2 - z_4)\vartheta_1(z_2 - z_3)\vartheta_1(z_1 - z_3)}{\vartheta_1(z_1 - z_2)\vartheta_1(z_3 - z_4)\vartheta_1(z_1 + z_2 - z_3 - z_4)} \tag{3.62}$$

Examining the periodicity properties and the positions of the zeros and the poles, the expression inside the square bracket may be shown to be identical to,

$$\vartheta_1'(0)\frac{\vartheta_1(z_1 - z_2)\vartheta_1(z_3 - z_4)\vartheta_1(z_1 + z_2 - z_3 - z_4)}{\vartheta_1(z_1 - z_4)\vartheta_1(z_2 - z_4)\vartheta_1(z_2 - z_3)\vartheta_1(z_1 - z_3)} \tag{3.63}$$

Using eq.(3.63), (3.62) reduces to,

$$\Lambda_1 + \Lambda_2 = 2\big(u_{(1)}\ \not{k}_4 u_{(2)}\big)\big(u_{(3)}\ \not{k}_1 u_{(4)}\big)\langle\prod_i e^{ik_i.X(z_i)}\rangle \tag{3.64}$$

On the other hand, (3.43), which receives contribution only from the tensors F and G in (3.43), gives,

$$\Lambda_3 = 2\Big[\big(u_{(3)}\ \not{k}_4 u_{(1)}\big)\big(u_{(4)}\ \not{k}_3 u_{(2)}\big) - \big(u_{(3)}\ \not{k}_4 u_{(2)}\big)\big(u_{(4)}\ \not{k}_3 u_{(1)}\big)\Big]\langle\prod_i e^{ik_i.X(z_i)}\rangle \tag{3.65}$$

Thus (3.38) may be written as,

$$\begin{aligned}\langle V_{-\frac{1}{2}}V_{-\frac{1}{2}}V_{\frac{1}{2}}V_{\frac{1}{2}}\rangle &= \Lambda_1+\Lambda_2+\Lambda_3\\ &= \langle\prod_i e^{ik_i.X(z_i)}\rangle K(u_{(1)},u_{(2)},u_{(3)},u_{(4)};k_i)\end{aligned} \tag{3.66}$$

where,

$$\begin{aligned}&K(u_{(1)},\ u_{(2)},\ u_{(3)},\ u_{(4)};k_i)\\ &=2\Big\{\big(u_{(1)}\not{k}_4 u_{(2)}\big)\big(u_{(3)}\not{k}_1 u_{(4)}\big)+\big(u_{(3)}\not{k}_4 u_{(1)}\big)\big(u_{(4)}\not{k}_3 u_{(2)}\big)\\ &\qquad -\big(u_{(3)}\not{k}_4 u_{(2)}\big)\big(u_{(4)}\not{k}_3 u_{(1)}\big)\Big\}\ .\end{aligned} \tag{3.67}$$

This may easily be seen to be totally antisymmetric in $1,2,3$ and 4. By γ–matrix manipulations this may also be shown to agree with the light-cone gauge result of ref. [11]. The final answer for the amplitude is:

$$\begin{aligned}A\ (1,\ 2,\ 3,\ 4) &= \int \frac{d^2\tau}{(Im\ \tau)^5}\\ &\int \prod_{i=2}^{4} d^2 z_i\ \left\{\left\langle\prod_i\ e^{ik_i\cdot X(z_i)}\right\rangle\ \bar{F}(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)\ \right\}\\ &K(u_{(1)},\ u_{(2)},\ u_{(3)},\ u_{(4)};k_i)\ \ .\end{aligned} \tag{3.68}$$

where $\bar{F}$ is the properly normalized contribution from the anti-analytic sector.

