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ADDENDUM

Correlation Functions of Spin Operators on a Torus*

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Note Added:

The attached note is to be inserted into SLAC–PUB–4083 as page 26(a).

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NOTE ADDED:

The expression (4.13) for the correlation function $\langle V_{\frac{1}{2}} V_{-\frac{1}{2}} V_{\frac{1}{2}} V_{-\frac{1}{2}} \rangle$ in the $k_i \to 0$ limit vanishes identically. This can be seen by using the Riemann theta identity (Eq. R5 of Ref. [a]),

$$\sum_{\nu} \delta_\nu \vartheta_\nu(x)\vartheta_\nu(y)\vartheta_\nu(u)\vartheta_\nu(v) = 2\vartheta_1(x_1)\vartheta_1(y_1)\vartheta_1(u_1)\vartheta_1(v_1)$$

where

$$x_1 = \frac{1}{2}(x + y + u + v)$$
$$y_1 = \frac{1}{2}(x - y + u - v)$$
$$u_1 = \frac{1}{2}(x + y - u - v)$$
$$v_1 = \frac{1}{2}(x - y - u + v)$$

A similar result can also be proved for $\langle V_{\frac{1}{2}} V_{-\frac{1}{2}} \rangle$. This is as expected from the non-renormalization theorems[b]. Thus our analysis provides an explicit verification of the results of Ref. [b], and at the same time gives a general prescription for calculating the $2n$-point fermion amplitude in the covariant formulation, which are expected to be non-vanishing for sufficiently large n.

We expect that the non-renormalization theorems on higher genus surfaces will be consequences of identities involving generalized ϑ-functions analogous to the Riemann theta identities[c]. The results will be published elsewhere[d].

References to Addendum

a. D. Mumford, Tata Lectures on Theta, Birkhauser, Basel, 1983.

b. E. Martinec, Phys. Lett. **171B** (1986) 189.

c. H. Clemens, A Scrapebook of Complex Curve Theory, Plenum Pub., 1980.

d. J. Atick and A. Sen, in preparation.

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ABSTRACT

An expression for the arbitrary n-point correlation function of $SO(2)$ spin operators on a torus is derived. From this expresssion one can calculate the n-point correlation function of any $SO(2N)$ spin operators. Application of our results to the calculation of one-loop scattering amplitudes involving $SO(16)$ and $SO(10)$ spin fields in superstring theories is also given.

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I. Introduction

Calculation of higher-loop amplitudes in closed string theories has been of interest in the recent past [1,2]. Most of the calculations that have been done so far, however, have focused on the calculation of the partition function of various string theories and on the issue of finiteness of the resulting expressions. The partition function in any string theory is expressed as an integral of an appropriate measure over the space of conformally inequivalent Riemann surfaces of a given genus (which is known as the moduli space). To calculate any scattering amplitude, one must multiply the integrand which appears in the partition function calculation by a correlation function of the appropriate vertex operators. In other words, once the string partition function is known, the problem of calculating any string scattering amplitude reduces to that of the calculation of correlation functions of the vertex operators.

The vertex operators in a string theory may be divided into two classes. Some vertices have simple expressions in terms of the basic variables of the theory. Examples of such operators are $e^{ik\cdot X}$, $\partial_{\bar{z}}X\partial_{z}X$, $\lambda(z)\ \partial_{\bar{z}}X\ldots$ etc. where X and λ are the basic bosonic and fermionic degrees of freedom in the theory. Correlation functions of such operators may be calculated on a given genus Riemann surface by using standard techniques such as path integrals. But there is also a class of vertex operators which cannot be expressed directly in terms of the basic variables of the theory. Examples of such operators are the spin fields which appear in the fermion emission vertex in the covariant formulation of the superstring theory [3,4], and the twist operators describing the emission of twisted states in string theories compactified on orbifolds [5,7]. The basic variables of the theory (λ or X) become multivalued on the Riemann surface in the presence of these fields. The calculation of correlation functions involving these vertex operators is usually more difficult. This is the problem we address in this paper. In particular, we study the correlation function of spin fields on a torus. This, for example, can be used to calculate one-loop fermion scattering amplitudes in the Ramond-Neveu-

Schwarz formulation of the string theory. The tree level (i.e., on the sphere) analog of this problem has been solved in Ref. [8] for the spin fields and in Ref. [6] for the twist operators on orbifolds, using the techniques of current algebra and conformal field theory [2,3,9]. One alternate approach that has also been implemented for the calculation of tree level twist field correlations is that of Ref. [7]. In such an approach one may try to calculate these correlation functions directly in the path integral by knowing how the presence of twist fields affect the behavior of the basic variables of the theory and implementing such behavior in the path integral.

The approach we shall take in calculating correlation functions of spin fields on a torus is entirely based on the techniques of conformal field theory. Our calculations are closely related to those of Ref. [6] where it was shown that, by knowing the expectation value of the stress tensor in the presence of vertex operators, one can derive a first order differential equation for the correlation function of these vertex operators, which may then be integrated to obtain the correlation function itself.

We start our calculation in Sec. II with a simple system, that of two real (or one complex) Weyl fermions on a torus. This system has $SO(2)$ symmetry and one can introduce the $SO(2)$ spin fields $S^{\pm}$. Using the techniques of conformal field theory we derive an explicit expression for the n-point correlation function of these $SO(2)$ spin operators. Since the $SO(2N)$ spin operators for a system of $2N$ real fermions may be constructed as products of spin operators of the N $SO(2)$ subgroups of $SO(2N)$, the n-point correlation function of the $SO(2)$ spin operators is sufficient to calculate the n-point correlation function of the $SO(2N)$ spin operators. We use this result in Sec. III to calculate the scattering of four gauge bosons in the $E_8 \times E_8$ heterotic string theory which belong to the spinor representation of the $SO(16)$ subgroup of $E_8 \times E_8$. This specific scattering amplitude requires the four point correlation of the $SO(16)$ spin operators. This is so, because in the fermionic formulation of the $E_8 \times E_8$ heterotic string theory only the $SO(16) \times SO(16)$ part of the gauge group is realized linearly [10], and the vertex operators for the emission of gauge bosons belonging to the spinor representation of $SO(16)$ are described by the spin operators of $SO(16)$.

In Sec. IV the results of Sec. II are applied to the calculation of the one-loop, four-point fermion scattering. The fermion vertex operators in this case involve a product of the $SO(10)$ spin operators and the spin operators associated with the world-sheet local supersymmetry ghost fields [3,4]. Thus, correlation functions of ghost spin fields need to be calculated. We derive in that section an explicit expression for the n-point correlation function of the ghost spin fields, and use this result to calculate the four-point correlation function of the fermion emission vertices. Sec. V summarizes our results and contains some speculations about the extension of these results to higher genus Riemann surfaces.

