SLAC – PUB – 4168
December 1986
(T/E)

Superstring Spectroscopy

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Lecture presented at the SLAC Summer Institute on Particle Physics
Stanford, California, July 28 – August 8, 1986

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

Introduction

For the past two years, a large part of the theoretical physics community has been locked in single-minded concentration one highly speculative approach to the fundamental structure of matter—the theory of relativistic strings. Proponents of this theory have claimed that it provides the basic laws unifying all known interactions and that it promises the solution to some of the deepest remaining questions about Nature, including the origin of the quark and lepton generations.

Such a fundamental description of Nature seems even more wonderful because it is built up of elementary entities of a simple and concrete structure. These basic entities, the elementary strings, can in fact be readily visualized by any physicist who has developed a proper quantum-mechanical intuition. The theory of superstrings has acquired a reputation as being mathematically abstruse and formidable in the extreme, but, while it is certainly true that some unfamiliar mathematical technology is needed to perform calculations in this theory, the basic elements of the theory are remarkably accessible. The purpose of this lecture is to set out these basic elements in terms which are as pictorial as possible. (Students who wish to study this subject in a serious way should consult one of the excellent technical reviews now available.[1–5])

Before beginning this explication, however, it is worth reviewing the main properties of string theories and, especially, of the supersymmetric version of the string theory which shows the most promise of making contact with the phenomena of elementary particle physics. This version of the theory was originally formulated in 1970 by Neveu and Schwarz,[6] Ramond,[7] and Thorn.[8] It underwent a second stage of development in the early 1980's, when Green and Schwarz[9] clarified many of its properties and pressed its interpretation as a unifying theory for all interactions.

The main properties of this superstring theory which bolster its interpretation as a fundamental theory of Nature are the following:

1. The theory requires that all particles—quarks, leptons, gauge bosons, gravitons, and their supersymmetric partners—are built of the same fundamental entities, the elementary strings. In this sense, string theories are the most elegant of all models of elementary particle substructure. One of my main tasks in this lecture will be to explain the origin of the various quantum numbers that these particles carry.

2. The theory requires that space-time be fundamentally supersymmetric. It also requires the existence of 10 space-time dimensions. This would be an excessive number if all of these dimensions were extended to the size of the 4 dimensions that are part of our everyday experience. However, the extra 6 dimensions may play a more subtle role, which the last two sections of this lecture should make clear. The most probable size for the compactified dimensions is the characteristic length of an elementary string; this is of order the Planck length, 10^{-33} cm or $(10^{19}\text{ GeV})^{-1}$.

3. The theory naturally contains as a part of its structure the gauge invariances of Yang-Mills theory and gravity. In fact, these invariances are realized as a small part of an enormous group of generalized gauge symmetries. This enormous gauge structure was first made clear in the work of Siegel.[10]

4. Although the theory contains within it a quantum theory of gravity, it is apparently free of ultraviolet divergences. The finiteness of the theory has been shown explicitly to 1-loop order[11,12] and a plausible intuitive argument has been given which extends this result to all orders.[13]

5. The theory restricts the possible choices for its Yang-Mills symmetry group to only two candidates: $O(32)$ and $E_8 \times E_8$, the latter involving the largest of the exceptional groups.[14] In fact, these two symmetry groups arise *geometrically* as solutions for the space-time structures allowed by the theory.[15] I will explain this point in some detail at an appropriate stage of my development.

This list of properties, some of which arise rather magically from the properties of the underlying strings, should be enough to attract anyone with a speculative bent. Still, it is worth noting that string theories also occupy a privileged position within theoretical physics, as the unification point for many strands of theoretical investigation which have been actively pursued in the past decade. First of all, strings are a natural generalization of point particles, since they are objects extended in a spatial direction as well as along a world-line. From this point of view, they have long been of interest to workers in the foundations of relativistic field theories. From the list given above of the properties of string theories, it should be clear that these theories provide a natural meeting ground for workers interested in Yang-Mills fields and grand unification, supersymmetry, models of quark and lepton substructure, and gravitation. In addition, string theories have provided quite nontrivial applications for more mathematical aspects of theoretical physics—the study of 2-dimensional model field theories, and the application of higher geometry and topology to field-theoretic problems. In a certain sense, it now seems that most of the developments in theoretical physics over the past ten years were really directed toward the solution of string theory. Small wonder, then, that this theory excites so much interest in so many quarters.

I would like to conclude this brief survey of the prospects for string theory by citing the major problems which must still be solved in order to bring this theory from the level of speculation to a point where it can make concrete predictions for experiment. The most pressing problems are those which concern the conversion of the 10-dimensional space-time of string theory into a form closer to experimental reality in which 6 of these 10 dimensions are curled up to a very small size. The geometry of this compactification of dimensions determines all of the detailed properties of the system of elementary particles which would be visible at energies accessible to experiment: the number of quark and lepton generations, the gauge group which results from breaking the grand unification symmetry, the values of the strong- and weak-interaction coupling constants, and the existence and number of supersymmetric partners. The most basic aspects of how the geometry of the compact 6 dimensions determines these parameters have been clarified by Candelas, Horowitz, Strominger, and Witten,[16] among others. (An elementary discussion of the physics of compactification may be found in my lectures at the 1985 SLAC Summer Institute.[17]) However, many issues, especially the mechanism of supersymmetry breaking and the relation of the weak-interaction scale to the fundamental string length scale, remain obscure. In addition, we still have no idea how Nature chooses a particular geometry for the compact 6 dimensions from among a wealth of possibilities.

In addition to these questions of quite direct physical importance, there are a number of absolutely fundamental formal questions about the superstring theory which have not yet been settled. We still do not know the complete equations of motion for the theory (though considerable progress has been made in the past year in understanding the more elementary, nonsupersymmetric case[18,19]). We still do not have a complete set of rules for computing the perturbation theory in string interactions (though, again, there has recently been some dramatic progress in this direction[20–22]). Finally, we have almost no idea of how to discuss string dynamics beyond perturbation theory. This last formal problem is a particularly important one, because it is known that many of the aspects of compactification that seem to us the most mysterious—which compact space is chosen, for example—are simply not determined at the level of the first perturbative loop corrections; quite plausibly, these questions can only be settled by looking beyond perturbation theory.[23,24] It is not an uncommon occurrence in physics that the most crucial phenomenological properties of a theory arise nonperturbatively; the appearance of solids in QED and the appearance of hadrons in QCD provide two examples. In both of those cases, the connection of the phenomena to the theory was forged by quite remarkable guesses about the correct treatment of the theory in the regime of strong coupling, guesses which were motivated crucially by the findings of experiment. If string theory is to be made a predictive theory which brings new and relevant information to the study of quarks and leptons and their interactions, this will only be done through a similarly remarkable conjecture about how strings determine the form of space-time.

Let us hope that our imaginations are worthy of this challenge.

The remainder of this lecture will concern the aspects of string theory which are well-understood, and which are most easy to visualize. I will develop the theory, to the level where it produces an observable spectrum of quarks and leptons, in eight easy lessons.

Lesson 1: The Basic Quantized String

A relativistic string is an idealized 1-dimensional extended object, an object whose sole property is that it lies along some curve in space. It is, then, the natural generalization of an idealized point particle. Just as the point particle can be viewed as sweeping out a world-line as it progresses through time, the string sweeps out a 2-dimensional space-time surface, a world-sheet. A string may or may not have endpoints; one refers to a string with or without endpoints as being open or closed (see Fig. 1).

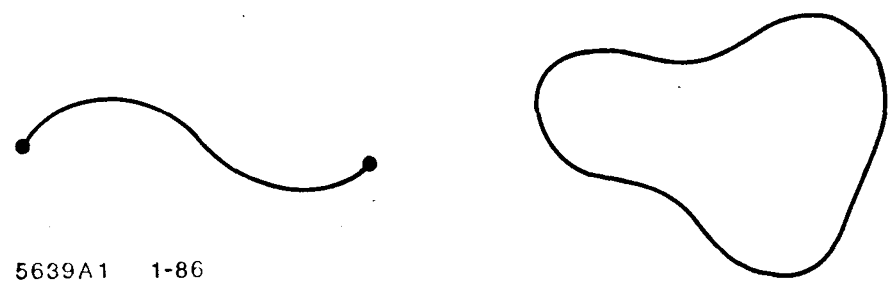

Figure 1. Typical configurations of open and closed strings.

Such an idealized object must have an extremely simple equation of motion. This equation must be relativistically invariant; it must also be local along the string, or, in a space-time view, across the world-sheet. There are two natural candidates for such an equation; these are illustrated in Fig. 2.

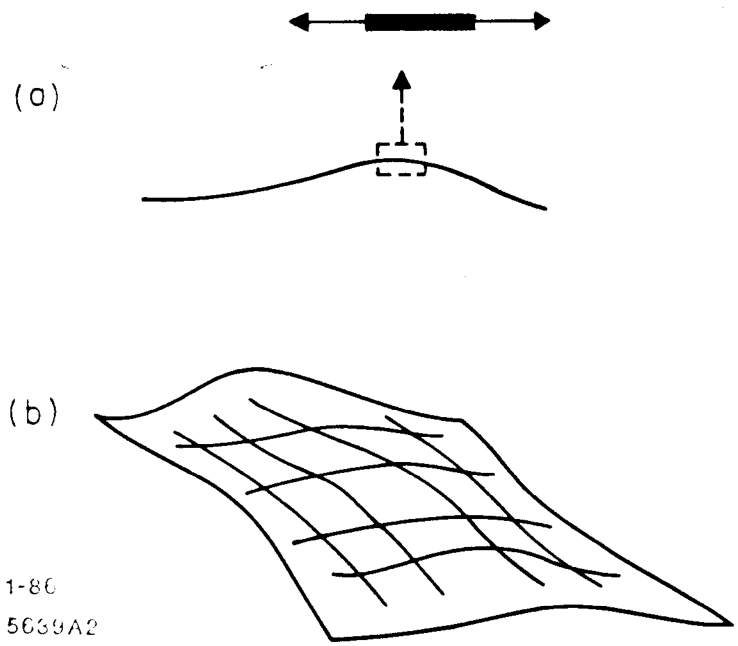

Figure 2. Rest-frame and space-time viewpoints on the equation of motion for a relativistic string.

