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NON-LINEAR σ-MODELS AND STRING THEORIES*

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ABSTRACT

Various applications of nonlinear σ-models to string theories are discussed.

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1. INTRODUCTION

In these lectures I shall discuss the connection between σ–models and string theories, and show how the σ–models can be used as important tools to prove various results in string theories.[1–21] Section 2 contains a very brief introduction to closed bosonic string theory in the light cone gauge. In sec. 3 I discuss closed bosonic string theory in the presence of massless background fields. I use the light-cone gauge and show that in order to obtain a Lorentz invariant theory, the string theory in the presence of background fields must be described by a two-dimensional conformally invariant theory. This in turn gives some constraints on the background fields which turn out to be the equations of motion of the string theory. Section 4 contains the extension of this analysis to the case of the heterotic string theory and the superstring theory in the presence of the massless background fields. In sec. 5 I show how to use these results to obtain nontrivial solutions to the string field equations. Section 6 contains another application of these results, namely to prove that the effective cosmological constant after compactification vanishes as a consequence of the classical equations of motion of the string theory.

2. CLOSED BOSONIC STRING IN THE LIGHT-CONE GAUGE

Free bosonic string in the light-cone gauge is described by 24 bosonic variables $X^i(\sigma,\tau)$ (i=1,...24) where σ, τ are the two-dimensional variables describing the string world-sheet.[22] Besides these there are other independent variables $x^{\pm}(\tau)$, which are functions of τ only. (The light-cone gauge constraints and the Virasoro constraints determine the σ–dependence of the variables $X^{\pm}$ in terms of the fields X^i.) The action for the string is given by

$$S = \frac{1}{2\alpha'}\left[\int d\tau x^+ x^- + \frac{1}{2\pi}\int d\tau \int_0^{\pi} d\sigma \partial_\alpha X^i \partial^\alpha X^i\right] \quad , \tag{2.1}$$

where ξ^α (α=0,1) denote the string coordinates σ and τ, respectively. From now on we shall set the inverse string tension α' to be 1/2, unless it is displayed explicitly. The various fields have the mode expansion

$$X^i(\sigma,\tau) = x_0^i + p^i\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^i e^{-2in(\tau+\sigma)} + \tilde{\alpha}_n^i e^{-2in(\tau-\sigma)}\right] \quad ,$$

$$x^{\pm}(\tau) = x_0^{\pm} + p^{\pm}\tau \quad , \tag{2.2}$$

with the standard commutation relations between the oscillators

$$[x^i, p^j] = i\delta_{ij} \quad ,$$

$$[\alpha_m^i, \alpha_n^j] = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta_{m,-n}\delta^{ij} \quad m \neq 0$$

$$[x^+, p^-] = [x^-, p^+] = i \quad , \tag{2.3}$$

all other commutators being zero.

Not all the states described by the action (eq. 2.1) are physical states. The physical states are defined by the constraints

$$p^+p^- - 4(L_0 - 1) = p^+p^- - 4(\tilde{L}_0 - 1) = 0 \quad , \tag{2.4}$$

where

$$\begin{aligned} L_0 &= \sum_{m>0} \alpha^i_{-m}\alpha^i_m + \frac{1}{8}p^i p^i \quad , \\ \tilde{L}_0 &= \sum_{m>0} \tilde{\alpha}^i_{-m}\tilde{\alpha}^i_m + \frac{1}{8}p^i p^i \quad . \end{aligned} \tag{2.5}$$

Generally, we define the operators

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m :\alpha^i_{n-m}\alpha^i_n: \quad , \\ \tilde{L}_n &= \frac{1}{2}\sum_m :\tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m: \quad , \end{aligned} \tag{2.6}$$

where in the above equation we must interpret α^i_0, $\tilde{\alpha}^i_0$ as

$$\alpha^i_0 = \tilde{\alpha}^i_0 = \frac{1}{2}p^i \quad . \tag{2.7}$$

Using the commutation relations (eq. 2.3) we may derive the commutation relations among the $L_m's$. They are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{2}(m^3 - m)\delta_{m,-n} \quad , \tag{2.8}$$

where

$$c = 4 \quad . \tag{2.9}$$

Thus, the L_ms satisfy a Virasoro algebra with central charge c=12. $\tilde{L}_m$s satisfy a similar commutation relation.

The action (eq. 2.1) has manifest SO(24) Lorentz invariance, which generates rotation among the 24 X^is. However, in order to get a sensible theory, we need full SO(25,1) Lorentz invariance. This is ensured by constructing the full set of SO(25,1) Lorentz generators in the theory in terms of the variables X^i, and verifying that they satisfy the correct commutation relations. These generators are given by

$$
\begin{aligned}
J^{ij} &= x^i p^j - x^j p^i - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n + \tilde{\alpha}^i_{-n} \tilde{\alpha}^j_n - \tilde{\alpha}^j_{-n} \tilde{\alpha}^i_n) \quad , \\
J^{i+} &= x^i p^+ - x^+ p^i \\
J^{+-} &= x^+ p^- - x^- p^+ \\
J^{i-} &= x^i p^- - x^- p^i - i (p^+)^{-1} \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^i_{-n} L_n - L_{-n} \alpha^i_n + \tilde{\alpha}^i_{-n} \tilde{L}_n - \tilde{L}_{-n} \tilde{\alpha}^i_n) \quad .
\end{aligned}
\tag{2.10}
$$

Using the commutation relations (eqs. 2.3 and 2.8) and the physical state condition (eq. 2.4), one can show that the Lorentz algebra closes on-shell. This shows that the theory has full SO(25,1) Lorentz invariance, although only the SO(24) subgroup of the full Lorentz group in manifest.

The spectrum of the theory described by the action (eq. 2.1) contains a whole tower of massive states, as well as a few massless states. For future discussions, we shall list here the massless states of the theory. They contain a symmetric rank 2 tensor, an antisymmetric rank 2 tensor, and a scalar, which we shall identify with the states created by a graviton field $G_{ij}(x)$, an antisymmetric tensor field $B_{ij}(x)$, and a dilaton field $\Phi(x)$, respectively. In the next section we shall show how to describe the propagation of the string in a background where the massless fields develop vacuum expectation values.

3. BOSONIC STRING IN MASSLESS BACKGROUND FIELDS

In this section we shall describe the propagation of the closed bosonic string in massless background fields.[1-12] The discussion will be carried out completely in the first quantized formulation. The situation is analogous to the motion of a first quantized Dirac particle in background classical electromagnetic field, where the background field is treated as a purely classical field, only the Dirac particle is treated quantum mechanically. However, unlike the point particle case, the propagation of a string in a given background field can be described consistently only if the background satisfies certain constraints. These constraints turn out to be equivalent to the dynamical equations of motion of the background fields, as we shall see shortly.