II. Correlations of $SO(2)$ Spin Fields

In this section we calculate correlation functions of spin operators on a torus. We restrict ourselves first to a simple field theory, namely that of a single complex Weyl fermion ψ. We shall show in the next section how the correlation functions of spin operators in theories with several fermions may be constructed in terms of these simple correlation functions. The torus we will be working on will be described by a parallelogram in the complex plane with sides given by 1 and τ with the identification

$$z \approx z + 1 \approx z + \tau \quad . \tag{2.1}$$

Here τ is the Teichmuller parameter. The fermion ψ is allowed to satisfy either periodic or antiperiodic boundary conditions in the 1 and τ directions. Thus, altogether, there are four different sectors (spin structures) [11] : (P,P), (P,A), (A,P) and (A,A), where the first index denotes periodicity along the direction 1 and the second index periodicity along τ and P(A) denotes periodic (antiperiodic) boundary condition.

The spin operator acting at a given point on the torus creates a branch point such that going around that point the fermion field ψ changes sign. One may also consider general types of spin operators, in particular, ones for which the field ψ changes by $e^{+2\pi ik/N}$ ($k < N$ are integers) as it goes around the point of insertion of the spin field. These operators do in fact arise in theories on orbifolds [6,7] as the superpartners of the bosonic twist operators. In this paper we restrict our attention to the regular spin fields. Our analysis, however, can be readily generalized to the fermionic twist fields encountered in orbifolds.

The simplest way to understand the spin operators is to bosonize the fermion fields [3] $\psi, \bar{\psi}$ as,

$$\psi \sim e^{+i\phi}, \quad \bar{\psi} \sim e^{-i\phi} \quad . \tag{2.2}$$

The spin operators are then given by,

$$S^{\pm} \sim e^{\pm i\phi/2} \quad . \tag{2.3}$$

Treating ϕ as a free bosonic field, we may calculate the operator product of the ψ 's and the spin fields [3]

$$\begin{aligned} \psi(z)S^-(w) &\sim \frac{1}{(z-w)^{1/2}}S^+(w) + \ldots, \\ \bar{\psi}(z)S^+(w) &\sim \frac{1}{(z-w)^{1/2}}S^-(w) + \ldots, \\ \psi(z)S^+(w) &\sim (z-w)^{1/2}\hat{S}^+(w) + \ldots, \\ \bar{\psi}(z)S^-(w) &\sim (z-w)^{1/2}\hat{S}^-(w) + \ldots \quad , \end{aligned} \tag{2.4}$$

where ... denotes less singular terms and $\hat{S}^{\pm}$ are excited spin fields. From Eq. (2.4) we see that $\psi, \bar{\psi}$ always have a branch point at w (i.e., at the point of insertion at the spin field). We may also calculate the operator product of the S's

$$S^+(z)S^-(w) \sim \frac{1}{(z-w)^{1/4}} + (z-w)^{3/4}\bar{\psi}\psi + \dots \tag{2.5}$$

and that of the ψ's

$$\bar{\psi}(z)\psi(w) \sim \frac{1}{(z-w)} + \text{nonsingular.} \tag{2.6}$$

In our calculation we shall only use the short distance expansion given in Eq. (2.4) and (2.6) and will not explicitly use the representation of the spin operators in terms of the bosonic fields. In fact, on a torus or higher genus surfaces, standard bosonization always gives the average over spin structures [12], hence it cannot be used to calculate correlation functions in a given sector. Instead we shall derive a differential equation for the correlator of the spin fields directly, on a torus using a technique originally developed in Ref. [6] for calculation of bosonic twist field correlators on the sphere. To implement this method, we define the stress tensor:

$$T(z) = \lim_{z\to w}\left[\frac{1}{2}\left(\partial_z\bar{\psi}(z)\psi(w) - \bar{\psi}(z)\partial_w\psi(w)\right) + \frac{1}{(z-w)^2}\right] \cdot \tag{2.7}$$

First we compute the two-point correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ in the (P,P) sector. The insertion of the operators $S^+(z_1)$ and $S^-(z_2)$ creates a cut on the torus between the points z_1 and z_2 (Fig. 1). Let us define

$$G(z,w;z_1,z_2) = \frac{\langle\bar{\psi}(z)\psi(w)S^+(z_1)S^-(z_2)\rangle}{\langle S^+(z_1)S^-(z_2)\rangle} \cdot \tag{2.8}$$

From Eqs. (2.4) and (2.6) we see that $G(z,w;z_1,z_2)$ must satisfy the following conditions:

$$
\begin{aligned}
&\lim_{z\to w} G(z,w;z_1,z_2) = \frac{1}{z-w} + \dots, \\
&\lim_{z\to z_1} G(z,w;z_1,z_2) \propto \frac{1}{(z-z_1)^{1/2}} + \dots, \\
&\lim_{z\to z_2} G(z,w;z_1,z_2) \propto (z-z_2)^{1/2} + \dots, \\
&\lim_{w\to z_1} G(z,w;z_1,z_2) \propto (w-z_1)^{1/2} + \dots, \\
&\lim_{w\to z_2} G(z,w;z_1,z_2) \propto \frac{1}{(w-z_2)^{1/2}} + \dots \,.
\end{aligned}
\tag{2.9}
$$

Finally, $G(z,w;z_1,z_2)$ must be periodic on the torus as a function of z and w. The unique Greens function which satisfies all these conditions is given by

$$
\begin{aligned}
G(z,w;z_1,z_2) = &\left[\frac{\vartheta_1'(0)}{\vartheta_1\left(\frac{z_2-z_1}{2}\right)}\right] \left[\frac{\vartheta_1(z-z_2)\vartheta_1(w-z_1)}{\vartheta_1(z-z_1)\vartheta_1(w-z_2)}\right]^{1/2} \\
&\times \left[\frac{\vartheta_1\left(z-w+\frac{z_2-z_1}{2}\right)}{\vartheta_1(z-w)}\right],
\end{aligned}
\tag{2.10}
$$

where ϑ_1 denotes the Jacobi ϑ-function.* Using various properties of the ϑ-functions (e.g., $\vartheta_1(z) \sim \vartheta_1'(0)z$ as $z \to 0$) one can verify that $G(z,w;z_1,z_2)$ given in Eq. (2.10) has all the right properties.