Note that in order to arrive at the final form (3.68) for the amplitude we had to explicitly evaluate the correlation function involving the X fields. Hence one might expect that the final form of the amplitude might be different if the vertex operators from the anti-analytic sector involve the field X. We shall now show that this is not the case. Let us, for example, consider the scattering amplitude of four gravitinos in the heterotic string theory. In this case the antianalytic sector will contribute terms of the form $\bar{\partial}X^{\nu}$ to the vertex operators, and the X correlator appearing in eq.(3.41) also involves these operators. The relevant correlator has been derived in eq. (C.6) in Appendix C which contains some extra terms. Note however that after we substitute this expression in (3.41) and subtract the term obtained by interchanging 3 and 4 (this interchange is only done for the analytic sector) the extra terms drop out, and we get back the kinematical structure given in eq. (3.67).

Finally, for completeness, we write down the answer for the scattering amplitude of four bosonic external legs. The answer has the same form as (3.35) or (3.68), except that the kinematic factor K is now given by,

$$\begin{aligned} &K(\varsigma^{(1)},\ \varsigma^{(2)},\varsigma^{(3)},\varsigma^{(4)};k_1,\ k_2,\ k_3,\ k_4) \\ &\quad = t_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} k_1^{\mu_1}(\varsigma^{(1)})^{\nu_1} k_2^{\mu_2}(\varsigma^{(2)})^{\nu_2} k_3^{\mu_3}(\varsigma^{(}3))^{\nu_3} k_4^{\mu_4}(\varsigma^{(4)})^{\mu_4}, \end{aligned} \tag{3.69}$$

where

$$
\begin{aligned}
t_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} = \\
& -\{\delta^{\mu_1\mu_2}\delta^{\nu_1\nu_2}\delta^{\mu_3\mu_4}\delta^{\nu_3\nu_4} + \delta^{\mu_1\mu_3}\delta^{\nu_1\nu_3}\delta^{\mu_2\mu_4}\delta^{\nu_2\nu_4} \\
& \quad + \delta^{\mu_1\mu_4}\delta^{\nu_1\nu_4}\delta^{\mu_2\mu_3}\delta^{\nu_2\nu_3} \\
& \quad + 9 \text{ more terms obtained by antisymmetrizing in} \\
& \quad (\mu_1\nu_1),(\mu_2\nu_2),(\mu_3\nu_3) \text{ and } (\mu_4\nu_4)\} \qquad (3.70) \\
& + \{\delta^{\nu_1\mu_2}\delta^{\nu_2\mu_3}\delta^{\nu_3\mu_4}\delta^{\nu_4\nu_1} + \delta^{\nu_1\mu_3}\delta^{\nu_3\mu_2}\delta^{\nu_2\mu_4}\delta^{\nu_4\mu_1} \\
& \quad + \delta^{\nu_1\mu_3}\delta^{\nu_3\mu_4}\delta^{\nu_2\mu_1}\delta^{\nu_4\mu_2} \\
& \quad + 45 \text{ more terms obtained by antisymmetrizing in} \\
& \quad (\mu_1\nu_1),(\mu_2\nu_2),(\mu_3\nu_3) \text{ and } (\mu_4\nu_4)\}.
\end{aligned}
$$

The scattering amplitudes in the type II superstring theory are calculated in the same fashion. As shown here the evaluation of the X correlators in the analytic and antianalytic sectors do not interfere with each other except in the evaluation of the overall factor $\langle\prod_i e^{ik_i\cdot X(z_i)}\rangle$. The answer for various scattering amplitudes are obtained by taking the direct product of the tensors K given in eq. (3.34), (3.67) and (3.69) in the left and the right sector.

This concludes our analysis of the four particle scattering amplitudes in the heterotic and superstring theories.

IV. Conclusion

In this paper we have calculated the 2 fermion 2 boson and the 4 fermion scattering amplitudes in the covariant formulation of the heterotic and superstring theories. The agreement of these results with the corresponding light-cone gauge answers provide an explicit verification of the validity of the covariant fermion emission vertex and the prescription of Friedan, Martinec and Shenker for calculation of scattering amplitudes at the one loop level. We feel that, aside from making covariant one-loop calculations as feasible as the light-cone gauge, this should shed some further insight on the structure of covariant fermionic strings.