The first step in our study is to generalize the action (eq. 2.1) in the presence of background fields. In order to do this we again turn to the example of the motion of a point particle in background fields. The hamiltonian derived from the action in this case has two parts, the free part, and the interaction part. Of this the interaction part has the property that if we replace the classical background by a plane wave, then the interaction hamiltonian reduces to the vertex operator for the emission of a photon from a Dirac particle. (The vertex operator V for a state γ of the photon is defined to be the operator such that $< m \mid V \mid n >$ gives the amplitude $m \rightarrow n + \gamma$, where m, n are any two first quantized states of the electron.) In the case of a string theory, the vertex operators for various massless states of the string are well-known.[22] The action of the string in massless background fields must satisfy the constraint that the interaction hamiltonian derived from this action reduces to the vertex operator upon replacing the background fields by plane waves. An action which satisfy these relations is

$$S = \frac{1}{2\alpha'} \left\{ \int d\tau x^+ x^- + \frac{1}{2\pi} \int d\tau \int_0^\pi d\sigma \left[G_{ij}(X) \partial_\alpha X^i \partial^\alpha X^j + B_{ij}(X) \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \right] \right\} \quad , \qquad (3.1)$$

where $G_{ij}(X)$ and $B_{ij}(X)$ are background gravitational and antisymmetric tensor fields, respectively. Here we have assumed that both G_{ij} and B_{ij} have nontrivial components in transverse directions only, and depends only on the transverse coordinates. Also we have set the background dilaton field *vev* to be zero. It turns out that the dilaton couples via a more complicated mechanism, which we shall discuss at the end of the section.

We shall now study the theory described by the action (eq. 3.1). Let us study a case where $G_{ij}(X)$ and $B_{ij}(X)$ acquire *vev* only along d of the 24 directions, and have nontrivial dependence only on these d directions (d$\leq$23). We may take these extra directions to be compact, but this is not necessary for our discussion. If we denote these directions by X^p (p=1,...d), and the free directions by X^M (M=d+1,...24), the action (eq. 3.1) may be split into two parts, S_0 and S_1, as follows,

$$S_0 = \frac{1}{2\alpha'} \left[\int d\tau x^+ x^- + \frac{1}{2\pi} \int d\tau \int_0^\pi d\sigma \partial_\alpha X^M \partial^\alpha X^M \right] \quad , \qquad (3.2)$$

$$S_1 = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left[G_{pq}(X) \partial_\alpha X^p \partial^\alpha X^q + B_{pq}(X) \epsilon^{\alpha\beta} \partial_\alpha X^p \partial_\beta X^q \right] \quad , \qquad (3.3)$$

where sum over repeated indices is understood. Only SO(24–d) Lorentz symmetry is manifest in this formalism, which consists of rotation of the (24–d) X^M's.

However, in order to get a consistent theory, we must get full SO(25–d,1) Lorentz invariance. (The rest of the Lorentz group is broken spontaneously by the *vev* of the background G_{pq} and B_{pq} fields.) Hence we must be able to construct the full set of Lorentz generators, and verify that they satisfy the correct commutation relations, as in sec. 2. It turns out that it is possible to do this if the nonlinear σ-model described by the action (eq. 3.3) describes a conformally invariant two-dimensional field theory with the central charge of the Virasoro algebra equal to $d/6$. To see how this can be done, note that the conformal invariance of the σ–model guarantees the existence of a set of Virasoro generators $L_n^{(1)}$, $\tilde{L}_n^{(1)}$ with the commutation relations:

$$\begin{aligned}
\left[\tilde{L}_m^{(1)}, \tilde{L}_n^{(1)}\right] &= (m-n)\tilde{L}_{m+n}^{(1)} + \frac{c^{(1)}}{2}(m^3-m)\delta_{m,-n} \quad , \\
\left[L_m^{(1)}, L_n^{(1)}\right] &= (m-n)L_{m+n}^{(1)} + \frac{c^{(1)}}{2}(m^3-m)\delta_{m,-n} \quad , \\
\left[L_m^{(1)}, \tilde{L}_n^{(1)}\right] &= 0 \quad ,
\end{aligned} \tag{3.4}$$

where $c^{(1)}$ is the central charge of the Virasoro algebra, which must take the value $d/6$ to make our construction work.

The theory described by the action S_0 is a free field theory and hence is also a conformally invariant theory. Its conformal generators $L_n^{(0)}$, $\tilde{L}_n^{(0)}$ satisfy commutation relations identical to those given by eq. 3.4 with $c^{(1)}$ replaced by $c^{(0)}$. $L_n^{(0)}$, $\tilde{L}_n^{(0)}$ can be explicitly constructed as in eqs. 2.5 and 2.6, with the sum over i replaced by sum over M. From this we may also calculate $c^{(0)}$ explicitly, which turn out to be $(24-d)/6$.

We may now write down the generators of the SO(25-d,1) Lorentz group in terms of the generators $L_m^{(1)}$, $\tilde{L}_m^{(1)}$, $L_m^{(0)}$, $\tilde{L}_m^{(0)}$, and the free oscillators α_m^M. They are

$$J^{MN} = x^M p^N - x^N p^M - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^M \alpha_n^N - \alpha_{-n}^N \alpha_n^M + \tilde{\alpha}_{-n}^M \tilde{\alpha}_n^N - \tilde{\alpha}_{-n}^N \tilde{\alpha}_n^M) \quad ,$$

$$J^{M+} = x^M p^+ - x^+ p^M \quad ,$$

$$J^{M-} = x^M p^- - x^- p^M - i(p^+)^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \left[\alpha_{-n}^M (L_n^{(0)} + L_n^{(1)}) - (L_{-n}^{(0)} + L_{-n}^{(1)} \right] \alpha_n^M$$

$$+ \tilde{\alpha}_{-n}^M \left[\tilde{L}_n^{(0)} + \tilde{L}_n^{(1)}) - (\tilde{L}_{-n}^{(0)} + \tilde{L}_{-n}^{(1)}) \tilde{\alpha}_n^M \right] \quad ,$$

$$J^{+-} = x^+ p^- - x^- p^+ \quad . \tag{3.5}$$

Notice that the Lorentz generators involve the free field oscillators x^M, p^M, α_n^M and $\tilde{\alpha}_n^M$ explicitly, but the oscillators of the interacting fields X^p appear only through the Virasoro generators $L^{(1)}$, $\tilde{L}^{(1)}$. As a result, we can calculate the commutators of the Lorentz generators knowing only the free field commutators, and the commutation relations (eq. 3.4). The algebra may be shown to close acting on the physical states satisfying

$$p^+ p^- = 4 \left(L_0^{(0)} + L_0^{(1)} - 1 \right) = 4 \left(\tilde{L}_0^{(0)} + \tilde{L}_0^{(1)} - 1 \right) \quad . \tag{3.6}$$

Thus, we see that the conformal invariance of the theory given by the action (eq. 3.3) is a sufficient condition for guaranteeing the Lorentz invariance of the corresponding string theory. Although classically the action (eq. 3.3) is invariant under the conformal transformation

$$X^i(\xi^+, \xi^-) \to X^i(f^+(\xi^+), f^-(\xi^-)) \quad , \tag{3.7}$$

where f^+ and f^- are arbitrary functions, this symmetry is generally broken in the quantum theory. If we demand that this theory remains a symmetry of the full quantum theory, we get certain set of constraints on the background fields. These are the constraints we shall study now.