From Eq. (2.7) we may now calculate the expectation value of the stress

* We use the notation of Whittaker and Watson [13]. In Mumford's [14] notation, the corresponding ϑ is ϑ_{11}.

tensor in the presence of spin fields,

$$\langle T(z)\rangle \equiv \frac{\langle T(z)S^+_(z_1)S^-(z_2)\rangle}{\langle S^+(z_1)S^-(z_2)\rangle}$$

$$= \lim_{z\to w}\left[\frac{1}{2}\partial_z G(z,w;z_1,z_2) - \frac{1}{2}\partial_w G(z,w;z_1,z_2)\right. \qquad (2.11)$$

$$\left. + \frac{1}{(z-w)^2}\right] .$$

The right-hand side of Eq. (2.11) can be calculated explicitly using the expression for $G(z,w;z_1,z_2)$ in Eq. (2.10). The result is

$$\langle T(z)\rangle = \left[\frac{1}{8}\left(\frac{\vartheta_1'(z-z_2)}{\vartheta_1(z-z_2)} - \frac{\vartheta_1'(z-z_1)}{\vartheta_1(z-z_1)}\right)^2\right.$$

$$- \frac{1}{2}\left(\frac{\vartheta_1'(z-z_1)}{\vartheta_1(z-z_1)} - \frac{\vartheta_1'(z-z_2)}{\vartheta_1(z-z_2)}\right)\frac{\vartheta_1'\left(\frac{z_2-z_1}{2}\right)}{\vartheta_1\left(\frac{z_2-z_1}{2}\right)} \qquad (2.12)$$

$$\left. + \text{nonsingular terms as } z\to z_1, z_2\right] ,$$

where "$\prime$" denotes differentiation with respect to the argument of the ϑ-function. As $z \to z_2$ the singular part of the above expression is given by:

$$\frac{1/8}{(z-z_2)^2} + \frac{1}{z-z_2}\left[-\frac{1}{4}\frac{\vartheta_1'(z_2-z_1)}{\vartheta_1(z_2-z_1)} + \frac{1}{2}\frac{\vartheta_1'\left(\frac{z_2-z_1}{2}\right)}{\vartheta_1\left(\frac{z_2-z_1}{2}\right)}\right] . \qquad (2.13)$$

On the other hand, the singular part of the operator product expansion for $T(z)S^-(z_2)$ is

$$T(z)S^-(z_2) = \frac{h}{(z-z_2)^2}S^-(z_2) + \frac{1}{z-z_2}\partial_{z_2}S^-(z_2), \qquad (2.14)$$

where h is the conformal weight of the field S^-. Thus the singular part of the left hand side of Eq. (2.11) is given by,

$$\frac{h}{(z-z_2)^2}+\frac{1}{z-z_2}\frac{\partial}{\partial z_2}\ln\langle S^+(z_1)S^-(z_2)\rangle \ . \tag{2.15}$$

Comparing Eqs. (2.13) and (2.15) we see that h is 1/8 and that,

$$\frac{\partial}{\partial z_2}\ln\langle S^+(z_1)S^-(z_2)\rangle = -\frac{1}{4}\frac{\vartheta_1'(z_2-z_1)}{\vartheta_1(z_2-z_1)}+\frac{1}{2}\frac{\vartheta_1'\left(\frac{z_2-z_1}{2}\right)}{\vartheta_1\left(\frac{z_2-z_1}{2}\right)} \ . \tag{2.16}$$

We could also derive another differential equation by analyzing the limit $z \to z_1$. This will fix the z_1 dependence of the correlation function. However, that is not necessary since by translation invariance on the torus the correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ must be a function of (z_2-z_1) only, then Eq. (2.16) integrated gives

$$\langle S^+(z_1)S^-(z_2)\rangle = K_1(\vartheta_1(z_2-z_1))^{-1/4}\vartheta_1\left(\frac{z_2-z_1}{2}\right) , \tag{2.17}$$

where K_1 is a normalization constant which will be determined later.

Before we go on, we want to stress the following two points:

(i) From Eq. (2.17) we see that:

$$\lim_{z_1\to z_2}\langle S^+(z_1)S^-(z_2)\rangle \sim (z_2-z_1)^{3/4} \ . \tag{2.18}$$

This may seem surprising at first since, according to Eq. (2.5), the most singular part in the operator product of $S^+(z_1)S^-(z_2)$ should go as $(z_1-z_2)^{-1/4}$. However, we should remember that we are calculating the correlation function in the sector with periodic boundary conditions, and the vaccuum expectation value of the identity operator (the partition function) vanishes in this sector due to the presence of fermion zero modes. Hence one expects the leading contribution to come from the next term in the operator product expansion in Eq. (2.5). This agrees with Eq. (2.18).

(ii) The function $\vartheta_1(z_2 - z_1)$ is quasiperiodic with periods 1 and τ (i.e., it changes by a multiplicative factor when we shift z_2 by 1 or by τ). However, $\vartheta_1\left(\frac{z_2-z_1}{2}\right)$ is not, since the argument changes only by $\frac{1}{2}$ or $\frac{1}{2}\tau$ as we change z_2 by 1 or τ. Instead under such a translation $\vartheta_1\left(\frac{z_2-z_1}{2}\right)$ goes to one of the other Jacobi ϑ-functions. This shows that the Greens function given in Eq. (2.17) is not even a quasiperiodic function on the torus as a function of z_2 (or z_1). To understand the reason for this let us go back to Fig. (1). If, in this figure, we take the point z_2 and translate it once around the torus, we get Fig. (2a), which is equivalent to Fig. (2b). Note that the branch cut now extends all the way across the torus, thus changing the spin structure. This clearly shows that one should not expect the correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ to be (quasi)-periodic under a translation of z_2 by 1 or τ, instead it should change into the correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ with a different spin structure.

We shall now calculate the correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ with different choices of the spin structure. (This could be done by starting from the correlation function in the periodic-periodic sector and replacing z_2 by $z_2+1, z_2+\tau$ and $z_2+\tau+1$ respectively, but we shall take a more direct approach.) We start with the Greens function $G(z, w, z_1, z_2)$, as defined in Eq. (2.8), but now impose different boundary conditions on G as z or w goes around a cycle. The Greens functions in the various sectors are given by:

$$G_\nu(z,w;z_1,z_2) = \left[\frac{\vartheta_1'(0)}{\vartheta_\nu\left(\frac{z_2-z_1}{2}\right)}\right]\left[\frac{\vartheta_1(z-z_2)\vartheta_1(w-z_1)}{\vartheta_1(z-z_1)\vartheta_1(w-z_2)}\right]^{1/2} \left[\frac{\vartheta_\nu\left(z-w+\frac{z_2-z_1}{2}\right)}{\vartheta_1(z-w)}\right] \tag{2.19}$$

where $\nu = 1,2,3,4$ corresponds to spin structure (P,P), (P,A), (A,A) and (A,P), respectively. $\vartheta_1, ...\vartheta_4$ denote the four Jacobi ϑ-functions [13]* . The same method