The first candidate is best formulated in the rest frame of an infinitesimal bit of string. This string-bit has a rest energy per unit length T_0. Stretching the string by dx would create more string, also at rest; this would cost energy $T_0 dx$. Thus, T_0 is also the (rest) tension in the string. We can then compute the net force exerted on each bit of string and, from this, deduce its motion. The second candidate is best formulated in a space-time approach. We know that the motion of a point particle in space-time is given by the geodesic principle that it should follow the shortest path between two points. The generalization of this statement to a string is that the string should sweep out a world-sheet of minimum area. Remarkably, these two formulations of the string equations of motion are equivalent.[25,26] Together, they give quite a clear picture of the classical mechanics of string.

To discuss a quantum-mechanical relativistic string, we should quantize the string's vibrational modes. Let us discuss this procedure first for the open string, moving in d space-time dimensions. Let σ be a coordinate along the string, running from $\sigma = 0$ at one end to $\sigma = 1$ at the other, and let $X^i(\sigma)$ be transverse displacement of the string as a function of σ $(i = 1, \ldots, d_\perp;\ d_\perp = (d-2))$. This function obeys the boundary condition $(\partial/\partial\sigma)X^i(\sigma) = 0$ at $\sigma = 0, 1$, that is, that there should be no unbalanced transverse tension acting on the endpoint. Then we may expand $X^i(\sigma)$ in a Fourier series as follows:

$$X^i(\sigma) \ = \ x^i \ + \sum_{n=1}^{\infty} X_n^i \cos n\pi\sigma \ . \tag{1}$$

In this expression, x^i is the position of the center of mass of the string. It should be no surprise that each X_n^i turns out to be the coordinate of a harmonic oscillator. Quantizing these oscillators, one finds an expression for the energy eigenvalues of the string in terms of the harmonic oscillator ladder operators a_n^i. These correspond to the energies of relativistic particles with masses

$$m^2 \ = \ (2\pi T_0) \cdot \left[\sum_{n=1}^{\infty} n\, a_n^{i\dagger} a_n^i \ + \ (d_\perp \cdot Z)\right] . \tag{2}$$

The last term in (2) represents the zero-point energy of the oscillators. This very simple equation summarizes all of the basic properties of the string. It is relativistic, since it is an equation which gives the rest mass of the string in terms of its internal structure. It is harmonic, since the internal states which appear are those of a set of simple oscillators. And, finally, it is geometrical in assigning energy only to transverse fluctuations of the string. Displacements of the string along the string itself are not physically observable and therefore should not affect the string energy levels.

The quantization of a closed string proceeds in a similar fashion. In this case, the only boundary condition to be satisfied is that of periodicity. It is convenient to Fourier-analyze the string displacement in terms of running waves which move to the left and to the right around the loop. Introducing a time coordinate τ on the world sheet,

$$X^i(\sigma) \;=\; x^i \;+\; \sum_{n=1}^{\infty} \left(X_n^i \, e^{2\pi i n(\sigma+\tau)} \;+\; \overline{X}_n^i \, e^{2\pi i n(\sigma-\tau)} \;+\; (c.c.) \right) . \qquad (3)$$

The left- and right-moving excitations form independent sets of harmonic oscillators. Denoting the ladder operators for the left- and right-moving oscillations by a_n^i, $\overline{a}_n^i$, respectively, we may write the mass formula for this case as:

$$m^2 \;=\; (4\pi T_0) \cdot \left[\sum_{n=1}^{\infty} n \left(a_n^{i\dagger} a_n^i \;+\; \overline{a}_n^{i\dagger} \overline{a}_n^i \right) \;+\; (2 \cdot d_\perp \cdot \mathcal{Z}) \right] . \qquad (4)$$

Again, this formula reflects the relativistic, quantum, harmonic, and geometrical aspects of the string.

Lesson 2: Zero-Point Energy

The last term in each of the formulae for the mass of a quantized string is the total zero-point energy of the harmonic oscillators. Normally in field theory we can simply throw away the zero-point energy, since it is not physically observable. In the analysis of the string, however, we have taken what is effectively a 2-dimensional field theory of transverse motions of the world-sheet and interpreted the eigenvalues of the Hamiltonian of this theory as the masses of particles. The zero-point energy of the field theory certainly contributes to these eigenvalues. Unfortunately, the zero-point energy of a field theory is usually infinite. We must regulate this infinity and try to make some sense of it.

Assigning a zero-point energy of $\frac{1}{2}\omega$ to each harmonic oscillator, we can write the total zero-point energy of the open string as

$$\mathcal{Z} \;=\; \frac{1}{2} \sum_{1}^{\infty} n \; . \qquad (5)$$

To define this infinite sum, add to the definition of Z a cutoff ϵ:

$$\begin{aligned} Z &= \frac{1}{2}\sum_{1}^{\infty} n e^{-n\epsilon} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial\epsilon}\right)\sum_{1}^{\infty} e^{-n\epsilon} \;=\; \frac{1}{2}\left(\frac{\partial}{\partial\epsilon}\right)\left[\frac{1}{1-e^{-n\epsilon}}\right] \\ &= \frac{1}{2}\left(\frac{\partial}{\partial\epsilon}\right)\left[\frac{1}{\epsilon}-\frac{1}{2}+\frac{1}{12}\epsilon-\frac{1}{90}\epsilon^2+\cdots\right] \\ &= \frac{1}{2}\left[\frac{1}{\epsilon^2}-\frac{1}{12}+\cdots\right] . \end{aligned} \tag{6}$$

The first term in this series is highly divergent as $\epsilon \to 0$. I propose that we ignore it. The next term gives a finite, cutoff-independent residual contribution:

$$Z = \frac{1}{2}\sum_{1}^{\infty} n \equiv -\frac{1}{24} . \tag{7}$$

Brink and Nielsen[27] have argued that the term we have omitted may be interpreted as a renormalization of the speed of light in this noncovariant calcucation. In any event, there are many cross-checks which insist that the theory of strings which we are constructing can be Lorentz-invariant and self-consistent only if we define the zero-point energy by the regulated expression (7).

For our future reference, it will be useful to perform another divergent sum:

$$Z(\alpha) = \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha) . \tag{8}$$

This quantity will arise as the zero-point energy of a string which runs around a region containing space-time curvature or magnetic field, as shown in Fig. 3.

In that case, the boundary condition of periodicity may become slightly more complicated: Write $X(\sigma) = X^1(\sigma)+iX^2(\sigma)$. Then the magnetic field or curvature

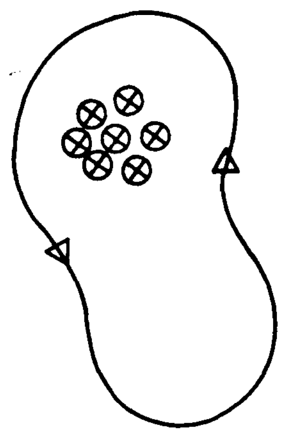

Figure 3. A closed string running around a region containing magnetic flux.

may be reflected in a phase factor:

$$X(\sigma = 1) \; = \; e^{2\pi i \alpha} X(\sigma = 0) \; . \tag{9}$$

When we Fourier expand $X(\sigma)$, we must use functions which obey this new boundary condition; for example, we must replace in (3)

$$e^{2\pi i n(\sigma+\tau)} \;\longrightarrow\; e^{2\pi i(n+\alpha)(\sigma+\tau)} \; . \tag{10}$$

Then all of the oscillator frequencies will be shifted by α, and the zero-point energy will be given by (8).

To evaluate $\mathcal{Z}(\alpha)$, we note that it obeys the functional equation

$$\mathcal{Z}(\alpha) \; = \; \frac{1}{2}\alpha \; + \; \mathcal{Z}(\alpha+1) \; . \tag{11}$$

Substituting for $\mathcal{Z}(\alpha)$ an arbitrary polynomial of α, we find that (11) can only be satisfied for

$$\mathcal{Z}(\alpha) = \frac{1}{4}\alpha(1-\alpha) + C . \tag{12}$$

We can determine the constant by noting that $\mathcal{Z}(0) = \mathcal{Z}(1) = \mathcal{Z}$. Thus

$$\mathcal{Z}(\alpha) = -\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha) . \tag{13}$$

The physical importance of the zero-point energy will be made clear by the central role that $\mathcal{Z}(\alpha)$ plays in our later discussion.

Lesson 3: The Bosonic String

Now that we have clarified all of the terms in (2) and (4), we should display the spectrum of possible string states that these equations predict. Let us begin with the case of the open string:

$$m^2 = (2\pi T_0) \cdot \left[\sum_{n=1}^{\infty} n\, a_n^{i\dagger} a_n^i - \frac{d_\perp}{24} \right] . \tag{14}$$

The ground state of the string is the state $|0\rangle$ annihilated by all of the a_n^i. Most regrettably, this state has $m^2 < 0$. The first excited state is almost as problematic:

$$a_1^{i\dagger} |0\rangle , \qquad m^2 = (2\pi T_0) \cdot (1 - \frac{d_\perp}{24}) . \tag{15}$$

The $d_\perp$ vector components are obviously trying to form a vector particle. However, this vector has only $d_\perp$ polarization states; the longitudinal polarization state is missing. This contradicts Lorentz invariance unless the vector particle is massless. Thus, we find that the open string theory we have constructed can be Lorentz-invariant only if $d_\perp = 24$, that is, if $d = 26$. As an integral part of this construction, we find a massless vector field. It can be seen that, once we have set $d = 26$, the Lorentz group automatically acts properly on all higher mass levels. For example, the next level contains states

$$a_1^{i\dagger} a_1^{j\dagger} |0\rangle \ , \quad a_2^{i\dagger} |0\rangle \ ; \qquad\qquad m^2 = (2\pi T_0) \tag{16}$$

of exactly the right number to form a 25×25 traceless symmetric tensor; this accounts for all of the components of a massive tensor field in 26 dimensions.