Let us first continue the time τ to the imaginary axis in the action (eq. 3.3), and define complex coordinates

$$z = e^{2(\tau+i\sigma)} \quad , \qquad \bar{z} = e^{2(\tau-i\sigma)} \quad . \tag{3.8}$$

Equation 3.3 may then be written as

$$S_1 = \frac{1}{\pi}\int dz d\bar{z}\Big[G_{mn}(X) + B_{mn}(X)\Big]\partial_z X^m \partial_{\bar{z}} X^n \quad . \tag{3.9}$$

There are various ways for checking the conformal invariance of the action (eq. 3.9). We shall mention each of them briefly.

a) Conformal invariance of a theory demands vanishing of all the β–functions of the theory. Equation 3.9 is the most general renormalizable action in two dimensions constructed out of the fields X^m. $G_{mn}(X)$ and $B_{mn}(X)$ are the coupling constants of the theory. Ultraviolet divergences of the theory will renormalize the coupling constants of the theory, from which we may define the β–functions β^G_{ij} and β^B_{ij}. In order to get a conformally invariant theory, both β^G_{ij} and β^B_{ij} must vanish, which, in turn, gives us some constraints on $G_{ij}(X)$ and $B_{ij}(X)$. The constraint $c^{(1)} = d/2$ cannot be implemented directly in this method.

b) Equation 3.9 gives us the following expression for the energy-momentum tensor:

$$T_{\alpha\beta} = G_{mn}\left[\partial_\alpha X^m \partial_\beta X^n - \frac{1}{2}\delta_{\alpha\beta}\partial_\gamma X^m \partial^\gamma X^n\right] \quad , \tag{3.10}$$

where α, β, γ denote z or $\bar{z}$. Conformal invariance of the theory tells us that the energy-momentum tensor must be traceless, *i.e.*, $T^\alpha_\alpha \equiv T_{z\bar{z}} = 0$. This condition is satisfied by the classical expression (eq. 3.10). However, because of ultraviolet divergences, we must regularize our theory, which, in general, spoils the tracelessness of the energy-momentum tensor. If, for example, we use dimensional regularization, working in the (2-ϵ) dimensions, T^α_α computed from eq. 3.10 will be proportional to $\epsilon/2G_{mn}\partial_\alpha X^m \partial^\alpha X^n$. The explicit power of ϵ may be cancelled by the $\frac{1}{\epsilon}$ poles coming from ultraviolet divergences, and give us a finite answer. The theory will be conformally invariant only when these terms vanish, which, in turn, will give us some constraints on the background fields.

As it stands, this prescription also does not tell us anything about the central charge of the theory. However, a generalization of this method may be used to compute the central charge. In this scheme we couple the σ–model to a background two-dimensional gravitational field. As a result, the trace anomaly now contains a new term proportional to the two-dimensional curvature. It can be shown that the coefficient of this term is precisely the central charge of the theory. Constraining this central charge to have the value $d/2$ gives further constraints on the background fields.

c) We may try to directly verify the Virasoro algebra (eq. 3.4). The $L_m^{(1)}$, $\tilde{L}_m^{(1)}$ are given by:

$$
\begin{aligned}
L_m^{(1)} &= \oint z^{m+1} T_{zz} \quad , \\
\tilde{L}_m^{(1)} &= \oint \bar{z}^{m+1} T_{\bar{z}\bar{z}} \quad ,
\end{aligned}
\tag{3.11}
$$

where $\oint$ denotes integration along a contour around the origin. It can be shown that verifying the algebra (eq. 3.4) is equivalent to verifying the following operator product relation

$$
T_{zz}(z)T_{zz}(w) = \frac{c^{(1)}}{(z-w)^4} + \frac{2T_{zz}(w)}{(z-w)^2} + \frac{\partial_w T_{zz}(w)}{(z-w)} + \text{finite terms} \quad , \tag{3.12}
$$

and a similar operator product relation involving $T_{\bar{z}\bar{z}}$. These operator products may be computed explicitly in the perturbation theory. Generally, the operator product will contain terms in addition to those on the right-hand side of eq. 3.12. Demanding that these terms should vanish gives us some constraints on the background fields. Furthermore, we can also calculate $c^{(1)}$ directly from this operator product, and get an extra constraint on the background fields by setting $c^{(1)}$ to $d/2$.

I shall not give the details of the calculation here, but only state the final result. The constraint equations on various fields to lowest order in: α' turn out to be

$$R_{pq} + S_{prs}S_q{}^{rs} = 0 \quad ,$$

$$D^r S_{pqr} = 0 \quad ,$$

$$R + \frac{1}{3}S_{pqr}S^{pqr} = 0 \quad , \tag{3.13}$$

where

$$S_{pqr} = \frac{1}{2}(\partial_p B_{qr} + \partial_q B_{rp} + \partial_r B_{pq}) \quad . \tag{3.14}$$

Let us now turn our attention to a somewhat different aspect of the string theory. As we have already mentioned, the massless states of the string theory may be described by states created by the fields $G_{ij}(x)$, $B_{ij}(x)$ and $\Phi(x)$. The three point functions, as well as the scattering amplitudes involving various massless states may be calculated using string perturbation theory. We may then try to write down an effective action involving these massless fields which reproduce the same scattering amplitude as calculated from the string theory. This action will be nonrenormalizable, but that will not bother us since we shall restrict our attention to the tree level of the string theory. The effective action will involve terms with arbitrary number of derivatives and arbitrary number of fields, each extra power of mass coming with the derivatives or the fields being compensated by a power of $\alpha'^{1/2}$. Thus, in the low energy limit only the terms containing lowest power of α' will be important. For closed-oriented bosonic string the action is given by

$$S_{eff} = \int \sqrt{g}\, d^{26}x\, e^{-2\Phi}\left[R + \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho} + 4(D\Phi)^2 - 4D^2\Phi\right] \quad , \tag{3.15}$$

where μ runs from 0 to 25. If we now derive the equations of motion of various fields from this effective action, and restrict ourselves to the background where

$G_{\mu\nu}$ and $B_{\mu\nu}$ takes *vev* only along the d transverse directions and depend only on these directions, and the dilaton field Φ is zero, these equations become identical to eq. 3.13. This correspondence has been shown to be true beyond the lowest order in α'. $O(\alpha')$ corrections to the effective action (eq. 3.15), coming from higher derivative terms seem to agree with the $O(\alpha')$ corrections to eq. 3.13 coming from higher loop corrections in the σ–model.