* In Mumford's [14] notation, the corresponding ϑ_ν's are $\vartheta_{11}, \vartheta_{10}, \vartheta_{00}$ and ϑ_{01} respectively.

can now be used to calculate the correlation function $\langle S^+(z_1)S^-(z_2)\rangle$ in different sectors. They are given by,

$$\begin{aligned}
\langle S^+(z_1)S^-(z_2)\rangle^{PA} &= K_2(\vartheta_1(z_2-z_1))^{-1/4}\vartheta_2\left(\frac{z_2-z_1}{2}\right) , \\
\langle S^+(z_1)S^-(z_2)\rangle^{AA} &= K_3(\vartheta_1(z_2-z_1))^{-1/4}\vartheta_3\left(\frac{z_2-z_1}{2}\right) , \\
\langle S^+(z_1)S^-(z_2)\rangle^{AP} &= K_4(\vartheta_1(z_2-z_1))^{-1/4}\vartheta_4\left(\frac{z_2-z_1}{2}\right) .
\end{aligned} \tag{2.20}$$

The K_ν's differ from each other only by constant phase factors. Note that, as expected, each of these correlation functions goes like $(z_2-z_1)^{-1/4}$ as $z_2 \to z_1$. Also, using properties of the ϑ functions one can verify that under $z_2 \to z_2+1$ or $z_2+\tau$, these correlation functions and the one given in Eq. (2.17) transform among themselves up to phase factors. As we shall see in the next two sections, the correlation functions of the physical vertex operators in string theory involve appropriate powers of the correlation functions given here which make them periodic after summing over the spin structures.

Finally, we comment on the normalization of the correlation functions. While calculating physical scattering amplitudes, we shall multiply suitable powers of these correlation functions by the partition function on a torus characterized by the Teichmuller parameter τ, and then integrate over τ. The relative normalization of various correlation functions in Eqs. (2.17) and (2.20) have been fixed in such a way that the spin structure dependent contribution to the partition function from the fermionic determinant coming from the integration over the variable ψ has already been included in Eqs. (2.17) and (2.20). It is for this reason that these functions transform among themselves under translations of z_2 by a period. The absolute normalization may be fixed by demanding that as $z_1 \to z_2$, the contribution to the Polyakov integral [15] from a given sector should be $\frac{1}{(z_1-z_2)^{1/4}}$ times the vaccuum functional [6]. This is a consequence of the operator product expansion (Eq. (2.5)), and will be explained further in Sec.III.

Next, we calculate correlation functions with more than two S's. First we consider the correlator $\langle S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle$. (Due to the U(1) charge conservation [3], only those correlators with an equal number of S^+ and S^- are nonvanishing.) Define,

$$G(z,w;z_1,z_2,z_3,z_4) = \frac{\langle\bar{\psi}(z)\psi(w)S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle}{\langle S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle} \tag{2.21}$$

Again, this Greens function can be written down in a given sector by examining the singularities as z or w goes to the various insertion points z_1,z_2,z_3,z_4. The explicit form of $G(z,w;z_1,z_2,z_3,z_4)$ for the various sectors is given by:

$$\begin{aligned} G_\nu(z,w;z_1,z_2,z_3,z_4) = &\left[\frac{\vartheta_1'(0)}{\vartheta_\nu\left(\frac{z_2+z_4-z_1-z_3}{2}\right)}\right] \\ &\times\left[\frac{\vartheta_1(z-z_2)\vartheta_1(z-z_4)\vartheta_1(w-z_1)\vartheta_1(w-z_3)}{\vartheta_1(z-z_1)\vartheta_1(z-z_3)\vartheta_1(w-z_2)\vartheta_1(w-z_4)}\right]^{1/2} \\ &\times\left[\frac{\vartheta_\nu\left(z-w+\frac{z_2+z_4-z_1-z_3}{2}\right)}{\vartheta_1(z-w)}\right] . \end{aligned} \tag{2.22}$$

Note that the spin structure dependence of G as a function of z,w is only through $\vartheta_\nu\left(z-w+\frac{z_2+z_4-z_1-z_3}{2}\right)$. From this we can compute

$$\frac{\langle T(z)S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle_\nu}{\langle S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle_\nu}$$

and in the same way as before derive first order differential equations for the correlator $\langle S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle_\nu$ in the variables z_1,z_2,z_3, and z_4. These equations may be integrated easily and the solution is

$$\begin{aligned} \langle S^+(z_1)S^-(z_2)S^+(z_3)S^-(z_4)\rangle_\nu = &K_\nu'\vartheta_\nu\left(\frac{z_2+z_4-z_1-z_3}{2}\right) \\ &\times\left[\frac{\vartheta_1(z_1-z_3)\vartheta_1(z_2-z_4)}{\vartheta_1(z_2-z_1)\vartheta_1(z_4-z_1)(z_2-z_3)\vartheta(z_4-z_3)}\right]^{1/4} \end{aligned} \tag{2.23}$$

where again $\nu = 1,2,3,4$ corresponding to the (P,P), (P,A), (A,A) and (A,P) sectors, respectively. One can verify that these correlation functions transform up to phase factors among themselves under a shift of one of the z_i's by 1 or τ. The overall normalization is again fixed by considering the limit $z_1 \to z_2, \quad z_3 \to z_4$ and using the operator product expansion (Eq. (2.5)). This will be illustrated in detail in the next section.

The generalization of Eq. (2.23) to arbitrary $2n$-point correlation is also straightforward. The Greens function in the presence of $2n$ spin fields is given by:

$$G_\nu(z, w; z_1, \dots z_n, w_1 \dots w_n) \equiv \frac{\langle \bar{\psi}(z)\psi(w) \prod_{i=1}^n S^+(z_i) S^-(w_i)\rangle_\nu}{\langle \prod_{i=1}^n S^+(z_i) S^-(w_i)\rangle_\nu}$$

$$= \left[\frac{\vartheta_1'(0)}{\vartheta_\nu \left(\frac{\sum_{i=1}^n w_i - \sum_{i=1}^n z_i}{2} \right)} \right] \left[\prod_i \frac{\vartheta_1(z - w_i)\vartheta_1(w - z_i)}{\vartheta_1(z - z_i)\vartheta_1(w - w_i)} \right]^{1/2}$$

$$\times \left[\frac{\vartheta_\nu \left(z - w + \frac{\sum_{i=1}^n w_i - \sum_{i=1}^n z_i}{2} \right)}{\vartheta_1(z - w)} \right] . \tag{2.24}$$

From which we derive,

$$\langle \prod_{i=1}^n S^+(z_i) S^-(w_i)\rangle_\nu = K_\nu^{(n)} \vartheta_\nu \left(\frac{\sum_{i=1}^n z_i - \sum_{i=1}^n w_i}{2} \right)$$
$$\times \left[\frac{\prod_{i<j} \vartheta_1(z_i - z_j)\vartheta_1(w_i - w_j)}{\prod_{i,j} \vartheta_1(z_i - w_j)} \right]^{1/4} . \tag{2.25}$$

This concludes our analysis of the $SO(2)$ spin model on a genus one Riemann surface. In the next two sections we shall use the results developed above to calculate correlation functions of physical vertex operators in string theories.