This construction works in a similar way for the closed string. In 26 dimensions, the mass formula is

$$m^2 \ = \ (4\pi T_0) \cdot \Big[\sum_{n=1}^{\infty} n \big(a_n^{i\dagger} a_n^i \ + \ \overline{a}_n^{i\dagger} \overline{a}_n^i \big) \ - \ 2 \Big] \ . \tag{17}$$

The ground state $|0\rangle$, and also the states $a_i^{1\dagger} |0\rangle$, $\overline{a}_1^{i\dagger} |0\rangle$, are tachyons, with $m^2 < 0$. But now the states

$$a_1^{i\dagger} \overline{a}_1^{j\dagger} |0\rangle \tag{18}$$

appear just at $m^2 = 0$. These states form a transverse symmetric tensor; this state would also be inconsistent with Lorentz invariance if it were not precisely massless.

It is tempting to speculate that the massless vector and tensor states that we have uncovered can be identified with the corresponding states that we see in Nature—gauge bosons and gravitons. This interpretation is surprisingly robust: when one introduces interactions into the string theory in the natural way, one finds that the low-energy scattering amplitudes for these particles agree with the predictions of Yang-Mills theory and general relativity.[28,29] One often hears theorists mutter that these gauge-invariant equations are the only possible equations for massless vector and tensor fields; still, it is amazing that this observation constrains a system that, at first sight, has nothing to do with local field theory.

I should note that the multiplet (18) actually contains other particles in addition to gravitons, since the states shown form a transverse tensor of arbitrary symmetry. Symmetrizing and antisymmetrizing, we find:

$$\begin{aligned} \text{symmetric, transverse, traceless} &\rightarrow h^{ij} \text{ (the graviton)} \\ \text{antisymmetric, transverse} &\rightarrow b^{ij} \\ \text{trace} &\rightarrow \phi \text{ (a scalar, the } \textit{dilaton}) \ . \end{aligned} \tag{19}$$

The antisymmetric tensor particle b^{ij} also appears in the string theory with appropriate gauge-invariant interactions.

Before we leave the subject of the closed string spectrum, I should correct one statement that I made above. In enumerating the low-mass states of the closed string, I listed the states $a_1^{i\dagger}\,|0\rangle$ and $\bar{a}_1^{i\dagger}\,|0\rangle$ as tachyons. But, in fact, these states do not exist in the interacting string theory. I would like to explain this statement, which reveals some subtle properties of string interactions.

Generalizations of this statement will play a crucial role in later stages of our analysis. The argument for this statement proceeds in three stages, indicated diagrammatically in Fig. 4. In this first stage, we note that the states such as $a_1^{i\dagger}\,|0\rangle$ with a preponderance of left-moving excitations have net momentum P running around the closed string. The ground state, and the states (18), have $P = 0$, so the state $a_1^{i\dagger}\,|0\rangle$ is distinguished from these by a conserved quantum number. We must next ask whether interactions can couple states with $P = 0$ to states with $P \neq 0$. To answer this question, note that $P = 0$ is exactly the criterion for the string state to be invariant under rotations of σ around the loop. Now study the picture of the 3-string interaction shown in Fig. 4(b). The three strings sweep out tubes in space-time. We can make them interact by connecting the tubes in a join; in an geometrically invariant theory, we should allow the join to form in all possible ways. In particular, we must integrate over the angle θ indicated in the figure. But if the string state coming in from the

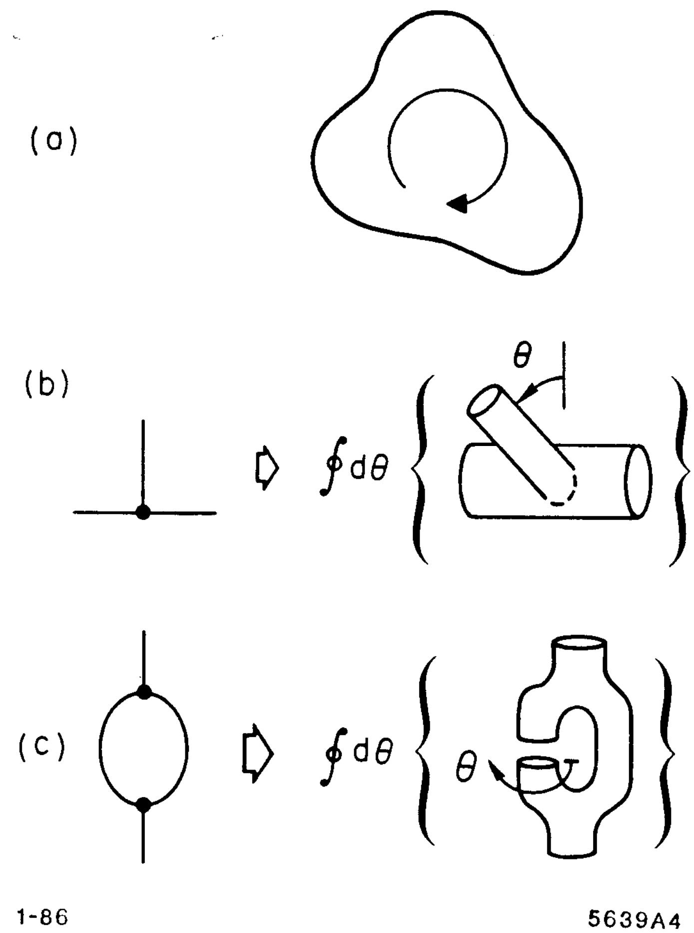

Figure 4. Three stages of the argument that states with net momentum around the string do not exist.

right is a rotionally invariant ($P = 0$) state, and the third state is joined on in a rotationally invariant way, the state going out to the left will rotationally invariant as well. This geometrical interaction, then, couples $P = 0$ states only to other $P = 0$ states. Finally, if states with $P \neq 0$ cannot be produced directly, they still might appear as intermediate states in closed-loop diagrams. This possibility is excluded by viewing string loop diagrams in the manner shown in Fig. 4(c). If we are to include all possible geometries in the join, we must sum over twists θ as indicated in the figure. The integral over θ projects out all states but those which are rotationally invariant ($P = 0$). We find, then, that states with $P \neq 0$, although apparently present as eigenstates of the mass operator, cannot be produced by string interactions, either as real or as virtual particles. For all practical purposes, then, they do not exist.

This argument greatly simplifies the spectrum of the closed string, leaving only a single tachyon, $|0\rangle$, the massless states given in (18), and the higher-mass states with $P = 0$, such as $a_1^{i\dagger} a_1^{j\dagger} \bar{a}_2^{k\dagger} |0\rangle$. The Lorentz group in 26 dimensions acts in a consistent way on the higher-mass states, as long as we consider only states with $P = 0$.

Lesson 4: Compactification on a Torus

We have now come roughly one-third of the way toward a theory which could be realistic. The simple, unadorned string discussed in the previous section contains vector bosons and gravitons; however, it includes no fermions. It has no well-defined gauge group. And it has a serious affliction—tachyons in both the open- and closed-string sectors. In the following two sections, I will explain how to remedy these difficulties. First, though, I would like to give a preliminary treatment of another issue which is raised by our results in the previous section: the proper interpretation of the extra spatial dimensions required for the consistency of the theory. Clearly, these extra dimensions must curl up into a compact space. Let us study, in a simple example, a possible consequence of this compactification.

Consider, then, the 26-dimensional closed string theory with one spatial dimension closed into a ring, so that space-time has the appearance of a cylinder of circumference R (Fig. 5).

To analyze the spectrum of string states in this geometry, let us Fourier-analyze $X^i(\sigma)$. This expansion is changed from (3) only in the compactified direction $X^c(\sigma)$, and there only in two simple respects: First, we must restrict

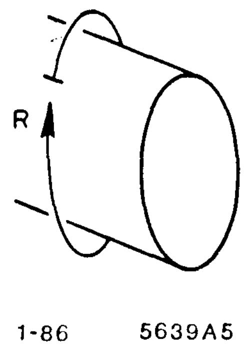

Figure 5. A space-time with one compactified dimension.

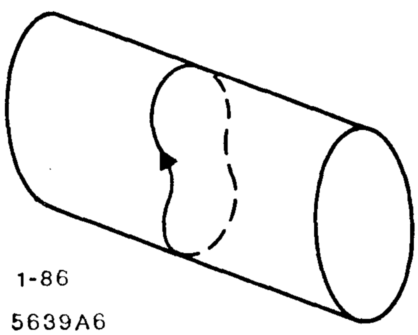

Figure 6. A closed string which winds around a compactified dimension.

the center-of-mass coordinate x^c to $0 \leq x^c < R$, and, second, we must allow closed string to wind around the c direction (Fig. 6).

Then

$$
\begin{aligned}
X^c(\sigma) \;&=\; x^c \;+\; \ell \cdot R\sigma \\
&\quad + \sum_{n=1}^{\infty} \left(X_n^c \, e^{2\pi i n(\sigma+\tau)} \;+\; \overline{X}_n^c \, e^{2\pi i n(\sigma-\tau)} \;+\; (c.c.) \right) \; ;
\end{aligned}
\tag{20}
$$

the integer ℓ is the number of times that the string winds around the cylinder.

To express the spectrum of string states, we should write the energy of a string as

$$E^2 = p^2 + m^2 = \tilde{p}^2 + p_c^2 + m^2 , \tag{21}$$

where $\tilde{p}$ is the momentum in uncompactified directions. Since the compactified dimension will be extremely small, the momentum in this direction will be physically relevant only as a part of the string energy. If we do not observe the extra dimension directly, we would say that the string states appear in $(d-1)$ dimensions with mass $\tilde{m}$ given by

$$\tilde{m}^2 = p_c^2 + m^2 . \tag{22}$$

Taking into account the new contribution to energy cost of winding, and the fact that p_c is quantized, we find for this effective mass:

$$\tilde{m}^2 = \left(\frac{2\pi k}{R}\right)^2 + T_0\big(\ell R\big)^2 + (4\pi T_0)\cdot\left[\sum(n+\overline{n}) - 2\right] . \tag{23}$$

Here k is an integer, and the last term is an abbreviation for eq. (17).