Finally, we shall indicate how we can couple background dilaton field to the string and recover the equations of motion derived from S_{eff} by demanding conformal invariance of the σ–model. It can be done in all the three schemes for studying conformal invariance of the σ–model, but we shall restrict ourselves to the scheme (c) here. The energy-momentum tensor given in eq. 3.10 is the one calculated from the σ-model action (eq. 3.9) by Noether prescription. However, we may define a new energy-momentum tensor

$$\hat{T}_{\alpha\beta} = T_{\alpha\beta} + \frac{\alpha'}{2\pi}(\partial_\alpha\partial_\beta - \delta_{\alpha\beta}\partial^2)\Phi(X) \quad , \tag{3.16}$$

where $\Phi(X)$ is an arbitrary scalar function of X. The extra term that we have added is conserved ($\partial^\alpha\hat{T}_{\alpha\beta} = \partial^\alpha T_{\alpha\beta} = 0$) and does not contribute to the total energy or the total momentum ($\int d\xi^1\hat{T}_{\alpha 0} = \int d\xi^1 T_{\alpha 0}$). Hence $\hat{T}$ is as good a definition of the stress tensor as T. $\Phi(X)$ may then be interpreted as a new coupling constant in the theory, which does not appear in the Lagrangian, but appears in the expression for the stress tensor. If we now calculate the operator product of $\hat{T}_{zz}(z)$ with $\hat{T}_{zz}(w)$, the terms involving Φ will also contribute to the singular parts of this operator product, and change the equations (eq. 3.13). The new equations to the lowest order in α' are

$$R_{pq} + S_{prs}S_q{}^{rs} - 2D_pD_q\Phi = 0 \quad ,$$

$$D^r S_{pqr} - 2S_{pqr}D^r\Phi = 0 \quad ,$$

$$R + \frac{1}{3}S_{pqr}S^{pqr} - 4D^2\Phi + 4(D\Phi)^2 = 0 \quad . \tag{3.17}$$

These are precisely the equations derived from the effective action (eq. 3.15). Thus we see that there is a one-to-one correspondence between the equations of motion in the string theory, and conformal invariance of the two-dimensional σ–models. As we shall see in the next section, this correspondence also holds for the superstring and the heterotic string theories, and can be used to obtain nontrivial solutions to the string field equations.

4. THE HETEROTIC STRING AND THE SUPERSTRING IN ARBITRARY BACKGROUND FIELDS

In this section I shall extend the analysis of the previous section to the heterotic string theory. The extension to the superstring theory is straightforward, and so I shall only mention it briefly at the end of the section. The free heterotic string in the light-cone gauge is described by the usual bosonic coordinates $x^{\pm}(\tau)$, $X^i(\sigma,\tau)$ (i=1,...8), as well as eight right-handed Majorana–Weyl fermions λ^i and 32 left-handed Majorana–Weyl fermions ψ^s. The free string action is given by [23]

$$S_0 = \frac{1}{2\alpha'}\Big[\int d\tau x^+ x^- + \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma[\partial_\alpha X^i \partial^\alpha X^i$$

$$+ i\lambda^i\partial_-\lambda^i + i\psi^s\partial_+\psi^s\Big] \quad , \tag{4.1}$$

where $\xi^{\pm} = \frac{1}{\sqrt{2}}(\xi^0 \pm \xi^1)$ are the light-cone coordinates on the world-sheet, and $\partial_{\pm}$ denote derivatives with respect to $\xi^{\pm}$. The fermion fields λ^i and ψ^s can have either periodic and anti-periodic boundary conditions under $\sigma \to \sigma + \pi$. The massless bosonic states in this theory consists of a symmetric tensor, an antisymmetric tensor, a scalar and 496 vectors. We shall associate them with the fields $G_{ij}(x)$, $B_{ij}(x)$, $\Phi(x)$, and $A_i^M(x)$, respectively. $A_i^M(x)$ are 496 gauge fields, the gauge group being either $E_8 \times E_8$ or SO(32), depending on the choice of possible boundary conditions on the left-handed fermions ψ^s. For the heterotic string theory with SO(32) gauge group, the fields ψ^s transform in the fundamental (32) representation of the gauge group. For the $E_8 \times E_8$ theory, only an SO(16)×SO(16) subgroup of the gauge group is realized linearly, and the fermions belong to the (16,1)+(1,16) representation of this group. While studying this theory in arbitrary background fields, we shall restrict the background gauge field for the $E_8 \times E_8$ heterotic string to lie in the SO(16)×SO(16) subgroup.

The action for the heterotic string in arbitrary background fields $G_{ij}(X)$, $B_{ij}(X)$ and $A_i^M(X)$ may again be written down by studying the vertex operators for various massless fields in the theory. The result is

$$S = \frac{1}{2\alpha'}\left\{\int d\tau x^+ x^- + \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma \Big[G_{ij}(X)\partial_\alpha X^i \partial^\alpha X^j\right.$$

$$+ \epsilon^{\alpha\beta} B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j + iG_{ij}(X)\{\lambda^i \partial_- \lambda^j + \lambda^i(\Gamma_k{}^j{}_\ell(X) - S_k{}^j{}_\ell(X))\lambda^\ell \partial_- X^k\}$$

$$\left. + \psi^s(i\partial_+ \psi^s + A_i^M(X)(T^M)_{st}\psi^t \partial_+ X^i) + \frac{i}{2} F_{ij}^M(X)\psi^s (T^M)_{st}\psi^t \lambda^i \lambda^j \Big]\right\} \quad , \quad (4.2)$$

where T^M is the generator of the gauge group and F_{ij}^M is the field strength constructed from the gauge field $A_i^M(X)$. Again we have taken the various fields to lie in the transverse directions only, and depend only on the transverse directions. The action (eq. 4.2) has a (1,0) supersymmetry

$$\delta X^i = i\epsilon\lambda^i \quad ,$$

$$\delta\lambda^i = -\epsilon\partial_- X^i \quad ,$$

$$\delta\psi^s = -\epsilon\lambda^i A_i^M(X)(T^M)_{st}\psi^t \quad . \quad (4.3)$$