III. Application to the Heterotic String

In this section we study the scattering of four gauge bosons in the $E_8 \times E_8$ heterotic string theory where the gauge particles belong to the spinor representation of the $SO(16)$ subgroup of one particular E_8 group. Since only the $SO(16)$ subgroup of E_8 is realized linearly in the fermionic formulation of the $E_8 \times E_8$ heterotic string theory, the vertex operators of the gauge bosons which belong to the spinor representation of $SO(16)$ are given by the spin operators of $SO(16)$. Although by a gauge transformation this amplitude may be transformed into an amplitude of scattering of four ordinary gauge bosons of $SO(16)$, and hence may be calculated without the use of spin operators, we illustrate it here to demonstrate the basic rules behind applying the results of the previous section to the $SO(16)$ spin model.

The one-loop partition function for the $E_8 \times E_8$ heterotic string may be written as

$$Z = \int d^2\tau f(\tau) \overline{\left\{ \vartheta_3^4(0,\tau) - \vartheta_4^4(0,\tau) - \vartheta_2^4(0,\tau) \right\}} \times \left\{ \vartheta_3^8(0,\tau) + \vartheta_4^8(0,\tau) + \vartheta_2^8(0,\tau) \right\} , \tag{3.1}$$

where the function $f(\tau)$ has been calculated by various authors [1]. In the integrand we have exhibited two more factors explicitly. The first factor in the curly bracket comes from the integration over the right-handed fermions (the spacetime fermions) and the supersymmetry ghosts, and vanishes due to an identity among the ϑ-functions [13,14]. The factor inside the second curly bracket, on the other hand, comes from the integration and sum over spin structures of 16 of the left-handed fermions (the fermions which couple to the external gauge bosons whose scattering ampliltude we are going to evaluate). There is of course a similar contribution from the other 16 fermions but this has been included in $f(\tau)$.

We shall now show that the four-point scattering that we are going to calculate is given by replacing the factors in the curly brackets by appropriately normalized correlation functions. The vertex operator for a gauge boson belonging to the spinor representation of $SO(16)$ is given by [10],

$$V(k,\epsilon,\alpha) = \int d^2z(\partial_{\bar{z}}X^\mu + \frac{i}{2}k_\nu\lambda^\mu\lambda^\nu)S^\alpha\epsilon_\mu e^{ik\cdot X} \ , \tag{3.2}$$

where ϵ and k are the polarization and momentum vectors, λ's are the space-time fermions, α is a gauge index, and S^α is a spin operator of $SO(16)$ of a definite chirality. To construct these operators, we first define eight complex fermions in terms of the 16 real gauge fermions and then define the spin operators $S_i^\pm$ associated with each of these complex fermions. A general $SO(16)$ spin operator will then be given by*

$$S_1^\pm \, S_2^\pm \, \ldots \, S_8^\pm \ . \tag{3.3}$$

This gives 256 operators. But the requirement of having a definite chirality restricts the total number of S^-'s to be odd or even. For definiteness, we take it to be even. This reduces the total number of spin operators to 128.

The one-loop scattering amplitude of four such particles is then given by,

* At this stage we are not being very careful about the cocycle factors which may produce overall constant phases in the correlation function. These phases however may be determined by demanding that the final amplitude has the appropriate symmetry. This is illustrated later.

$$
\begin{aligned}
Z(k^1,k^2,k^3,k^4) = \int d^2\tau f(\tau) \int & d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \\
& \Big\langle \left(\partial_{\bar z_1} X^{\mu_1}(z_1) + \frac{i}{2} k^1_{\nu_1} \lambda^{\mu_1}(z_1)\lambda^{\nu_1}(z_1)\right) e^{ik^1\cdot X(z_1)} \\
& \left(\partial_{\bar z_2} X^{\mu_2}(z_2) + \frac{i}{2} k^2_{\nu_2} \lambda^{\mu_2}(z_2)\lambda^{\nu_2}(z_2)\right) e^{ik^2\cdot X(z_2)} \\
& \left(\partial_{\bar z_3} X^{\mu_3}(z_3) + \frac{i}{2} k^3_{\nu_3} \lambda^{\mu_3}(z_3)\lambda^{\nu_3}(z_3)\right) e^{ik^3\cdot X(z_3)} \\
& \left(\partial_{\bar z_4} X^{\mu_4}(z_4) + \frac{i}{2} k^4_{\nu_4} \lambda^{\mu_4}(z_4)\lambda^{\nu_4}(z_4)\right) e^{ik^4\cdot X(z_4)} \Big\rangle \\
& \Big\langle \ S^\alpha(z_1) S^\beta(z_2) S^\gamma(z_3) S^\delta(z_4) \Big\rangle \ \epsilon^1_{\mu_1}\epsilon^2_{\mu_2}\epsilon^3_{\mu_3}\epsilon^4_{\mu_4}
\end{aligned}
\quad . \qquad (3.4)
$$

The correlation function involving the X's and $\lambda's$ may be calculated by using the standard Greens function for the free fermionic and bosonic fields on a torus, and their normalization may be fixed exactly in the same way as the spin operators.