Let us examine this formula for the particular choice $R = (2\pi/T_0)^{\frac{1}{2}}$. In that case, the mass formula becomes

$$\tilde{m}^2 = (2\pi T_0)\cdot\left(k^2 + \ell^2\right) + (4\pi T_0)\cdot\left[\sum(n+\overline{n}) - 2\right] . \tag{24}$$

The ground state is still a tachyon: $|0\rangle$, with $k = \ell = 0$ still has $m^2 = (-2)\cdot(4\pi T_0)$. However, we now have a more interesting spectrum of zero-mass states. Let us denote the state composed of the oscillator ground state plus winding quanta k, ℓ by $|k,\ell\rangle$ This state has $P = k\cdot\ell$. We can then enumerate the states with $\tilde{m}^2 = 0$ and $P = 0$. These include, of course, the states of (18) with both indices in uncompactified dimensions, plus

$$\begin{array}{cc} \overline{a}_1^{i\dagger}\,|1,1\rangle & a_1^{i\dagger}\,|1,-1\rangle \\ \overline{a}_1^{i\dagger}a_1^{c\dagger}\,|0,0\rangle & a_1^{i\dagger}\overline{a}_1^{c\dagger}\,|0,0\rangle \\ \overline{a}_1^{i\dagger}\,|-1,-1\rangle & a_1^{i\dagger}\,|-1,1\rangle \end{array} \tag{25}$$

These new states form two triplets of massless vector bosons. It is extremely tempting to conjecture that these are the gauge bosons of $SU(2) \times SU(2)$, and that the complete string theory has an $SU(2) \times SU(2)$ symmetry group. A detailed analysis shows that this is indeed the case. (For values of R other than this special choice, one finds only one zero-mass state on each side of (25) and thus a lower symmetry, $U(1) \times U(1)$.)

We can describe this phenomenon more generally as follows: Consider compactifying some number n of the extra dimensions into rings. The result is a generalized torus. A set of motions $\vec{\ell}$ carry us around the compact manifold and back to the same point. These motions form a lattice in n-dimensional space. We can, in fact, view the torus as being the full n-dimensional space, but with points related by lattice translations identified. This correspondence is illustrated in Fig. 7.

Certain lattices are closely connected to Lie groups, since the quantum numbers associated with the finite-dimensional representations of Lie groups fall at points of a lattice, called the *root lattice*⋆ . A trivial example is given by $SU(2)$: the values of I^3 for all (tensor) representations are integers; thus, the root lattice of $SU(2)$ is the 1-dimensional lattice shown in Fig. 7(a). Physicists will recognize the lattice of Fig. 7(b) as the root lattice of the group $SU(3)$. Using this language, we can state the generalization of our result above to any "simply laced" group (a class which includes $SU(N)$ and $O(2N)$ for all N and also the exceptional groups E_6, E_7, E_8): If one compactifies the closed string on a torus whose associated lattice is the root lattice of G, there is a special value of the radius of the compact space at which the compactified theory has a $G \times G$ gauge symmetry! The dimension of the compact space gives the rank of the symmetry group, the number of generators which can be simultaneously diagonalized. This number equals $(N-1)$ for $SU(N)$ (as in the two examples given), N for $O(2N)$, and k for the exceptional groups E_k.

⋆ Properly, the root lattice includes only the quantum numbers of representations which can be built up as products of the adjoint representation. For $SU(2)$, this lattice includes the quantum numbers of the tensor, but not the spinor, representations.

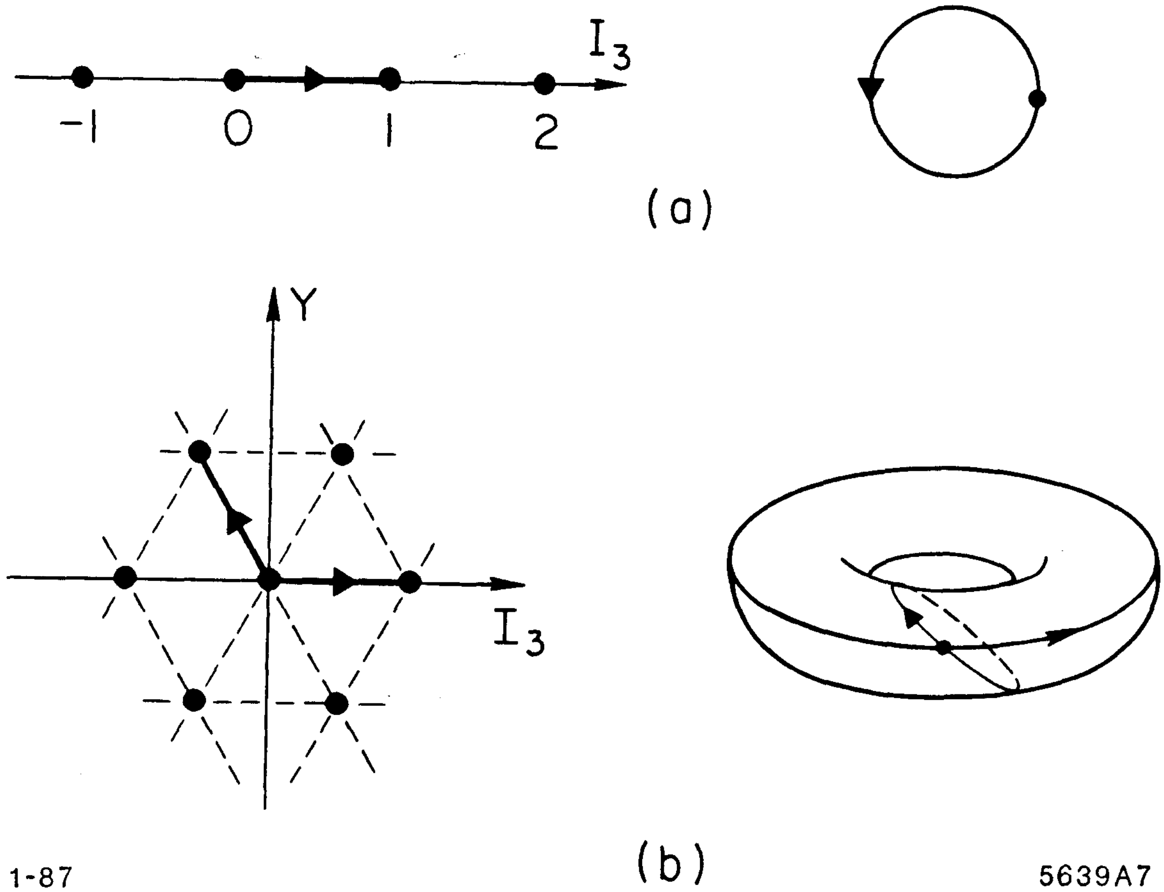

Figure 7. Examples of the correspondence between lattices in n-dimensional space and n-dimensional tori.

Apparently, closed-string theories compactified on tori can give rise to gauge symmetries in a way that is completely geometrical; this mechanism generates the gauge bosons and the gravitons in exactly the same fashion, as zero-mass, $P = 0$ closed-string eigenstates, so that these closed-string theories represent a true unification of Yang-Mills theory with gravity. It seems appropriate, then, to discard the open-string theory and pursue the theory of closed strings alone.

Lesson 5: The Superstring

Now let us begin a search for solutions to the difficulties of the bosonic string theory listed at the beginning of the previous section. Let us take up first the question of how to introduce fermionic states of string. In the theoretical climate of the 1980's, a natural suggestion is to replace the geometrical theory of world-sheets (2-dimensional gravity) by 2-dimensional supergravity (Fig. 8).* The practical effect of this change is to replace the transverse displacement field $X^i(\sigma)$ by a supermultiplet $(X^i(\sigma), \psi^i(\sigma))$. The new field $\psi^i(\sigma)$ is a fermion on the world-sheet and cannot be directly interpreted as a space-time fermion. Its influence on the theory is, as we will see, considerably more subtle.

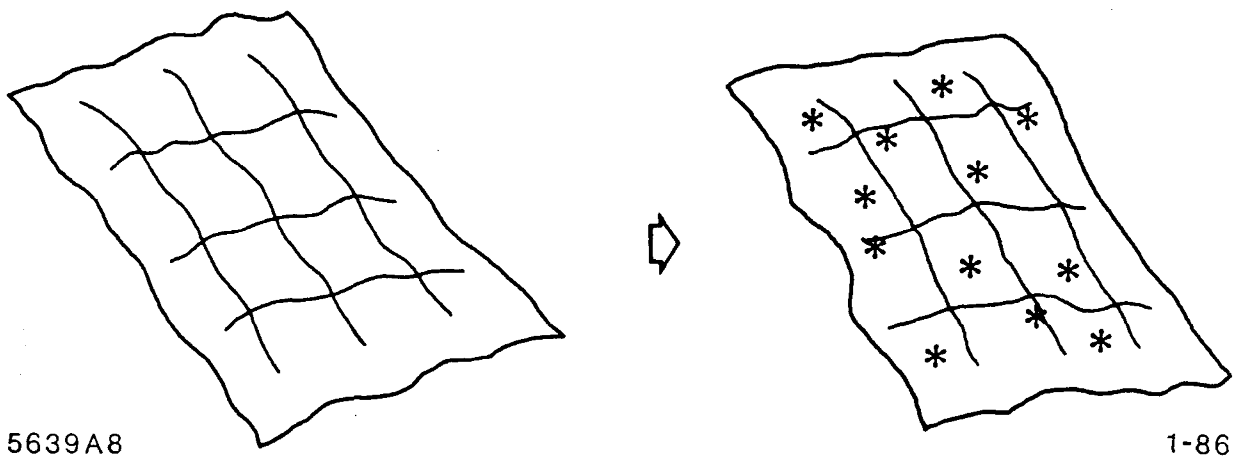

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Figure 8. Conversion of the bosonic string world-sheet to the superstring world-sheet.

Let us, then, compute the influence of $\psi^i(\sigma)$ on the closed-string spectrum. In performing this analysis, I will treat the left-moving modes of the string in isolation from the right-moving modes. At the very end of the analysis, we can add the right-moving excitations and impose the condition $P = 0$.