In order to formulate a consistent string theory in such a background the action (eq. 4.2) must again be conformally invariant. To understand the reason for this we restrict the background fields to take *vev* only in d of the eight directions (which we denote by X^p (p=1,...d), and depend only on these directions. Let X^M (M=d+1,...8) denote the free directions. The action (eq. 4.1) then may be split into two parts, S_0 and S_1, as follows:

$$S_0 = \frac{1}{2\alpha'}\left[\int d\tau x^+ x^- + \frac{1}{2\pi}\int d\tau \int_0^\pi d\sigma \partial_\alpha X^M \partial^\alpha X^M + \lambda^M \partial_- \lambda^M\right] \quad ,$$

$$\begin{aligned} S = \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \Big[& G_{pq}(X)\partial_\alpha X^p \partial^\alpha X^q + \epsilon^{\alpha\beta} B_{pq}(X)\partial_\alpha X^p \partial_\beta X^q \\ & + iG_{pq}(X)\{\lambda^p \partial_- \lambda^q + \lambda^p(\Gamma_m{}^q{}_n(X) - S_m{}^q{}_n(X))\lambda^n \partial_- X^m\} \\ & + \psi^s(i\partial_+ \psi^s + A_i^M(X)(T^M)_{st}\psi^t \partial_+ X^i) \\ & + \frac{i}{2}F_{ij}^M(X)\psi^s(T^M)_{st}\psi^t \lambda^i \lambda^j \Big] \quad . \end{aligned} \tag{4.4}$$

From now on, we shall again set $\alpha' = 1/2$. The action (eq. 4.4) possesses a (1,0) supersymmetry,[24] given by replacing X^i, λ^i by X^p, λ^p in eq. 4.3. It turns out that if the action (eq. 4.4) also posseses conformal invariance with the correct central charge, then it is possible to write down the full set of SO(9–d,1) Lorentz generators, and verify that their commutator closes on-shell. To see how this can be done, note that conformal invariance and supersymmetry implies that the action (eq. 4.4) has a full (1,0) superconformal symmetry. Although the theory contains sectors with periodic as well as antiperiodic boundary conditions on the λ^i's, we shall restrict our discussion to the Ramond sector with periodic boundary conditions on the λ^i's. The extension of this analysis to the other sector would be straightforward. In the Ramond sector the superconformal symmetry is generated by the generators $\tilde{L}_m^{(1)}$, $L_m^{(1)}$, $G_m^{(1)}$ satisfying the (anti-)commutation relations:

$$[\tilde{L}_m^{(1)}, \tilde{L}_n^{(1)}] = (m-n)\tilde{L}_{m+n}^{(1)} + \frac{\tilde{c}^{(1)}}{4}(m^3 - m)\delta_{m,-n} \quad ,$$

$$[L_m^{(1)}, L_n^{(1)}] = (m-n)L_{m+n}^{(1)} + \frac{c^{(1)}}{4}(m^3 - m)\delta_{m,-n} \quad ,$$

$$[L_m^{(1)}, G_n^{(1)}] = (\frac{m}{2} - n)G_{m+n}^{(1)} \quad ,$$

$$[G_m^{(1)}, G_n^{(1)}] = 2L_{m+n} + c^{(1)}(m^2 - \frac{1}{4})\delta_{m,-n} \quad ,$$

$$[\tilde{L}_m^{(1)}, L_n^{(1)}] = [\tilde{L}_m^{(1)}, G_n^{(1)}] = 0 \quad . \tag{4.5}$$

In order to construct the Lorentz generators we also need the oscillators of the free fields X^M, λ^M. Hence, we write down their mode expansion:

$$X^M = x_0^M + p^M\tau + \frac{i}{2}\sum_{n\neq 0}\left[\alpha_n^M e^{-2in(\tau+\sigma)} + \tilde{\alpha}_n^M e^{-2in(\tau-\sigma)}\right] \quad ,$$

$$x^{\pm} = x_0^{\pm} + p^{\pm}\tau \quad ,$$

$$\lambda^M = \sum_{n=-\infty}^{\infty} b_n^M e^{2in(\tau+\sigma)} \quad . \tag{4.6}$$

The various oscillators satisfy the commutation relations:

$$[x^M, p^N] = i \quad ,$$

$$[\alpha_m^M, \alpha_n^N] = [\tilde{\alpha}_m^M, \tilde{\alpha}_n^N] = m\delta_{m,-n}\delta^{MN} \quad ,$$

$$\{b_m^M, b_n^N\} = \delta^{MN}\delta_{m,-n} \quad . \tag{4.7}$$

From these oscillators we can construct the superconformal generators for the free field theory described by the action S_0. They are

$$\tilde{L}_m^{(0)} = \frac{1}{2}\sum_n : \tilde{\alpha}_{m-n}^M \tilde{\alpha}_n^M : \quad ,$$

$$L_m^{(0)} = \frac{1}{2}\sum_n \left\{ : \alpha_{m-n}^M \alpha_n^M : + \left(n - \frac{m}{2}\right) : b_{m-n}^M b_n^M : \right\} \quad , \tag{4.8}$$

satisfying the commutation relations:

$$\left[\tilde{L}_m^{(0)}, \tilde{L}_n^{(0)}\right] = (m-n)\tilde{L}_{m+n}^{(0)} + \frac{\tilde{c}^{(0)}}{4}(m^3 - m)\delta_{m,-n} \quad ,$$

$$\left[L_m^{(0)}, L_n^{(0)}\right] = (m-n)L_{m+n}^{(0)} + \frac{c^{(0)}}{4}(m^3 - m)\delta_{m,-n} \quad ,$$

$$\left[\tilde{L}_m^{(1)}, L_n^{(1)}\right] = 0 \tag{4.9}$$

where

$$\tilde{c}^{(0)} = \frac{8-d}{3} \quad ,$$

$$c^{(0)} = \frac{8-d}{2} \quad . \tag{4.10}$$

The SO(9–d,1) Lorentz generators may now be expressed as

$$J^{MN} = x^M p^N - x^N p^M - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^M_{-n}\alpha^N_n - \alpha^N_{-n}\alpha^M_n + \tilde{\alpha}^M_{-n}\tilde{\alpha}^N_n - \tilde{\alpha}^N_{-n}\tilde{\alpha}^M_n\right)$$

$$- i\sum_{n=1}^{\infty}\left(b^M_{-n}b^N_n - b^M_{-n}b^M_n\right) - \frac{i}{2}\left(b^M_0 b^N_0 - b^N_0 b^M_0\right) \quad ,$$

$$J^{M+} = x^M p^+ - x^+ p^M \quad ,$$

$$J^{M-} = x^M p^- - x^- p^M - i(p^+)^{-1}\sum_{n=1}^{\infty}\frac{1}{n}\Big[\alpha^M_{-n}(L^{(0)}_n + L^{(1)}_n) - (L^{(0)}_{-n} + L^{(1)}_{-n})\alpha^M_n$$

$$+ \tilde{\alpha}^M_{-n}\left(\tilde{L}^{(0)}_n + \tilde{L}^{(1)}_n\right) - (\tilde{L}^{(0)}_{-n} + \tilde{L}^{(1)}_{-n})\alpha^M_n\Big] - i(p^+)^{-1}\sum_{n=-\infty}^{\infty}\alpha^N_n : b^M_{-n-r}b^N_r :$$

$$- i(p^+)^{-1}\sum_{n=-\infty}^{\infty} b^M_{-n}G^{(1)}_n \quad ,$$

$$J^{+-} = x^+ p^- - x^- p^+ \quad . \tag{4.11}$$

The commutators of these generators may be calculated by knowing the superconformal algebra (eq. 4.5) and the free field commutators. They reduce to the standard commutation relations among the Lorentz generators acting on the physical states defined by

$$(p^+p^-) \mid phys >= 4\left(L^{(1)}_0 + L^{(0)}_0\right) \mid phys >= 4\left(\tilde{L}^{(1)}_0 + \tilde{L}^{(0)}_0\right) \mid phys > \quad , \tag{4.12}$$

if the central charges $c^{(1)}$ and $\tilde{c}^{(1)}$ take values,

$$c^{(1)} = \frac{d}{2} \quad ,$$

$$\tilde{c}^{(1)} = \frac{d+16}{3} \quad , \tag{4.13}$$

respectively.