We now demonstrate how to calculate the correlation function of the S's with proper normalization. Using Eq. (3.3), we may reduce the correlation function of the S^α's into a product of eight correlation functions of the form given in Eq. (2.23). Let us assume that m of these correlations are of the form $\langle S^+(z_1)S^-(z_2)S^-(z_3)S^-(z_4)\rangle$ or $\langle S^-(z_1)S^+(z_2)S^+(z_3)S^+(z_4)\rangle$, n of these are of the form $\langle S^\pm(z_1)S^\mp(z_3)S^\pm(z_2)S^\mp(z_4)\rangle$, and $8-m-n$ are of the form $\langle S^\pm(z_1)S^\mp(z_2)S^\pm(z_4)S^\mp(z_3)\rangle$. The correlation function $\langle S^\alpha(z_1)S^\beta(z_2)S^\gamma(z_3)S^\delta(z_4)\rangle$, after summing over spin structures, then looks like,

$$
\begin{aligned}
K''\varepsilon(m,n)\sum_{\nu=1}^{4}&\left[\vartheta_\nu\left(\frac{z_1+z_3-z_2-z_4}{2}\right)\right]^m \\
&\left[\frac{\vartheta_1(z_3-z_1)\vartheta_1(z_2-z_4)}{\vartheta_1(z_2-z_1)\vartheta_1(z_4-z_1)\vartheta_1(z_2-z_3)\vartheta_1(z_4-z_3)}\right]^{m/4} \\
&\cdot\left[\vartheta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)\right]^n \\
&\left[\frac{\vartheta_1(z_1-z_2)\vartheta_1(z_3-z_4)}{\vartheta_1(z_3-z_1)\vartheta_1(z_4-z_1)\vartheta_1(z_3-z_2)\vartheta_1(z_4-z_2)}\right]^{n/4} \\
&\cdot\left[\vartheta_\nu\left(\frac{z_1+z_4-z_2-z_3}{2}\right)\right]^{8-m-n} \\
&\cdot\left[\frac{\vartheta_1(z_1-z_4)\vartheta_1(z_2-z_3)}{\vartheta_1(z_2-z_1)\vartheta_1(z_3-z_1)\vartheta_1(z_2-z_4)\vartheta_1(z_3-z_4)}\right]^{\frac{8-m-n}{4}}
\end{aligned}
\quad . \tag{3.5}
$$

where K'' is a normalization constant and $\varepsilon(m,n)$ are some constant phases to be determined. It can be easily seen that the restriction on the spin operators to be of positive chirality forces m, n to be even. Using this fact and the transformation laws of ϑ functions under translations of its argument one can show that Eq. (3.5) is invariant under translation of any of z_i 's, $z_i \to z_i + 1$ or $z_i \to z_i + \tau$.

The relative normalizations as well as the relative phases between terms coming from different spin structures for fixed m, n have been fixed in the above expression by demanding periodicity in z_i. The overall normalization of Eq. (3.5) can be fixed as follows: Let us look at a fixed spin structure (say $\nu = 3$) and at $m = n = 0$. If we now consider the limit $z_1 \to z_2$ and $z_3 \to z_4$, we get the singular part as,

$$
K''(\vartheta_3(0))^8 \, \frac{1}{(\vartheta_1'(0))^4(z_1-z_2)^2(z_3-z_4)^2} \; . \tag{3.6}
$$

Here we have chosen $\varepsilon(0,0)$ to be 1 for convenience. On the other hand, with the choice $m = n = 0$, the most singular part of the operators products of S^α 's are given by,

$$S^\alpha(z_1)S^\beta(z_2) \sim \frac{1}{(z_1-z_2)^2}$$
$$S^\gamma(z_3)S^\delta(z_4) \sim \frac{1}{(z_3-z_4)^2} \tag{3.7}$$

as can be readily calculated using Eq. (2.5). Thus, in this limit, a properly normalized correlation function should be given by,

$$(\vartheta_3(0))^8 \frac{1}{(z_1-z_2)^2(z_3-z_4)^2} , \tag{3.8}$$

where the factor $(\vartheta_3(0))^8$ reflects the fact that according to our convention the correlation function of the S's replaces an explicit factor of $(\vartheta_3(0))^8$ in Eq. (3.1) in this given spin structure. Comparing Eqs. (3.6) and (3.8) we get,

$$K'' = (\vartheta_1'(0))^4 . \tag{3.9}$$

This still leaves an ambiguity in the phase of the correlation function depending on m, n, which we have denoted by $\varepsilon(m,n)$. It manifests itself as the ambiguity in determining the signs of the square roots which occur in Eq.(3.5). These ambiguities may be resolved by demanding that the final amplitude must be expressible in an $SO(16)$ invariant form. We write down the most general $SO(16)$ invariant tensor structure with arbitrary coefficients, and then determine these coefficients from the calculation of the correlation functions with fixed choice of the indices $\alpha, \beta, \gamma, \delta$. Since the number of $SO(16)$ invariant tensor structures is less than the possible number of combinations of m and n, there is a self consistency requirement which this amplitude must satisfy. This dictates the choice of phases. We shall illustrate this in the context of $SO(10)$ invariance of the four fermion scattering amplitude in the next section.

Finally, we discuss the modular invariance of the amplitude (Eq. (3.4)). Since the partition function is modular invariant, we only have to verify that the properly normalized correlation functions transform under modular transformations in the same way as the quantities they replace. Using standard transformation laws of the ϑ functions under $\tau \to \tau + 1$ we can show that Eq. (3.5) is invariant. This is as it should be, since $(\vartheta_2)^8 + (\vartheta_3)^8 + (\vartheta_4)^8$ is invariant under this transformation. On the other hand, to study the behavior of Eq. (3.5) under $\tau \to -\frac{1}{\tau}$, we use the standard relations between $\vartheta_\nu(z,\tau)$ and $\vartheta_\nu(-\frac{z}{\tau}, -\frac{1}{\tau})$. If we define Eq. (3.5) as $f(z_1, z_2, z_3, z_4, \tau)$, we can show that,

$$f(z_1, z_2, z_3, z_4, \tau) = f\left(\frac{z_1}{\tau}\ \frac{z_2}{\tau}\ \frac{z_3}{\tau}\ \frac{z_4}{\tau}, -\frac{1}{\tau}\right)\tau^{-8}\,, \tag{3.10}$$

remembering to include the modular transformation of the normalization constant K'' given by Eq. (3.9). Then, schematically,

$$\begin{aligned} &\int dz_1\ dz_2\ dz_3\ dz_4\ f(z_1,\ z_2,\ z_3,\ z_4, \tau) = \\ &= \int dz_1'\ dz_2'\ dz_3'\ dz_4'\ f(z_1',\ z_2',\ z_3',\ z_4',\ -\frac{1}{\tau})\tau^{-4} \end{aligned} \tag{3.11}$$

where $z_i' = z_i/\tau$. This is precisely the way the modular form $(\vartheta_2(0,\tau))^8 + (\vartheta_3(0,\tau))^8 + (\vartheta_4(0,\tau))^8$ transforms under $\tau \to -\frac{1}{\tau}$. The actual vertex operators involve integrals over z_i as well as $\bar{z}_i$, and there will be a factor of $\bar{\tau}^{-4}$ coming from the change of the variables $\bar{z}_i$. But this factor should be combined with the correlation function in the right-handed sector to show that the resulting expression transforms in the same way as $\overline{(\vartheta_3(0,\tau)^4 - \vartheta_4(0,\tau)^4 - \vartheta_2^4(0,\tau))}$ under the modular transformation $\tau \to -\frac{1}{\tau}$.