⋆ Historically, though, this construction was invented first,[6–8] and supersymmetry arose from attempts to understand its structure.[30]

Begin by choosing periodic boundary conditions (Ramond[7] boundary conditions) for ψ^i. Then ψ^i has a Fourier expansion analogous to (3). We can quantize the $n \neq 0$ modes of ψ^i with (anticommuting) ladder operators b^i_n to find the mass formula

$$m^2 \;=\; (4\pi T_0)\cdot\Big[\sum_{n=1}^{\infty} n\big(a^{i\dagger}_n a^i_n \;+\; b^{i\dagger}_n b^i_n\big) \;-\; \frac{d_\perp}{24} \;+\; \frac{d_\perp}{24}\Big]\;. \tag{26}$$

The last term denotes the corresponding right-moving contributions. Note that the fermionic contribution to the zero-point energy has just the opposite sign from the bosonic contributions, so that these two terms cancel. (This is a familiar consequence of supersymmetry.) The constant terms of $X^i(\sigma)$ and its conjugate momentum form the center-of-mass position and momentum, which satisfy $[x^i, p^j] = i\delta_{ij}$. Similarly, the constant term of ψ^i plays a special role. The constant pieces of the ψ^i naturally satisfy the anticommutation relations:

$$\{\psi^i_0, \psi^j_0\} \;=\; 2\delta^{ij}\;. \tag{27}$$

This is exactly the defining algebra for Dirac matrices. Thus, we may represent the ψ^i_0 as Dirac matrices; the string ground state must then be a zero-mass spinor. By the connection between spin and statistics, this state and all states built by applying to it the (space-time vector) operators $a^{i\dagger}_n$, $b^{i\dagger}_n$ should be fermionic particles.

It is, however, equally valid to begin with antiperiodic boundary conditions (Neveu-Schwarz[6] boundary conditions):

$$\psi^i(\sigma = 1) \;=\; -\psi^i(\sigma = 0)\;. \tag{28}$$

This condition also insures that bosonic quantities built out of the ψ^i are periodic around the loop. We can analyze the effect of this boundary condition by noting that it is just the condition (9) for $\alpha = \frac{1}{2}$. The quantization of the ψ^i oscillators is then shifted by $\frac{1}{2}$. The constant mode ψ^i_0 no longer satisfies the boundary conditions and so disappears. The zero-point energy of the ψ^i oscillators is given by

$$-d_\perp \cdot Z(\alpha = \frac{1}{2}) \;=\; d_\perp \cdot (\frac{1}{24} - \frac{1}{16}) \; . \tag{29}$$

Note that we have reversed the sign, as is appropriate for fermions. Then the mass formula reads

$$m^2 \;=\; (4\pi T_0) \cdot \Big[\sum_{n=1}^{\infty} \big(n\, a_n^{i\dagger} a_n^i \;+\; (n - \frac{1}{2})\, b_{n-\frac{1}{2}}^{i\dagger} b_{n-\frac{1}{2}}^i \big) \;-\; \frac{d_\perp}{16} \Big] \; . \tag{30}$$

The spectrum of this theory is as follows: The ground state $|0\rangle$ is a tachyon of mass $m^2 = (4\pi T_0) \cdot (-\frac{1}{2})$. Since there are no ψ_0^i operators, this state is a spinless boson. The first excited state is

$$b_{\frac{1}{2}}^{i\dagger} |0\rangle \;, \qquad m^2 \;=\; (4\pi T_0) \cdot (\frac{1}{2} \;-\; \frac{d_\perp}{16}) \; . \tag{31}$$

This state is a transverse vector; as we saw for the state (15), the present of this state is inconsistent with Lorentz invariance unless the state has precisely zero mass. This implies that the superstring with Neveu-Schwarz boundary conditions cannot be Lorentz-invariant unless $d_\perp = 8$, or $d = 10$. As with the bosonic string, imposition of the condition $d = 10$ makes (31) a massless gauge boson (or, after adding the right-moving excitations, a massless graviton) with gauge-invariant couplings.

We have now studied the supersymmetric string with two different boundary conditions for the fields ψ^i. Each has its advantages: Ramond boundary conditions produce massless fermions; Neveu-Schwarz boundary conditions produce the massless vector bosons. Clearly, we want to include both sets of boundary conditions in our theory. In geometrical terms, we would like to sum over world-sheets with the two sets of boundary conditions shown in Fig. 9(a) and (b).

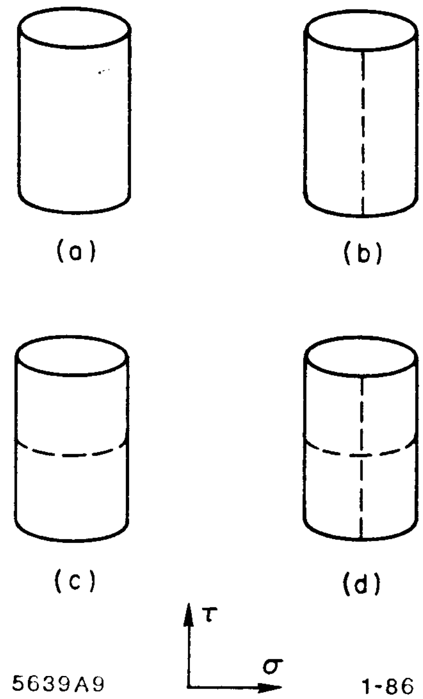

Figure 9. Possible boundary conditions which one might impose on superstring world-sheets. Each dotted line represents an antiperiodic boundary condition: Identify ψ^i on one side of the line with $-\psi^i$ on the other.

This unification of the Neveu-Schwarz and Ramond theories has a serendipitous effect, first noted by Gliozzi, Scherk, and Olive.[31] The prescription of summing over the first two sets of boundary conditions in Fig. 9 violates geometrical invariance unless we also sum over the remaining two sets of boundary conditions shown in that figure. These latter two conditions introduce an antiperiodic boundary condition in the τ direction. Adding Fig. 9(a) and (c) or (b) and (d) is equivalent to inserting the operator

$$\mathcal{P}_{\mathrm{GSO}} = \left(1 - (-1)^F\right) , \tag{32}$$

where F is the fermion number. This operator, called the GSO projector, removes from the spectrum states of even fermion number. Thus, the spectrum of the

Neveu-Schwarz sector

$$|0\rangle\ ,\ b^{i\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\ a^{i\dagger}_{1}\,|0\rangle\ ,\ b^{i\dagger}_{\frac{1}{2}}b^{j\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\ b^{i\dagger}_{\frac{3}{2}}\,|0\rangle\ ,\ a^{i\dagger}_{1}b^{j\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\ \ldots \tag{33}$$

is reduced to

$$b^{i\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\qquad\qquad b^{i\dagger}_{\frac{3}{2}}\,|0\rangle\ ,\ a^{i\dagger}_{1}b^{j\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\ \ldots \tag{34}$$

All states at half-integer mass levels disappear, including the tachyon. The effect of (32) on the states of the Ramond sector is to pick out one chirality for the spinors. (Recall that γ^i, which flips the chirality of a spinor, is identified with the fermion operator ψ^i_0.) If we denote the left- and right-handed massless spinors by $|La\rangle$, $|Ra\rangle$, the action of (32) leaves the states

$$|La\rangle\ ,\quad a^{i\dagger}_{1}\,|La\rangle\qquad b^{i\dagger}_{1}\,|Ra\rangle\ ,\quad\ldots \tag{35}$$

The states eliminated by this GSO projection disappear from the theory, in just the way that the $P \neq 0$ states disappeared in our argument of Lesson 3. Summing over all possible boundary conditions in the join between three strings prevents these states from being produced in scattering processes. Summing over all possible boundary conditions on the figures associated with closed-loop diagrams (as indicated in Fig. 10) keeps these states from appearing in loops. For all practical purposes, then, the states removed by the GSO projection simply do not exist in the theory.

Let us now add back the right-moving string excitations. It can be seen that it is consistent to impose Neveu-Schwarz or Ramond boundary conditions, and to perform the GSO projection, independently for the left- and right-moving parts of $\psi^i(\sigma)$. Summing over boundary conditions in this way, we find the following zero-mass states:

$$b^{i\dagger}_{\frac{1}{2}}\,|0\rangle\otimes\bar{b}^{j\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\quad |La\rangle\otimes\bar{b}^{j\dagger}_{\frac{1}{2}}\,|0\rangle\ ,\quad b^{i\dagger}_{\frac{1}{2}}\,|0\rangle\otimes|La\rangle\ ,\quad |La\rangle\otimes|La\rangle\ . \tag{36}$$

The first multiplet of states contains h^{ij}, b^{ij}, and ϕ in just the manner indicated in eq. (19). The next two multiplets provide two vector-spinors; these act as

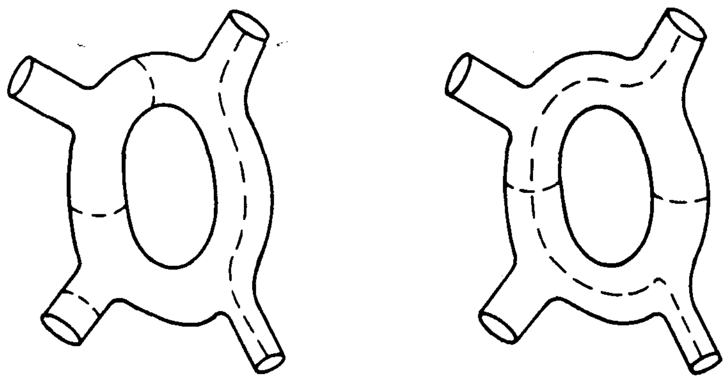

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Figure 10. Some typical contributions to the sum over all possible boundary conditions for a closed-loop diagram.

gravitini, the supersymmetric partners of the graviton. The last multiplet is bosonic, and contains an array of tensor fields. All of these fields together form the content of $N = 2$ supergravity in 10 dimensions. Apparently, the theory we have constructed has not only 10-dimensional fermions but also 10-dimensional supersymmetry.