Thus, we see that in order to get a Lorentz invariant string theory we need a supersymmetric conformally invariant two-dimensional field theory, with correct values of the central charge. The conformal invariance of the theory given by the action (eq. 4.4) may be checked in the perturbation theory by any of the three methods mentioned in the previous section. Equation 4.13 seems to contain two equations. But $c^{(1)} - \tilde{c}^{(1)}$ may be identified to the two-dimensional gravitational anomaly when we couple the σ–model to a background two-dimensional gravitational field, and hence is expected to remain unrenormalized from the free field value (which is the one loop result) due to Adler–Bardeen theorem. Hence the equation for $c^{(1)} - \tilde{c}^{(1)}$ is automatically satisfied, and we need only to check that $c^{(1)} + \tilde{c}^{(1)}$ has the proper value given by eq. 4.13. To the lowest order the constraints on the background fields obtained by demanding superconformal invariance are given by

$$\begin{aligned}
R_{pq} + S_{prs} S_q{}^{rs} = 0 \quad , \\
D^r S_{pqr} = 0 \quad , \\
D^q F^M_{pq} - S_p{}^{qr} F^M_{qr} = 0 \quad , \\
R + \frac{1}{3} S^2 = 0 \quad ,
\end{aligned} \tag{4.14}$$

in the absence of the the background dilaton field. The dilaton field may be coupled to the string exactly in the same way as in the bosonic string theory. These equations are derivable from an effective action

$$\begin{aligned}
S_{eff} = \int d^{10}x e^{-2\Phi} \Big[R + \frac{1}{3} S^2 - 4D^2\Phi \\
+ 4(D\Phi)^2 + \frac{\alpha'}{8} F^M_{\mu\nu} (F^M)^{\mu\nu} \\
+ \frac{\alpha'}{12} S^{\mu\nu\rho} (\Omega_3(A))_{\mu\nu\rho} \Big] \quad ,
\end{aligned} \tag{4.15}$$

to lowest order in α'. Here $\Omega_3(A)$ is the Chern–Simons three form for the gauge field A_μ^M. Again, this is the same action which reproduces the scattering amplitude for massless particles in the string theory at the tree level to lowest order in α'. Higher loop corrections in the σ–model correspond to including higher derivative terms in the string effective action. The correspondence between the string effective action and the equations describing the conformal invariance of the σ–model has been verified beyond one loop order. In the next section we shall assume that this correspondence holds to all orders in the perturbation theory, and show how we may obtain nontrivial solution to the string field equations using this result.

The analysis for the type II closed superstring theory may be done exactly in the same way. In this case the 32 left-handed Majorana–Weyl fermions are replaced by eight left-handed Majorana–Weyl fermions transforming as a vector under the SO(8) Lorentz group. In the presence of general massless background fields the theory reduces to a nonlinear σ–model with (1,1) supersymmetry. Again we can show that the theory is Lorentz invariant if the corresponding σ–model has (1,1) superconformal invariance with the correct central charge. The constraints imposed on the background fields by demanding (1,1) superconformal invariance again turns out to be identical to the equations of motion derived from the effective action in the string theory.

5. COMPACTIFICATION ON CALABI–YAU MANIFOLDS

In this section we shall show how the results obtained in the previous sections may be used to obtain nontrivial solutions to the string equations of motion. We shall be interested in the solutions where six of the ten dimensions are compactified to form an internal space K, and the other directions remain flat. We shall also restrict ourselves to the case where the background $B_{ij}(x)$ vanishes, and the background gauge field takes *vev* in a particular SO(6) subgroup of SO(32) or SO(16)×SO(16). As a result only six of the 32 left-handed fermions become interacting, others remain free. The action for the resulting two-dimensional field theory then splits into a free part and an interacting part. The interacting part of the theory is given by

$$\begin{aligned}\frac{1}{4\pi\alpha'}\int d\sigma d\tau\Big[&G_{mn}(X)\partial_\alpha X^m\partial^\alpha X^n + i\{\lambda^a\partial_-\lambda^a + \lambda^a\omega^{ab}(X)\lambda^b\partial_- X^m\}\\&+\{i\psi^s\partial_+\psi^s + A_m^M(X)(T^M)_{st}\psi^s\psi^t\partial_+ X^m\}\\&+\frac{i}{2}F_{ab}^M(X)\psi^s(T^M)_{st}\psi^t\lambda^a\lambda^b\Big]\quad ,\end{aligned}\tag{5.1}$$

where a, b are the tangent space indices, introduced through the vielbeins $e_m^a(X)$ satisfying

$$e_m^a(X)e_n^a(X) = G_{mn}(X)\quad ,\tag{5.2}$$

$$\lambda^a = e_m^a(X)\lambda^m\quad ,\tag{5.3}$$

and $\omega_m^{ab}(X)$ is the spin connection constructed from the Christoffel symbol $\Gamma_{m\ b}^{\ n}$. In eq. 5.1 the sum over m as well as s runs from 1 to 6. We shall now further restrict the background gauge fields in such a way that when written down as a 6×6 matrix it is equal to the spin connection at every point x. The index s may then be identified with the index a, $A_m^M(X)(T^M)_{ab}$ with $i\omega_m^{ab}(X)$ and $F_{ab}^M(T^M)_{cd}$ with iR_{abcd}, where R is the Riemann tensor. If we further define the two-component spinor

$$\chi^a = \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix} \quad , \tag{5.4}$$

$$\chi^m = E_a^m(X)\chi^a \quad , \tag{5.5}$$

where E_a^M is the inverse of the vielbein, then eq. 5.1 may be written as

$$\frac{1}{4\pi\alpha'}\int d\sigma d\tau G_{mn}(X)\Big[\partial_\alpha X^m \partial^\alpha X^n + i\bar{\chi}^m \ \not{\partial}\chi^n$$

$$+ i\bar{\chi}^m \Gamma_{\ell\ p}^{\ n}\rho^\alpha\chi^p\partial_\alpha X^\ell - \frac{1}{4}R_{mnpq}\bar{\chi}^m\rho^\alpha\chi^n\bar{\chi}^p\rho_\alpha\chi^q\Big] \quad , \tag{5.6}$$