The first logical thing one should try to do next is to understand how the results of this paper extend to higher genus. In any attempt along that direction one is immediately confronted with several new problems that were absent on the torus. For one, the supermoduli now exist and play an important role, for another , the background ghost charge of the superconformal ghosts is nonzero and hence one has to find a prescription for soaking up this charge.

An ansatz for handling the above subtleties was presented in ref [7]. Within this prescription amplitudes with less than 4 external states vanish identically as they should because of the nonrenormalization theorems[11,18]. Moreover the 2 fermion 2 boson calculation can be carried through without major obstacles. The kinematical factor that we get coincides with the one obtained in section III. However we cannot normalize the correlation functions since the partition function is not known. The 4 fermion calculation is harder to extend to higher loops. This calculation entails the use of new generalized ϑ–function identities analogous to the one that was needed at the one loop in section III. Currently these are not fully known.

Finally we wish to mention that the techniques developed here may also be used to calculate various amplitudes involving untwisted fields on orbifolds. In particular one may verify by explicit string calculation [19] the existence of the Fayet–Illiopoulos D term predicted from low energy considerations in ref [20].

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APPENDIX A: γ–matrix conventions

In this appendix we shall give a convenient representation for the $SO(10)$ γ–matrices. The γ–matrices give the Clebsch-Gordon coefficients for $16 \times 16 \to 10$ in $SO(10)$, and are denoted by $\gamma^{\mu}_{\alpha\beta}$, where μ is the vector index and α, β are spinor indices. We shall group the 10 real vector indices into 5 complex ones, $1, ..., 5$ and their complex conjugates $\bar{1}, ..., \bar{5}$. We choose the γ–matrices as,

$$
\begin{aligned}
\gamma^{1} &= -\frac{(1-\sigma^{3})}{2} \otimes \sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{\bar{1}} &= -\frac{(1+\sigma^{3})}{2} \otimes \sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{2} &= -\sigma^{2} \otimes \frac{(1-\sigma^{3})}{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{\bar{2}} &= -\sigma^{2} \otimes \frac{(1+\sigma^{3})}{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{3} &= -\sigma^{2} \otimes \sigma^{1} \otimes \frac{(1-\sigma^{3})}{2} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{\bar{3}} &= -\sigma^{2} \otimes \sigma^{1} \otimes \frac{(1+\sigma^{3})}{2} \otimes \sigma^{2} \otimes \sigma^{1} \\
\gamma^{4} &= -\sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \frac{(1-\sigma^{3})}{2} \otimes \sigma^{1} \\
\gamma^{\bar{4}} &= -\sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \frac{(1+\sigma^{3})}{2} \otimes \sigma^{1} \\
\gamma^{5} &= -\sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \otimes \frac{(1-\sigma^{3})}{2} \\
\gamma^{\bar{5}} &= -\sigma^{2} \otimes \sigma^{1} \otimes \sigma^{2} \otimes \sigma^{1} \otimes \frac{(1+\sigma^{3})}{2}.
\end{aligned}
\tag{A.1}
$$

These γ–matrices as is easily verified are real and symmetric.

A spinor of $SO(10)$ will be represented by a direct product of 5 $SO(2)$ spinors. We shall denote the $SO(2)$ spinor $\binom{1}{0}$ by $+$ and $\binom{0}{1}$ by $-$. Thus for example the $SO(10)$ spinor $(+-++-)$ corresponds to

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}.$$

This is known as the helicity basis.