Lesson 6: The Heterotic String

The only element missing from the theory constructed in the previous section is a grand unification gauge symmetry group. In this section, we will see how to modify that theory so that a gauge group is naturally generated dynamically. The required modification is a bit bizarre: One must consider a string whose left-moving components are those of the supersymmetric string, but whose right-moving components are those of the bosonic string! This hybrid forms the *heterotic string* of Rohm, Martinec, Harvey, and Gross.[15]

To say that this heterotic construction is problematical is something of an understatement. The supersymmetric string can be consistent only if it has 10 space-time coordinates. The bosonic string requires 26 space-time coordinates. How can these be made consistent? The solution is to find a physical interpretation for the extra 16 purely right-moving coordinate fields. To begin, write the Fourier expansion of a purely right-moving field:

$$\vec{X}(\sigma-\tau) \;=\; \vec{v}\cdot(\sigma-\tau) \;+\; \sum_{n=1}^{\infty}\left(\vec{X}^i_n\, e^{2\pi i n(\sigma-\tau)} \;+\; (c.c.)\right) . \tag{37}$$

The σ-dependence of the first term has the form of a winding; this term can be present only if we compactify these extra dimensions. For compactification to a torus, $\vec{v}$ will be a vector of the associated lattice. The τ-dependence of this term indicates that $\vec{v}$ is also proportional to the center-of-mass momentum $\vec{p}$; if we compactify to a torus, this momentum will be quantized. The precise quantization rule is the following: Let $\{\vec{e}_a\}$ denote the basic periodicities of the compact space, that is, the elementary vectors of the associated lattice. Then $\vec{v} = \ell_a\vec{e}_a$. The momentum $\vec{p}$ must then satisfy $\vec{p}\sim k_a\vec{E}_a$, where $\vec{e}_a\cdot\vec{E}_b = \delta_{ab}$. If $\vec{p}$ is also to be identified with $\vec{v}$, then the $\vec{e}_a$ and the $\vec{E}_a$ must coincide, that is, the lattice must be self-dual. In that case, for an appropriate choice of the radius,

$$\vec{X}(\sigma-\tau) \;=\; \left(\frac{2\pi}{T_0}\right)^{\frac{1}{2}}\cdot\ell_a\vec{e}_a\cdot(\sigma-\tau) \;+\; \sum_{1}^{\infty}(\,\cdots\,) . \tag{38}$$

Following the steps in the derivation of eq. (24), but inserting a factor $\frac{1}{2}$ because the winding contributions come only from the left-moving components, we find

$$\tilde{m}^2 \;=\; \frac{1}{2}\,(2\pi T_0)\cdot\left((\ell_a\vec{e}_a)^2 \;+\; (\ell_a\vec{e}_a)^2\right) \;+\; (4\pi T_0)\cdot\left[\sum(n+\overline{n}) - \cdots\right] . \tag{39}$$

A factor $\frac{1}{2}$ also appears in the equation for P:

$$P \;=\; -\frac{1}{2}\,(\ell_a\vec{e}_a)\cdot(\ell_a\vec{e}_a) . \tag{40}$$

We can see from these formulae that the (40) will give integers and the first term of (39) will give integer multiples of $(4\pi T_0)$ only if $(\vec{e}_a)^2 = 2$. A standard cubic lattice is self-dual, but it does not satisfy this additional condition, so we must seek a more exotic lattice to use in our compactification. The simplest self-dual lattice in which the lattice vectors have length $\sqrt{2}$ occurs in 8 dimensions; Fig. 11 gives some idea of its structure.

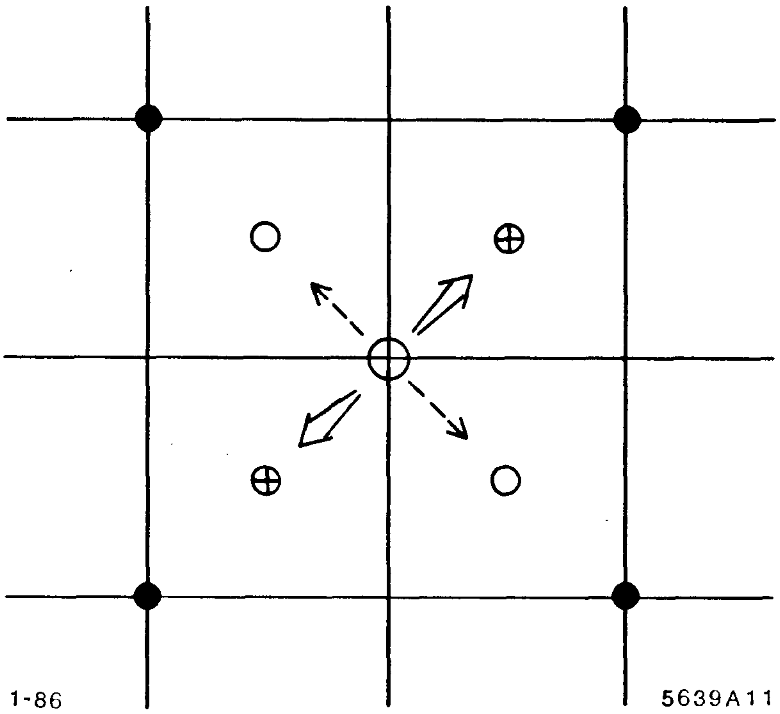

Figure 11. A representation of the 8-dimensional self-dual lattice used to compactify the heterotic string. The arrows point to the centers of hypercubes; the bold arrows point out of the paper, the dotted arrows into the paper.

The elementary vectors of this lattice consist of the points

$$\vec{e}_a = (0,0,\pm1,0,\ldots,0,\pm1,0) \tag{41}$$

at the opposite corners of squares from the origin, plus the points

$$\vec{e}_a = (\pm\frac{1}{2},\pm\frac{1}{2},\ldots,\pm\frac{1}{2}) \tag{42}$$

at the centers of hypercubes (such that the product of the signs is $(+1)$). The first set of vectors generate the root lattice of $O(16)$. The second set of vectors correspond to the quantum numbers of the left-handed spinor representation of $O(16)$. Together, these multiplets comprise exactly the adjoint representation of E_8. Compactifying the 16 right-moving dimensions using two copies of this lattice yields a gauge theory with gauge group $E_8 \times E_8$. Repeating the above construction directly in 16 dimensions yields a second self-dual lattice with $(\vec{e}_a)^2$ even; compactifying with this lattice gives an $O(32)$ gauge group.

The zero-mass, $P = 0$ states of the compactified heterotic string theory are obtained as products

$$\begin{pmatrix} b^{i\dagger}_{\frac{1}{2}} \left|0\right\rangle \\ \\ \left|La\right\rangle \end{pmatrix} \otimes \begin{pmatrix} \overline{a}^{j\dagger}_1 \left|\ell_a = 0\right\rangle \\ \\ \vec{a}^{\dagger}_1 \left|\ell_a = 0\right\rangle \\ \\ \left|\ell_a = \delta_{ab}\right\rangle \end{pmatrix} . \tag{43}$$

The product of the left-moving Neveu-Schwarz vector with the top state on the left gives the graviton multiplet (19). (This is also the bosonic content of $N = 1$ supergravity in 10 dimensions.) The product of the Neveu-Schwarz vector with the states of the other two forms yields a multiplet of vector bosons in the adjoint representation of $E_8 \times E_8$ or $O(32)$. The products involving the Ramond spinor give the supersymmetric partners of these states. We now have a theory with gauge bosons and fermions interacting through a large grand unification gauge group, unified in a most beautiful way with a supersymmetric theory of gravity. The theory even has a natural handedness, which will eventually be translated into the chirality of the electroweak interactions. What more can we ask of a unifying theory of Nature?

Lesson 7: Field Compactification on an Orbifold

What more, indeed! We can hardly consider this grand theory more than a philosophy unless it can give some insight into the outstanding problems of elementary particle physics. We should certainly hope that this theory will have something to say about the origin of the quark and lepton generations and the calculation of quark and lepton masses. Though it is premature to give precise predictions, I believe that the superstring theory has the power to give insight into these questions. In these last two sections, I will try to demonstrate this by discussing some physical consequences of compactification from 10 to 4 dimensions. In this section, we will warm up by compactifying ordinary field theories. In the next section, we will discuss some additional issues which arise when we compactify strings.

Onto what kind of space should we compactify the extra 6 spatial dimensions of the superstring theory? We should properly make this decision by solving the theory, but at the present level of our understanding there seem to be many possibilities. In particular, the possibility that the string prefers 10 extended dimensions has not been ruled out. Assuming, however, that the true solution to the string theory will involve compactification, several approaches have been proposed for choosing a particular form for the compact space. The first of these, due to Candelas, Horowitz, Strominger, and Witten,[16] involves simplifying the problem by assuming that the compact space is larger than the natural length scale set by T_0, deriving the string Einstein equations in this limit (where they reduce essentially to the equations of supergravity), and then looking for solutions. This procedure led to the Calabi-Yau spaces, 6-dimensional spaces with $R_{\mu\nu} = 0$, containing $SU(3)$ gauge fields which trace the curvature. These spaces gave some appealing qualitative features, including chiral fermion generations, but they are unfortunately quite complicated to deal with, since they have no symmetries and only topological properties of these spaces are known explicitly. The second method is to choose the compact space to be a torus. This approach suffers from just the opposite difficulty—it has too much symmetry In particular, neither the supersymmetry nor the $E_8 \times E_8$ gauge symmetry can be broken.

Fortunately, an elegant compromise between these two approaches was discovered by Dixon, Harvey, Vafa, and Witten.[32] These authors recommend compactifying on an *orbifold*, a torus with a further identification of points related by the action of a discrete symmetry.

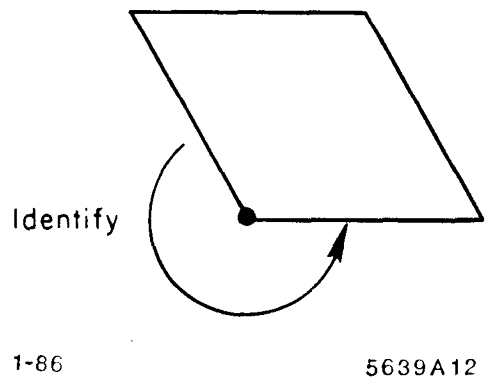

Figure 12. An example of an orbifold, obtained from the torus associated with the $SU(3)$ lattice by identifying points related by 120° rotations.

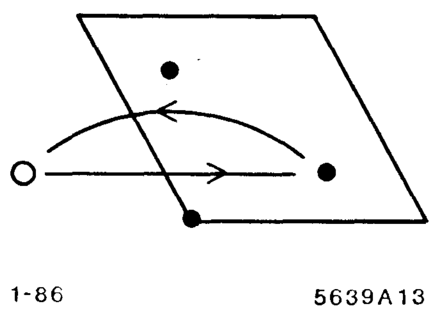

Figure 13. Fixed points of the orbifold shown in Fig. 12.