where ρ^α $(\alpha = 0, 1)$ are the two-dimensional γ matrice

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{5.7}$$

The action (eq. 5.6) is the action for an (1,1) supersymmetric nonlinear σ–model. Thus we see that by restricting the background fields to have some specific form we may get extra supersymmetries in the model. In fact, for our purpose we need to restrict the background fields further in order to get a (2,2) supersymmetric model. This is done by taking the internal manifold to be complex and Kahler. We may then introduce complex coordinates on the internal manifold. Let us denote them by z^m and $\bar{z}^{\bar{m}}$, respectively. In this coordinate system the various components of the metric are given by

$$G_{mn} = G_{\bar{m}\bar{n}} = 0 \quad ,$$
$$G_{m\bar{n}} = \partial_m \partial_{\bar{n}} K \quad , \tag{5.8}$$

where K is a function of the internal coordinates, and known as the Kahler potential. K is only defined locally, as we go from one coordinate patch to another K does not remain invariant, but changes as

$$K'(z', \bar{z}') = K(z, \bar{z}) + f(z) + g(\bar{z}) \quad , \tag{5.9}$$

where f and g are holomorphic and anti-holomorphic functions of the internal coordinates, respectively. Neither f nor g contribute to $G'_{m\bar{n}}$ calculated from eq. 5.8 and hence $G_{m\bar{n}}$ transforms as a tensor field as we go from one coordinate patch to the other.

Under these restrictions on the internal manifold, the σ–model described by eq. 5.6 has a (2,2) supersymmetry. The easiest way to see this is to use the superfield notation. We introduce four Grassman parameters θ_R, $\bar{\theta}_R$, θ_L, $\bar{\theta}_L$ and define

$$D_L = \frac{\partial}{\partial \theta_L} + i\bar{\theta}_L \partial_+ \quad ,$$

$$\bar{D}_L = \frac{\partial}{\partial \bar{\theta}_L} - i\theta_L \partial_+ \quad ,$$

$$D_R = \frac{\partial}{\partial \theta_R} + i\bar{\theta}_R \partial_- \quad ,$$

$$\bar{D}_R = \frac{\partial}{\partial \bar{\theta}_R} - i\theta_R \partial_- \quad , \tag{5.10}$$

$$\Phi^m = Z^m + \theta_L \chi_R^m + \theta_R \chi_L^m + \theta_L \theta_R F^m + \text{Higher } \theta \text{ terms} \quad ,$$

$$\bar{\Phi}^m = \bar{Z}^{\bar{m}} + \bar{\theta}_L \chi_R^{\bar{m}} + \bar{\theta}_R \chi_L^{\bar{m}} + \bar{\theta}_L \bar{\theta}_R F^{\bar{m}} + \text{Higher } \theta \text{ terms} \quad , \tag{5.11}$$

where $\chi_R^m = \lambda^m$ and $\chi_L^m = \psi^m$ are the right and left moving components of χ^m, respectively. F^m is an auxiliary field, and higher θ terms are determined in terms of the lower θ terms by solving the constraint equations:

$$\bar{D}_L\Phi^m = \bar{D}_R\Phi^m = D_L\bar{\Phi}^{\bar{m}} = D_R\bar{\Phi}^{\bar{m}} = 0 \quad . \tag{5.12}$$

The action (eq. 5.6) may then be written as

$$\frac{1}{4\pi\alpha'}\int d\theta_L d\theta_R d\bar{\theta}_L d\bar{\theta}_R d\sigma d\tau K(\Phi,\bar{\Phi}) \quad . \tag{5.13}$$

This may be verified by expanding $K(\Phi,\bar{\Phi})$ in terms of the component fields, and performing the θ, $\bar{\theta}$ integrals explicitly using the rules of Grassman integration, and using expression eq. 5.8 for the metric. The (2,2) superfield formulation (there are two left-handed and two right-handed Grassman coordinates) clearly shows that the action has (2,2) supersymmetry. Furthermore, eq. 5.13 is the most general dimension 2 operator in the theory consistent with (2,2) supersymmetry. Hence, if we regularize the theory maintaining N=2 supersymmetry (for example, by using N=2 superfields) the most general divergent counterterm has the same form as the action (eq. 5.13). The effect of renormalization may then be summarized into a single β–function β^K.

On the other hand, we may regard the model (eq. 5.13) as a (1,1) supersymmetric model with coupling constants $G_{mn}(=0)$, $G_{\bar{m}\bar{n}}(=0)$ and $G_{m\bar{n}}$. Hence, we may also describe the effect of renormalization by the β-function of these coupling constants. These are of course related to β^K. The relationship is the same as that between G and K, namely,

$$\beta^G_{mn} = \beta^G_{\bar{m}\bar{n}} = 0, \quad \beta^G_{m\bar{n}} = \partial_m\partial_{\bar{n}}\beta^K \quad . \tag{5.14}$$

We shall now see that there exists a special class of internal manifolds for which $\beta^G_{m\bar{n}}$ vanishes identically. First we shall investigate the one loop result. At this order β^K is given by $\alpha' c Tr\ \ln G$ where c is a numerical constant and the trace is taken over the indices m,$\bar{n}$, taking G as a 3×3 complex matrix. $\beta^G_{m\bar{n}}$ calculated from β^K is proportional to $cR_{m\bar{n}}$ where $R_{m\bar{n}}$ is the Ricci tensor.[25] Thus, the vanishing of the one loop β–function requires a Ricci flat metric. There is a special class of manifolds admitting such metric, known as Calabi–Yau manifolds. Thus, to one loop order we may get a supersymmetric σ–model with vanishing β–function by choosing the internal manifold to be a Calabi–Yau manifold with Ricci flat metric.

What about the higher loop corrections? I shall now give a general proof that on a Calabi–Yau manifold we may always choose a metric which gives us vanishing β–function to a given loop order. To see how this can be done, let us assume that $\alpha'^2 \Delta\beta^K(G)$ is the total contribution to β^K from beyond one loop order. Then we need to solve the equation:

$$\alpha' c \partial_m \partial_{\bar{n}} Tr \ln G + \alpha'^2 \partial_m \partial_{\bar{n}} \Delta\beta^K(G) = 0 \quad . \tag{5.15}$$

We shall look for a solution where the metric is Kahler, *i.e.*,

$$G_{m\bar{n}} = \partial_m \partial_{\bar{n}} K \quad . \tag{5.16}$$

Let $\tilde{G}_{m\bar{n}}$ be the Ricci flat metric on the Calabi–Yau manifold, satisfying

$$R_{m\bar{n}}(\tilde{G}) = 0 \quad , \tag{5.17}$$

and $\tilde{K}$ be the corresponding Kahler potential

$$\tilde{G}_{m\bar{n}} = \partial_m \partial_{\bar{n}} \tilde{K} \quad . \tag{5.18}$$