As a check on our procedure, the correlation function in (3.5) may be compared to the result derived in Ref.[10] for the one loop four point scattering of gauge bosons in the heterotic string theory. It is not difficult to verify that Eq.(3.5) agrees with the result given in Eq. (6.12) in the last reference in [10].

IV. Correlation Functions of the Fermion Emission Vertex

The original motivation for introducing spin operators in string theory was to describe the fermion emission vertex [3,4,16]. Indeed, in the Ramond-Neveu-Schwarz [17] formulation of the superstring or the heterotic string theory, the fundamental fields are the ten right-handed Majorana-Weyl fermions λ^μ transforming in the fundamental representation of $SO(10)$ Lorentz group. The fermion emission vertices in this theory are described in terms of the spin operators of $SO(10)$. These operators may be constructed as follows. Let us combine the ten real fermions into five complex fermions. We may then in the standard fashion introduce five sets of spin operators $(S_i^+, S_i^-), (i = 1, \dots 5)$. The $SO(10)$ spin operators are given by

$$S_1^{\pm}\ S_2^{\pm}\ \dots\ S_5^{\pm}\ . \tag{4.1}$$

There are 32 such operators. As in the $SO(16)$ case, we may divide these into two sets according to their chiralities. We take the convention that operators with an even number of S^-'s are positive chirality and those with an odd number of S^-'s are negative chirality.

The correlation functions involving these $SO(10)$ spin operators may be calculated in the same way as we did in Sec. III. The fermion emission vertices, however, also contain spin operators of the ghost fields and hence we also need to calculate correlation functions involving these fields. Let b, c denote the ghost fields due to reparametization invariance and β, γ be the local world-sheet supersymmetry ghost fields. One can bosonize the ghost fields β, γ as [3]:*

$$\gamma(z) = e^{+\phi(z)}\eta(z), \quad \beta(z) = e^{-\phi(z)}\ \partial\xi(z)\,, \tag{4.2}$$

where ϕ is a scalar field and η and ξ are two fermionic fields. The fermion

* To be consistent with the analysis of the previous section one should replace z by $\bar{z}$ everywhere in this section, since the Lorentz fermions and the gauge fermions have opposite chirality.

emission vertex in the theory is then given by,

$$V_{-1/2}(u,\ k,\ z) = u^{\alpha}(k)e^{-\frac{1}{2}\phi(z)}S_{\alpha}(z)e^{ik\cdot X(z)}\ . \tag{4.3}$$

(This is actually only half of the vertex. It must be multiplied by an operator of dimension (0,1) in the left-handed sector (e.g., $\partial_{\bar{z}}X^{\mu}$) to form a complete vertex operator).

As we can see the operator given in Eq. (4.3) carries a non-zero ghost charge $(-\frac{1}{2})$. Since the ghost number current is conserved on a torus, the total ghost charge of all the operators in a correlation function must add up to zero. (On a surface of genus g they should add up to $2g-2$). Hence, correlation functions involving only the $V_{-1/2}$ always vanish. The solution to this problem was given in Ref. [3] where an infinite set of vertex operators for the same fermion state was introduced. In particular, another vertex operator describing the emission of the same state as Eq. [4.3] is,

$$V_{+1/2}(u,\ k,\ z) = u^{\alpha}\left[e^{+\frac{1}{2}\phi}\left\{\partial_z X^{\mu} + \frac{i}{4}\ (k\cdot\lambda)\lambda^{\mu}\right\}(\gamma_{\mu})_{\alpha\beta}S^{\beta}\ +\ e^{\frac{3}{2}\phi}\eta bS\right]e^{ik\cdot X} \tag{4.4}$$

The prescription for calculating an n-point correlation function was to calculate the correlation of these vertex operators using any one of the infinite set of operators at a given point. The operators are to be judiciously chosen so that their total ghost number adds up to $2g-2$ for a surface of genus g. The result was proven to be independent of actually which set of operators are chosen to calculate a given correlation function. In particular, on a torus for a $2n$-point correlation function we may choose n of the vertices to be $V_{+1/2}$ and n of them to be $V_{-1/2}$. In such a case, the term in Eq. (4.4) proportional to $e^{+\frac{3}{2}\phi}$ drops out due to the conservation of the reparametrization ghost charge $(N_b - N_c)$, and we only need to evaluate ghost correlation functions of the form:

$$\left\langle e^{-\frac{1}{2}\phi(z_1)}e^{+\frac{1}{2}\phi(w_1)}\ \ldots\ e^{-\frac{1}{2}\phi(z_n)}\ e^{+\frac{1}{2}\phi(w_n)}\right\rangle\ . \tag{4.5}$$

The above correlation function may be calculated in the same way as in Sec. II.

The energy momentum tensor for the β, γ ghost system is given by,

$$T_g(z) = \lim_{z\to w}\left[-\frac{3}{2}\,\beta(z)\partial_w\gamma(w) - \frac{1}{2}\partial_z\beta(z)\gamma(w) - \frac{1}{(z-w)^2}\right]. \tag{4.6}$$

we also can calculate the operator product expansions from Eq. (4.2):

$$\begin{aligned}
\beta(z)e^{\frac{1}{2}\phi(z_1)} &\sim (z-z_1)^{1/2}\; e^{-\frac{1}{2}\phi(z_1)}\partial\xi(z_1)\,,\\
\beta(z)e^{-\frac{1}{2}\phi(z_1)} &\sim (z-z_1)^{-1/2}\; e^{-\frac{3}{2}\phi(z_1)}\partial\xi(z_1)\,,\\
\gamma(z)e^{+\frac{1}{2}\phi(z_1)} &\sim (z-z_1)^{-1/2}\; e^{+\frac{3}{2}\phi(z_1)}\eta(z_1)\,,\\
\gamma(z)e^{-\frac{1}{2}\phi(z_1)} &\sim (z-z_1)^{1/2}\; e^{+\frac{1}{2}\phi(z_1)}\eta(z_1)\,.
\end{aligned} \tag{4.7}$$