A useful rule to remember about the above γ-matrices is the following : Any entry in these γ-matrices say $\gamma^i_{\alpha\beta}(\gamma^{\bar{i}}_{\alpha\beta})$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, is nonvanishing if $\alpha_i = \beta_i = -$ (if $\alpha_i = \beta_i = +$) and $\alpha_j \neq \beta_j$ for $j \neq i$. For a nonvanishing entry the answer is either $+1$ or -1. The phase is given by the product of all α_j's preceding the ith slot in α times the product $\beta_2\beta_4$ *i.e.*, the phase is $(\alpha_1...\alpha_{i-1})(\beta_2\beta_4)$. For example

$$\gamma^3_{-+-++,+----} = (-)(+)(-)(-)1 = -1.$$

With this in mind we never need to use the explicit representation in (A.1).

Although the representation looks $2^5 = 32$ dimensional, we restrict the spinors belonging to the 16 representation by the chirality projection *i.e.*, by demanding that the total number of $(-)$ in the spinor must be even. Thus for example the spinor $(+++--)$ belongs to the 16 representation, whereas the spinor $(++---)$ belongs to $\bar{1}6$.

Finally we introduce the concept of raising and lowering indices which takes us from a spinor in the 16 representation to the one in the $\bar{1}6$ representation in the 32 dimensional space. This is done by the tensor $\epsilon^{\alpha\beta}$, given by,

$$\epsilon = -\sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1, \qquad (A.2)$$

with this we may define ,

$$(\gamma^\mu)_\alpha^{\ \beta} = (\gamma^\mu)_{\alpha\gamma}\epsilon^{\gamma\beta}. \qquad (A.3)$$

Multiplication of two γ–matrices is defined as,

$$(\gamma^{\mu}\gamma^{\nu})_{\alpha}{}^{\gamma} \equiv (\gamma^{\mu})_{\alpha}{}^{\beta}(\gamma^{\nu})_{\beta}{}^{\gamma} \equiv (\gamma^{\mu})_{\alpha\beta}\epsilon^{\beta\delta}(\gamma^{\nu})_{\delta\eta}\epsilon^{\eta\gamma}, \qquad (A.4)$$

with this rule, it is easy to see that the γ–matrices in $(A.1)$ satisfy the anti-commutation relations:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \delta^{\mu\nu}. \qquad (A.5)$$

APPENDIX B : A Theta Identity

We define the Jacobi ϑ–function as [21],

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\,|\,\tau)=\sum_{n\epsilon Z} exp[i\pi(n+\alpha)^2\tau+2\pi i(n+\alpha)(z+\beta)]$$
$$= exp[i\pi\alpha^2\tau+2\pi i\alpha(z+\beta)]\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(z+\alpha\tau+\beta) \qquad (B.1)$$

and [22]

$$\vartheta_1(z\,|\,\tau)=\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z\,|\,\tau)$$
$$\vartheta_2(z\,|\,\tau)=\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(z\,|\,\tau)$$
$$\vartheta_3(z\,|\,\tau)=\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(z\,|\,\tau) \qquad (B.2)$$
$$\vartheta_4(z\,|\,\tau)=\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(z\,|\,\tau)$$

For $\alpha,\beta=0$ or $\frac{1}{2}$ $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ satisfy the periodicity properties :

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+1\,|\,\tau)=exp[2\pi i\alpha]\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\,|\,\tau),$$
$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+\tau\,|\,\tau)=exp[-i\pi\tau-2\pi i(z+\beta)]\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\,|\,\tau). \qquad (B.3)$$

From now on and in the text we shall suppress the τ dependence of ϑ–functions.

The ϑ–functions also satisfy the "Riemann Identity" :

$$\sum_{\nu} \delta_\nu \vartheta_\nu(z_1)\vartheta_\nu(z_2)\vartheta_\nu(z_3)\vartheta_\nu(z_4)$$
$$= 2\vartheta_1(\frac{z_1+z_2+z_3+z_4}{2})\vartheta_1(\frac{z_1+z_2-z_3-z_4}{2})\vartheta_1(\frac{z_1-z_2-z_3+z_4}{2}) \qquad (B.4)$$
$$\vartheta_1(\frac{z_1-z_2+z_3-z_4}{2}),$$

where,

$$\delta_1 = \delta_3 = +1, \qquad \delta_2 = \delta_4 = -1. \qquad (B.5)$$

It is important to note that ϑ_1 is an odd function of z while $\vartheta_2, \vartheta_3, \vartheta_4$, are even. Finally the zeros of $\vartheta\left[{\alpha \atop \beta}\right](z \mid \tau)$ as a function of z are at

$$z = (\alpha - \frac{1}{2})\tau + (\beta - \frac{1}{2}) + (m + n\tau), \qquad (B.6)$$

where m and n are integers.