Finally, let us define

$$\delta K = K - \tilde{K}, \quad \delta G_{m\bar{n}} = G_{m\bar{n}} - \tilde{G}_{m\bar{n}} = \partial_m \partial_{\bar{n}} \delta K \quad . \tag{5.19}$$

Equation 5.15 is satisfied by a G which satisfies

$$\alpha' c Tr \ln G + \alpha'^2 \Delta\beta^K(G) = \alpha' c Tr \ln \tilde{G} \quad . \tag{5.20}$$

This equation may be written as

$$\tilde{G}^{m\bar{n}} \partial_m \partial_{\bar{n}} \delta K = -\alpha' c^{-1} \Delta\beta^K(G) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} Tr(\tilde{G}\delta G)^n \quad . \tag{5.21}$$

We shall now show that there always exists a solution to the above equation on a Calabi–Yau manifold to any given order in α'. For this we need to use a special property of $\Delta\beta^K(G)$, namely that it is a globally-defined scalar function on the manifold if G is a globally-defined tensor. Note that the one loop contribution to β^K does not satisfy this property, since $Tr \ln G$ changes by a term proportional to $\left[Tr \ln(\partial z'^i/\partial z'^j) + Tr \ln(\partial \bar{z}'^i/\partial \bar{z}'^j)\right]$ as we go from the coordinate system z to z'. However, a direct examination of Feynman graphs show that beyond one loop order $\Delta\beta^K(G)$ is always a scalar function of G.[26,27] In fact, $\Delta\beta^K$ vanishes at the two and three loop order, and is proportional to the Euler density at the four loop order.[28–30]

We may now solve eq. 5.21 iteratively.[26] Since δK is expected to be of order α', we may replace $\Delta\beta^K(G)$ by $\Delta\beta^K(\tilde{G})$, and ignore the $Tr(\tilde{G}\delta G)^n$ $(n \geq 2)$ terms on the right-hand side of (eq. 5.21) in the lowest order. The equation then reduces to

$$\tilde{\sqcap} \delta K = -\alpha' c^{-1} \Delta\beta^K(\tilde{G}) \quad , \tag{5.22}$$

where $\tilde{\Box}$ is the Laplacian operator with metric $\tilde{G}$. Since $\Delta\beta^K(\tilde{G})$ is a globally-defined scalar function, it can be split into two parts, one proportional to the zero mode of $\tilde{\Box}$ (a constant) and the other orthogonal to the zero mode, on which $\tilde{\Box}$ is invertible. In other words,

$$\Delta\beta^K(\tilde{G}) = a_0 + \tilde{\Box} b_0 \quad , \tag{5.23}$$

where a_0 is a constant, and b_0 is another globally-defined scalar function. Now note that,

$$\tilde{\Box}\tilde{K} = \tilde{G}^{m\bar{n}}\partial_m\partial_{\bar{n}}\tilde{K} = \tilde{G}^{m\bar{n}}\tilde{G}_{m\bar{n}} = 3 \tag{5.24}$$

for a manifold of complex dimension 3. Hence a solution to eq. 5.22 is given by

$$\delta K = -\alpha' c^{-1}(\frac{a_0}{3}\tilde{K} + b_0) \quad . \tag{5.25}$$

Note that $\tilde{K}$ is not a globally-defined scalar function on the manifold. However,

$$\delta G_{m\bar{n}} = \partial_m\partial_{\bar{n}}\delta K = -\alpha' c^{-1}(\frac{a_0}{3}\tilde{G}_{m\bar{n}} + \partial_m\partial_{\bar{n}} b_0) \tag{5.26}$$

is a globally-defined tensor, since b_0 is a globally-defined scalar field. Thus, the new metric $G_{m\bar{n}} = \tilde{G}_{m\bar{n}} + \delta G_{m\bar{n}}$ is an admissible metric on the Calabi–Yau manifold.

We may now substitute the value of δG on the right-hand side of eq. 5.21. Since $\delta G_{m\bar{n}}$ is a globally-defined tensor field, the right-hand side of eq. 5.21 remains a globally-defined scalar field. Since this is the only constraint that we needed to arrive at the solution (eq. 5.26), the new equations may also be solved as before. This process may be repeated indefinitely until we get the solution to the desired order in α'.

This shows that we can formulate a (2,2) supersymmetric field theory on a Calabi–Yau manifold, at least perturbatively. What about the central charge of the Virasoro algebra? Unfortunately, no direct proof of the nonrenormalization of the central charge has been given. Another way to study this problem is to study the equations of motion of the graviton and the dilaton fields derived from the string effective action and trying to see if one can find solutions of these equations on Calabi–Yau manifolds. This has been shown to be true to order α'^4, but again, no general result exists beyond this order.[31] However, there are indirect arguments showing that this must be the case.[32]

6. VANISHING OF THE COSMOLOGICAL CONSTANT

In this section we shall show how the results obtained in secs. 3 and 4 may be used to prove another important result in string theory, namely, that at the tree level of the string theory, the four-dimensional cosmological constant always vanishes after compactification.[33] A more precise statement is the following: Let us look for a solution of the string field equations where the ten- (or 26-) dimensional manifold is of the form $M_4 \times K$, M_4 being a maximally symmetric four-dimensional space and K an internal six-dimensional space. The fields G_{ij}, B_{ij} and A_i^M are allowed to take nontrivial *vev* only in the internal directions and are allowed to depend only on the internal coordinates. Then we shall show that as long as the classical equations of motion of the string theory are satisfied, M_4 is always the Minkowski space (as opposed to di Sitter or anti-di Sitter space). This is equivalent to the vanishing of the four-dimensional cosmological constant.

The proof of this statement goes as follows. If we write down the string theory in the background field of the type mentioned above, the two-dimensional field theory splits into two parts, one involving the internal coordinates corresponding to the compactified dimensions, and the other involving the coordinates of the maximally symmetric space M_4. In order for the theory to be conformally invariant each of these two theories must be separately conformally invariant. The theory involving the coordinates of M_4 describe an O(4) (or more precisely O(3,1)) σ–model with radius proportional to the inverse of the cosmological constant. This theory becomes a conformally invariant free field theory in the limit of infinite radius, *i.e.*, for zero cosmological constant. This shows the vanishing of the cosmological constant as a consequence of the string field equations.*

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* We should note that the metric that appears in the σ-model is not the physical ten-dimensional metric, but is related to the physical metric through multiplication by e^{ϕ} where ϕ is the dilaton field. Thus a flat four-dimensional metric $\eta_{\mu\nu}$ in the σ-model will correspond to a *vev* of the physical metric of the form $e^{\phi}\eta_{\mu\nu}$. This has been called the warp factor in ref. 34. However, since ϕ does not depend on the four-dimensional coordinates, the effective cosmological constant in four dimensions still vanishes.

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