A simple example of an orbifold is shown in Fig. 12. A good way to visualize the geometry of this orbifold is to identify the points which are fixed under the combined action of the discrete symmetry and the lattice translation. In the orbifold of Fig. 12 there are 3 such points, shown in Fig. 13. Triples of points, related by 120° rotations about the fixed point, are being identified; this process causes the neighborhood of the fixed point to be folded up into a cone, with the fixed point at its apex.

Now imagine that 2 dimensions of space are curled up into the form of the orbifold in Fig. 12. Let us study the components of fields in such a space which would be visible at low energies. We saw in our earlier study that particles which are not massless in the first approximation receive very large masses: of order $(T_0)^{\frac{1}{2}} \sim 10^{19}$ GeV. We will, then look for particles which are left massless after compactification; these particles will then receive GeV-scale masses from $SU(2) \times U(1)$ breaking and supersymmetry breaking effects at the weak scale.

To begin, define the effective mass after compactification in the same way that we did in the discussion of Lesson 4:

$$\tilde{m}^2 = p_{(2)}^2 + m^2 , \tag{44}$$

where $p_{(2)}$ is the momentum in the compactified dimensions, and look for modes of the field for which the effective mass $\tilde{m}$ vanishes. If we start with fields which are massless in the original 10-dimensional space, we can satisfy this condition only for modes for which $p_{(2)}$ vanishes, that is, modes which are constant over the orbifold.

For a scalar field $\phi(x)$, this criterion is easily satisfied: Field configurations which are *constant* over the compactified dimensions will be viewed as massless scalar fields after compactification. For fields with spin, however, some subtleties arise. To explain them, I will introduce the following notation: Let indices in capital letters (M, N) run over the full 10 dimensions, indices in lower-case (m, n) run over compactified dimensions, and indices in greek letters (μ, ν) run over extended, visible dimensions. In this notation, the massless scalar field modes we have just described have the form:

$$\phi(x^M) = \phi(x^\mu) . \tag{45}$$

Now let us try to generalize this condition for massless modes to a vector field $A^M(x^N)$. For the components of A^M which point into the uncompactified directions, the mode which is constant over the orbifold gives a massless vector field after compactification,

$$A^\mu(x) = A^\mu(x^\nu) . \tag{46}$$

However, this observation fails for the components of A^M which point into the compactified dimensions.

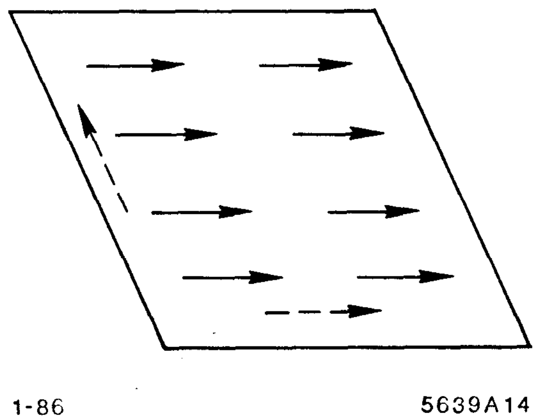

Figure 14. If one identifies the boundaries of this figure to form an orbifold, one must also identify the dotted tangent vectors.

The reason is shown in Fig. 14. When one identifies points related by a 120° rotation, one must also identify the directions of the vectors between these points. An A^m configuration constant on the orbifold is represented in the figure by the solid arrows. These arrows are tangent to the lower boundary but lie at an angle to the boundary on the left. Thus, this mode of A^m does not satisfy the boundary condition which is imposed when we identify these two boundary lines.

It is useful to formulate the boundary condition required by the orbifold a bit more abstractly. A field on the left boundary of Fig. 14 must be rotated by 240° to bring it into coincidence with a field on the lower boundary. If this rotation is implemented by an operator $\mathcal{R}(240°)$, the field will be smooth across the join if it obeys

$$\varphi = \mathcal{R}(240°) \cdot \varphi . \tag{47}$$

To find modes with $\tilde{m}^2 = 0$, we must find constant fields which satisfy this criterion. The explicit form of $\mathcal{R}(240°)$ depends on the spin and spin direction:

for a vector field A^μ:

$$\mathcal{R}(240°) = 1$$

for a vector field A^m:

$$\mathcal{R}(240°) = e^{\pm 4\pi i/3} \qquad \text{on the combinations } A^1 \pm A^2$$

for a spinor field Ψ:

$$\mathcal{R}(240°) = e^{\pm 2\pi i/3} \qquad \text{depending on the chirality}$$

Apparently, in this simple example, eq. (47) can be solved only by scalars and vectors oriented normal to the compact dimensions.

It is possible to obtain a much more interesting result, however, by studying a slightly more complex generalization of this structure. Consider, then, the 6-dimensional torus shown in Fig. 15, consisting of three copies of the $SU(3)$ torus in three orthogonal planes.

We can turn this space into an orbifold (the *Z- orbifold* of ref. 32) by identifying points related by simultaneous 120° rotations in the three planes. In addition, let us introduce $SU(3)$ gauge fields into the model, and allow a quantum of magnetic flux to point upward through one of the corners. The boundary condition

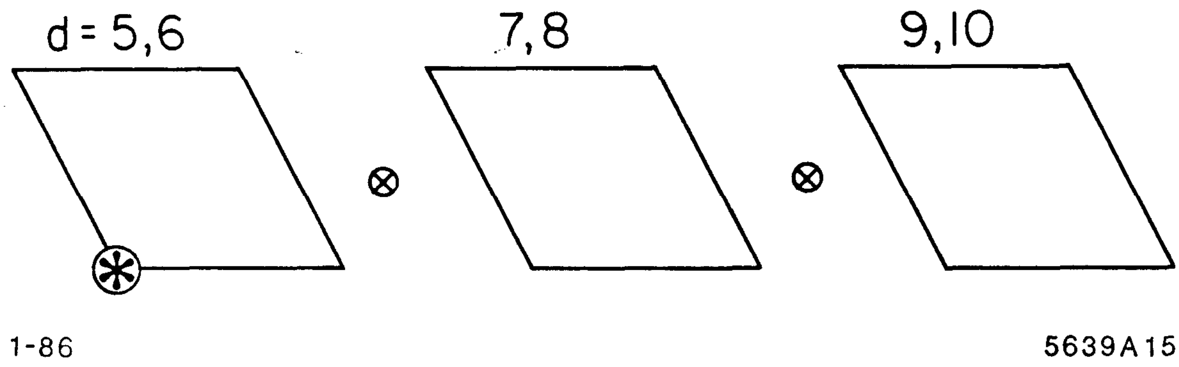

Figure 15. The 6-dimensional torus used to form the Z-orbifold.

for a field compactified on this space reads:

$$\varphi \;=\; [\mathcal{R}(240^\circ)]_{5,6}\cdot[\mathcal{R}(240^\circ)]_{7,8}\cdot[\mathcal{R}(240^\circ)]_{9,10}\cdot\mathcal{G}\;\cdot\;\varphi\;, \tag{48}$$

where the three rotation operators implement the rotations in the three orthogonal planes and $\mathcal{G}$ is the Bohm-Aharonov phase associated with the magnetic flux

$$\mathcal{G} \;=\; e^{ig\oint dx\cdot A}\;. \tag{49}$$

Let us assume, further, that $\mathcal{G}$ is quantized so that, for the fundamental representations of $SU(3)$,

$$\mathcal{G} \;=\; e^{2\pi i/3} \quad \text{on the } \mathbf{3} \qquad \mathcal{G} \;=\; e^{-2\pi i/3} \quad \text{on the } \overline{\mathbf{3}}\;. \tag{50}$$

On representations in the product of **3** and $\overline{\mathbf{3}}$, or generally on representations of zero triality, $\mathcal{G} = 1$. This association of a threefold gauge group element with a threefold rotation axis realizes the proposal of Candelas, Horowitz, Strominger, and Witten that gauge fields should trace the curvature of the compactification space.

Combining all the factors in eq. (48), we can find a myriad of nontrivial solutions to this boundary condition, depending on the spin and $SU(3)$ representation of the field. Some examples are:

$$
\begin{array}{rcccccccccc}
A^\mu \textit{ in the } \mathbf{1} \text{ or } \mathbf{8}: & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & = & 1 \\
A^m \textit{ in the } \mathbf{3}: & e^{4\pi i/3} & \cdot & 1 & \cdot & 1 & \cdot & e^{2\pi i/3} & = & 1 \\
\Psi \textit{ in the } \mathbf{1} \text{ or } \mathbf{8}: & e^{2\pi i/3} & \cdot & e^{2\pi i/3} & \cdot & e^{2\pi i/3} & \cdot & 1 & = & 1 \\
\Psi \textit{ in the } \mathbf{3}: & e^{2\pi i/3} & \cdot & e^{-2\pi i/3} & \cdot & e^{-2\pi i/3} & \cdot & e^{2\pi i/3} & = & 1
\end{array} \tag{51}
$$

The second and fourth lines show one of three possible solutions; the others are obtained by permuting the first three elements of the product. The solution for A^m appears as a scalar in the uncompactified dimensions. In compactification of a 10-dimensional chiral fermion, the observed 4-dimensional chirality equals the product of the chiralities evident in (51). Both of the solutions shown, then, would be observed as positive-chirality spinors in 4 dimensions. Note that, for a Ψ in the **3**, there is no solution which gives a negative-chirality spinor in 4-dimensions. Thus, this orbifold is capable of producing a spectrum of light chiral fermions resulting from compactification.

Let us now apply this construction specifically to the massless sector of the $E_8 \times E_8$ heterotic string theory. The content of this sector is a multiplet of gauge bosons A^M and gauginos Ψ in the **248**-dimensional adjoint representation of each E_8 group. E_8 has a maximal subgroup $E_6 \times SU(3)$, and under this subgroup the **248** transforms as

$$
\mathbf{248} \rightarrow (\mathbf{78}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) + (\mathbf{1}, \mathbf{8}) . \tag{52}
$$

I should remind you that E_6 has often been proposed as a grand unification group, since it contains $SU(5)$ and $O(10)$ as natural subgroups, and that, with this identification, one **27** of E_6 contains 1 generation of quarks and leptons.