We propose the following ϑ–identity,

$$\sum_{\nu} \delta_\nu \vartheta_\nu(z_1)\vartheta_\nu(z_2)\vartheta_\nu(z_3)\vartheta_\nu(z_4)\vartheta_\nu(z_5)\vartheta_\nu^{-1}(z_1+z_2+z_3+z_4+z_5)$$
$$= -2\vartheta_1(z_1+z_2+z_3+z_4)\vartheta_1(z_2+z_3+z_4+z_5)\vartheta_1(z_1+z_5+z_3+z_4)$$
$$\vartheta_1(z_1+z_2+z_4+z_5)\vartheta_1(z_1+z_2+z_3+z_5)\vartheta_1^{-1}(2(z_1+z_2+z_3+z_4+z_5)). \qquad (B.7)$$

This identity is crucial in the calculation of the 4 fermion amplitude. Its proof goes as follows:

i) As a function of z_1, the right and the left hand sides of eq. (B.7) may be shown to have the same periodicity properties.

ii) The only poles of the left hand side as a function of z_1 are at,

$$z_1 = -\sum_{j\neq 1} z_j + \alpha + \beta\tau, \tag{B.8}$$

where $\alpha = 0, \frac{1}{2}$, $\beta = 0, \frac{1}{2}$. This gives,

$$2z_1 = -2\sum_{j\neq 1} z_j + m + n\tau, \tag{B.9}$$

where m, n are integers. As can be easily seen, the right hand side of (B.7) has poles precisely at the same points.

iii) Consider the point in the z_1 plane :

$$z_1 = -z_2 - z_3 - z_4. \tag{B.10}$$

The right hand side of (B.7) vanishes at this point. Substituting $z_1 = -z_2 - z_3 - z_4$ on the left hand side of (B.7) and using eq. (B.4) we see that the left hand side also vanishes at this point. Similarily one can show that the left hand side vanishes identically whenever the right hand side does.

Let us now consider the ratio of the left and right hand side of eq.(B.7) , and consider it as a function of z_1. From our discussion so far we conclude that this function is periodic in z_1, and has no poles in the z_1 plane, hence it must be independent of z_1. But since the function is symmetric in all z_i it must be independent of each z_i and hence must be a constant. Comparing the residues at the pole $z_1 = -z_2 - z_3 - z_4 - z_5$ on the two sides of eq.(B.7) we recover the factor of -2 appearing on the right hand side of this equation. This establishes the identity (B.7).

APPENDIX C: Evaluation of the correlators of the X fields

In this appendix we shall evaluate an expression for the correlator:

$$\langle \partial X^\mu(z) \prod_{j=1}^{N} \bar\partial X^{\nu_j}(w_j) \prod_{i=1}^{4} e^{ik_i.X(z_i)} \rangle \equiv F^{\mu\nu_1\ldots\nu_N}(z, w_j, z_i, k_i) \qquad (C.1)$$