From (51), we can see that this set of fields $A^M + \Psi$ transforming according to (52) yields the following set of massless particles: a 4-dimensional vector and chiral fermion in the $(\mathbf{78}, \mathbf{1})$, which form the gauge bosons and gauginos of E_6; a 4-dimensional vector and chiral fermion in the $(\mathbf{1}, \mathbf{8})$, which form the gauge bosons and gauginos of an extra $SU(3)$; three 4-dimensional scalars and three 4-dimensional chiral fermions in the $(\mathbf{27}, \mathbf{3})$; and their antiparticles in the $(\overline{\mathbf{27}}, \overline{\mathbf{3}})$. This exercise has been a bit complex, but the result is worth it: From a simply visualized compactification geometry, we have seen quark and lepton generations—of definite chirality—emerge in a natural way as part of an effective supersymmetric grand unified theory. We have, of course, found too many generations (and more will appear in the next section), but this problem is less severe for other choices of the 6-dimensional geometry.[33]

Lesson 8: String Compactification on a Orbifold

The generalization of the argument just given to string compactification introduces some further complications, which I would like to discuss only briefly. Closed strings can actually wrap around the compactified geometry, giving rise to new configurations which appear as light particles in 4 uncompactified dimensions.

Fig. 16 illustrates the various possibilities. Fig. 16(a) shows a string in a trivial configuration; all of the solutions described in the previous section correspond to this situation. Fig. 16(b) shows a string which winds around the torus. These states are analogous to the winding states discussed in Lesson 4, though for the Z-orbifold one finds no new massless particles in this way. Finally, Fig. 16(c) shows a new configuration: a string runs around a fixed point from one point of the torus to a second point identified with the first under the discrete symmetry. This configuration is actually a closed loop on the orbifold. Strings in such a configuration are said to form a twisted sector.

The twisted sectors of the string theory can contain additional massless states.

(a)

(b)

(c)

1-86

5639A16

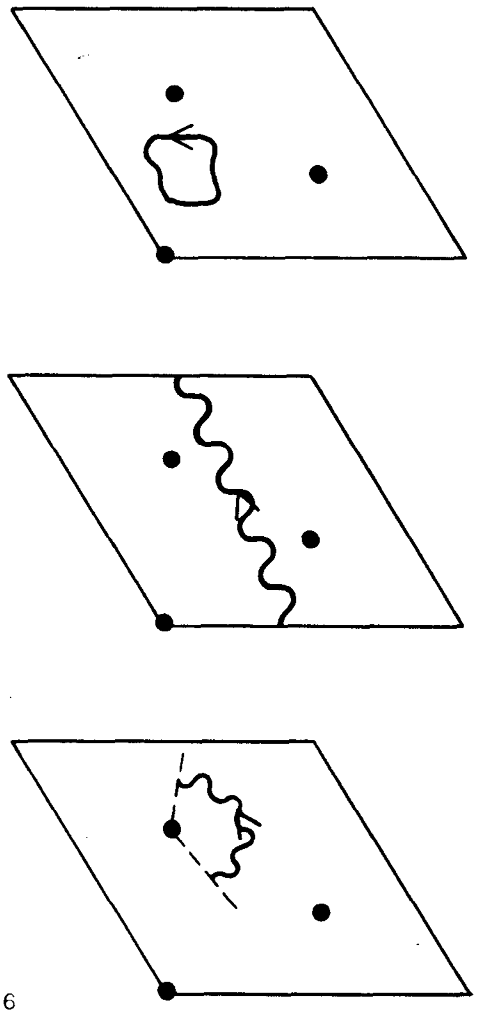

Figure 16. Three possible dispositions for a closed string on an orbifold: (a) trivial; (b) winding; (c) twisted.

To understand how this can occur, let me sketch the computation of the zero-point energy for the right-moving fields of the heterotic string in such a geometry. The nontrivial boundary conditions lead to a shift in the quantization of the oscillators in compactified directions by $\alpha = \frac{1}{3}$. The flux quantum (50) is implemented by insisting that the two identified ends of the string be shifted from one another on the E_8 lattice, so that a fixed point of the orbifold has associated with it a dislocation in the lattice of the 16 purely right-moving dimensions. For a string in the $(\mathbf{27},\mathbf{1})$ of $E_6 \times SU(3)$, the total zero-point energy is:

$$6 \cdot \mathcal{Z}(\frac{1}{3}) \ + \ 18 \cdot \mathcal{Z}(0) \ + \ \frac{2}{3} \ ; \tag{53}$$

the last term is the energy cost of the shift on the E_8 lattice. The three terms in (53) sum to 0, so we find one **27** of fermions and one **27** of scalars in the twisted sector about each fixed point. This gives a second mechanism for producing chiral fermion generations, and their supersymmetric partners.

Some of the scalars we have uncovered will become the Higgs bosons of the electroweak theory which is derived from the string theory. The couplings of these scalars to pairs of fermions thus provide the Yukawa couplings which are responsible for the quark and lepton masses. Since both the fermions and the Higgs bosons may be visualized as string configurations on the orbifold, the couplings of these particles may be evaluated by considering transitions from one string configuration to another. In some simple models, these calculations have been carried out explicitly.[34,35] One remarkable feature of which has been uncovered in that analysis is shown in Fig. 17. It is simplest to think of the Yukawa coupling as the amplitude for two fermion strings and one Higgs boson string to combine and disappear into the vacuum, as illustrated in Fig. 17(a). Figs. 17(b) and (c) look inside the vertex for two different cases: If the three strings being coupled belong to the same twisted sector, the strings can combine and annihilate by the relatively simple process shown in Fig. 17(b). This produces a sizable Yukawa coupling: $\lambda \sim 1$. However, if the three strings belong to three different twisted sectors, their annihilation requires a nontrivial physical process on the world-surface, shown in Fig. 17(c). Essentially, one string must tunnel quantum-mechanically across the compact torus. For this case, the annihilation amplitude contains a barrier penetration factor and so is suppressed by

$$\lambda \ \sim \ e^{-\alpha T_0 R^2} \ , \tag{54}$$

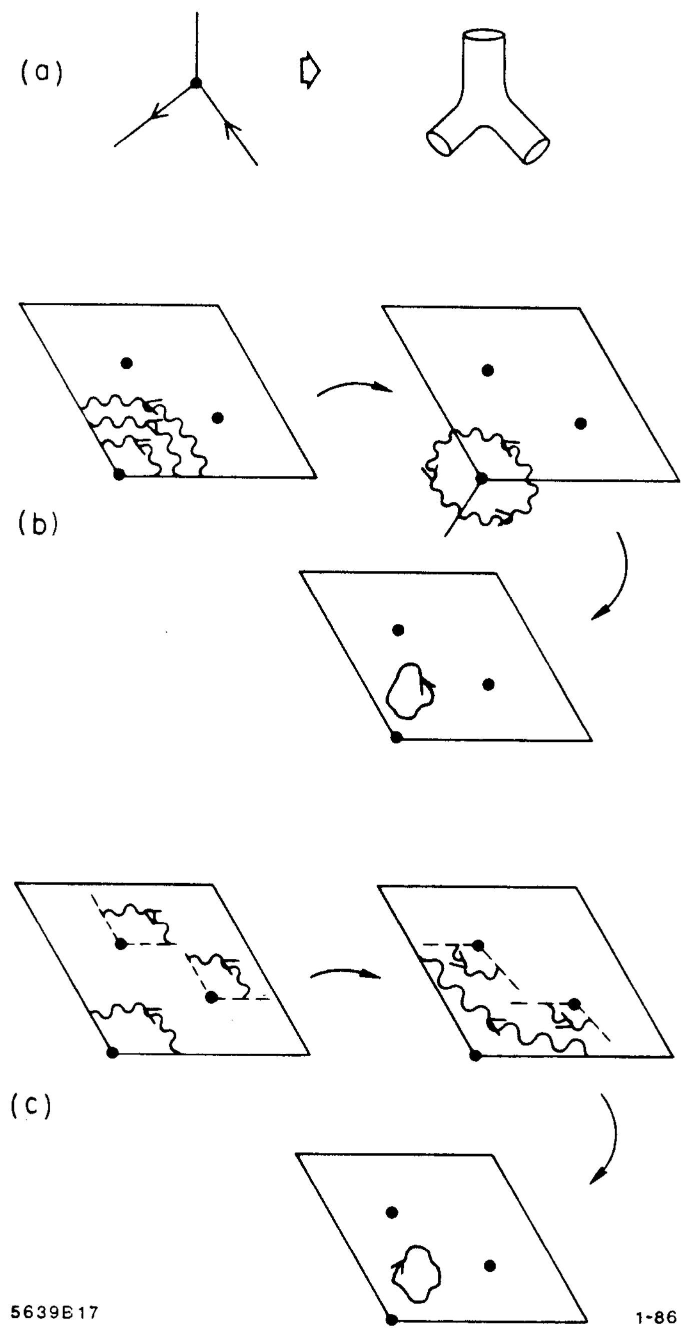

Figure 17. Illustration of the physical processes which determine the fermion Yukawa couplings.

where R is the physical size of the compactified dimensions. Thus, the hierarchical pattern of Yukawa couplings observed in Nature may have an intuitive physical origin, even if this physical picture must be visualized at distances very close to the Planck scale.

Conclusion

In this course of lessons, I have tried to explain how string theory builds up all of the types of particles which appear in physics—quarks, leptons, gauge bosons, gravitons, and others—from the same basic elements. These elements, the fundamental strings, have a dynamics which one can grasp and visualize, even if some of its mathematical features appear magical. The theory gives a geometrical unification of all known interactions. But it also gives a concrete picture of what is happening behind the unification, a picture of the internal structure of quarks and leptons and their interactions.

Critics of string theory often complain that the theory is predictive only for quantities observable at the Planck scale. I have tried to argue here that this is an overly pessimistic view. Because string theories treat quarks and leptons as dynamical entities, they allow explicit calculations of the quark and lepton Yukawa couplings. In ordinary field theory, Yukawa couplings normally cannot be computed as a matter of principle, and those models with sufficient structure to allow such computations often require a complex array of new interactions and undefined parameters. But string theory gives the promise of making definite predictions about the structure of the fermion mass spectrum, inviting a direct and nontrivial confrontation with experiment. This is, of course, an extravagant promise, but it seems to me not an unrealistic one. We will soon see whether it can be fulfilled.

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