In order to evaluate this we start with the correlator,

$$\begin{aligned} & f(z, w_j, z_i, k_i, \epsilon, \ell, \ell_i) \\ & = \langle e^{i\epsilon\ell.X(z)} \Big(\prod_{j=1}^{N} e^{i\epsilon\ell_j.X(w_j)} \Big) \prod_{i=1}^{4} e^{ik_i.X(z_i)} \rangle \end{aligned} \qquad (C.2)$$

where ℓ, ℓ_i are light-like vectors and ϵ is a small number. Note that,

$$\begin{aligned} & \partial_z \partial_{\bar w_1} \ldots \partial_{\bar w_N} f(z, w_j, z_i, k_i, \epsilon, \ell, \ell_i) \\ & = i^{N+1} \epsilon^{N+1} \ell_\mu (\ell_1)_{\nu_1} \ldots (\ell_N)_{\nu_N} F^{\mu\nu_1\ldots\nu_N}(z, w_j, z_i, k_i) + O(\epsilon^{N+2}) \end{aligned} \qquad (C.3)$$

Thus we may know the expression for F if we know the expression for f.[*] f is obtained from the formula[11],[†]

$$\begin{aligned} \langle \prod_{i=1}^{M} e^{ik_i.X(z_i)} \rangle = & C \prod_{i<j} \Big[exp\Big(-2\pi k_i.k_j \frac{(Im(z_i - z_j))^2}{Im\ \tau} \Big) \\ & \Big(\vartheta_1(z_i - z_j) \bar\vartheta_1(\bar z_i - \bar z_j) \Big)^{k_i.k_j} \Big] \end{aligned} \qquad (C.4)$$

---

* ℓ_i's may be taken to be linearly independent by making small shifts of order ϵ in the k_i's, which does not affect the right hand side of eq.(C.3) to order ϵ^{N+1}.

† We are using a convention where the slope parameter α' has been set to unity.

The relevant part of F that contributes to the left hand side of eq.(C.3) to order ϵ^{N+1} is given by,

$$
\begin{aligned}
&f(z, w_j, z_i, k_i, \epsilon, \ell, \ell_i) \\
&= C \prod_{1\le i<j\le 4} \left[exp\Big(-2\pi k_i.k_j \frac{(Im(z_i - z_j))^2}{Im\ \tau} \Big) \Big(\vartheta_1(z_i - z_j) \bar{\vartheta}(\bar{z}_i - \bar{z}_j) \Big)^{k_i.k_j} \right] \\
&\prod_{i=1}^{4} \left[exp\Big(-2\pi\epsilon\ell.k_i \frac{(Im(z - z_i))^2}{Im\ \tau} \Big) \Big(\vartheta_1(z - z_i) \Big)^{\epsilon\ell.k_i} \right] \\
&\prod_{j=1}^{N}\prod_{i=1}^{4} \left[exp\Big(-2\pi\epsilon\ell_j.k_i \frac{(Im(w_j - z_i))^2}{Im\ \tau} \Big) \Big(\bar{\vartheta}(\bar{w}_j - \bar{z}_i) \Big)^{\epsilon\ell_j.k_i} \right] \\
&\prod_{j=1}^{N} \left[exp\Big(-2\pi\epsilon^2\ell.\ell_j \frac{(Im(z - w_i))^2}{Im\ \tau} \Big) \right]
\end{aligned}
\tag{C.5}
$$

Substituting this in the left hand side of eq.(C.3) and comparing the two sides we obtain,

$$
\begin{aligned}
&F^{\mu\nu_1\ldots\nu_N}(z, w_j, z_i, k_i) \\
&= \sum_{i=1}^{4} k_i^{\mu} \Big[-i\partial_z \ln\vartheta(z - z_i) + \frac{2\pi}{Im\ \tau} Im(z - z_i) \Big] \langle \prod_{i=1}^{4} e^{ik_i.X(z_i)} \prod_{j=1}^{N} \bar{\partial} X^{\nu_j}(w_j) \rangle \\
&- \frac{2\pi}{Im\ \tau} \sum_{j=1}^{N} \delta^{\mu\nu_j} \langle \prod_{i=1}^{4} e^{ik_i.X(z_i)} \prod_{1\le k\ne j\le N} \bar{\partial} X^{\nu_k}(w_k) \rangle
\end{aligned}
\tag{C.6}
$$

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