Using these operator product expansions, we may write down the Greens function

$$G(z,w;z_1,\,\ldots\,z_n,\,w_1,\,\ldots\,w_n) = \left\langle\beta(z)\gamma(w)e^{-\frac{1}{2}\phi(z_1)}e^{+\frac{1}{2}\phi(w_1)}\ldots e^{-\frac{1}{2}\phi(z_n)}e^{+\frac{1}{2}\phi(w_n)}\right\rangle \tag{4.8}$$

for a given spin structure. This will be given by an expression identical to Eq. (2.24). From this we can calculate the correlation function $\langle T_g(z)\prod_i \left(e^{-\frac{1}{2}\phi(z_i)}e^{+\frac{1}{2}\phi(w_i)}\right)\rangle$ using Eq. (4.6) and derive a differential equation for the correlation function Eq. (4.5) the same way we did for the $SO(2)$ spin operators in Sec. II. The solution to the differential equation in a given spin structure labeled by ν turns out to be,

$$\left\langle\prod_{i=1}^{n} e^{-\frac{1}{2}\phi(z_i)}e^{\frac{1}{2}\phi(w_i)}\right\rangle_\nu = \tilde{K}_\nu^{(n)}\left[\vartheta_\nu\left(\frac{\sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i}{2}\right)\right]^{-1} \left[\frac{\prod_{i>j}\vartheta_1(z_i-z_j)\vartheta_1(w_i-w_j)}{\prod_{i,\,j}\vartheta_1(z_i-w_j)}\right]^{-1/4}, \tag{4.9}$$

where $\tilde{K}_\nu^{(n)}$ is a normalization factor.

We can now proceed to calculate the correlation function of any number of the operators $V_{1/2}$ and $V_{-1/2}$. We shall focus for definiteness on the four-point correlation function and carry out the computatation in the soft k limit in which case the $k \cdot \lambda\lambda^{\mu}$ term drops out from $V_{1/2}$. The correlation function of two $V_{1/2}$ and two $V_{-1/2}$'s reduces to

$$\left\langle e^{-\frac{1}{2}\phi(z_1)} e^{\frac{1}{2}\phi(z_2)} e^{-\frac{1}{2}\phi(z_3)} e^{\frac{1}{2}\phi(z_4)} \right\rangle \left\langle S_{\alpha_1}(z_1)\ S^{\alpha_2}(z_2) S_{\alpha_3}(z_3)\ S^{\alpha_4}(z_4) \right\rangle . \qquad (4.10)$$

The S_α, S^α are spin operators of $SO(10)$ with positive and negative chiralities, respectively. (The S^α's appearing in $V_{1/2}$ carries opposite chirality from the S_α's appearing in $V_{-1/2}$, due to the presence of the γ_μ factor in $V_{+1/2}$.) The correlation of the S's may be calculated in the same way as Eq. (3.5). Let us define the analogs of m, n in this case. Remembering that we must multiply the two correlation functions in Eq. (4.10) before summing over the spin structure (the spin structure of β and γ must be identified with that of λ) we get the answer,

$$\begin{aligned}
K'''\varepsilon'(m,n) \sum_\nu \delta_\nu & \left[\vartheta_\nu\left(\frac{z_1+z_3-z_2-z_4}{2}\right)\right]^{m-1} \\
& \left[\frac{\vartheta_1(z_3-z_1)\vartheta_1(z_2-z_4)}{\vartheta_1(z_2-z_1)\vartheta_1(z_4-z_1)\vartheta_1(z_2-z_3)\vartheta_1(z_4-z_3)}\right]^{\frac{m-1}{4}} \\
\cdot & \left[\vartheta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)\right]^{n} \\
& \left[\frac{\vartheta_1(z_1-z_2)\vartheta_1(z_3-z_4)}{\vartheta_1(z_3-z_1)\vartheta_1(z_4-z_1)\vartheta_1(z_3-z_2)\vartheta_1(z_4-z_2)}\right]^{n/4} \\
\cdot & \left[\vartheta_\nu\left(\frac{z_1+z_4-z_2-z_3}{2}\right)\right]^{5-m-n} \\
& \left[\frac{\vartheta_1(z_1-z_4)\vartheta_1(z_2-z_3)}{\vartheta_1(z_2-z_1)\vartheta_1(z_3-z_1)\vartheta_1(z_2-z_4)\vartheta_1(z_3-z_4)}\right]^{\frac{5-m-n}{4}}
\end{aligned} \qquad (4.11)$$

where $\delta_1, \delta_2, \delta_3, \delta_4$ are 1, -1, 1 and -1 respectively. Again, these relative phases between the contributions from different spin structures have been fixed by demanding periodicity in z_i. Using the fact that $S_{\alpha_1}(z_1)$ and $S_{\alpha_3}(z_3)$ have the same chirality, whereas $S^{\alpha_2}(z_2)$ and $S^{\alpha_4}(z_4)$ carry chiralities opposite to that of $S^{\alpha_1}(z_1)$, we can show that m must be odd, while n must be even. With this fact, it is now very easy to demonstrate that the expression in Eq. [4.11] is invariant under the translation of any one of the z_i's by 1 or τ, provided the δ_ν's are fixed as above. The normalization constant K''' is determined as before by looking at the limit $z_1 \to z_2$, $z_3 \to z_4$. To calculate the correlation function for arbitrary momentum, we need to calculate correlation functions involving the operator $k \cdot \lambda\lambda^\mu(\gamma_\mu)_{\alpha\beta}S^\beta$. This can be done using the same method, if we define the operator through point splitting and suitable subtractions. We shall not carry out these calculations here.

Finally, we would like to clarify the issue of covariance alluded to earlier as well as fix the phases $\varepsilon'(m,n)$. As mentioned in Sec. III, our method allows us to calculate the correlation function for arbitrary polarizations, $\alpha, \beta, \gamma, \delta$, but does not directly give the answer in a Lorentz invariant fashion. However, we may find the Lorentz invariant answer by writing down the most general Lorentz structure for a given correlation function and determining the various Lorentz invariant coefficients from the calculation of correlation functions with fixed polarizations. To see how this is done, let us write down the general expression for:

$$\Big\langle V_{\alpha_1}(z_1)V^{\alpha_2}(z_2)V_{\alpha_3}(z_3)V^{\alpha_4}(z_4)\Big\rangle = G_1(z_i)\delta_{\alpha_1}{}^{\alpha_2}\delta_{\alpha_3}{}^{\alpha_4} + G_2(z_i)\delta_{\alpha_1}{}^{\alpha_4}\delta_{\alpha_3}{}^{\alpha_2}$$
$$+ G_3(z_i)(\gamma^\mu)_{\alpha_1\alpha_3}(\gamma^\mu)^{\alpha_2\alpha_4} , \qquad (4.12)$$

G_1 can readily be seen to be given by Eq. (4.11) with $m = 3$, $n = 0$, G_2 with $m = 3$, $n = 2$, and G_3 with $m = 1$, $n = 2$. There are some ambiguities in determining the relative phases but those can be determined by demanding proper antisymmetry under interchange of the points 1, 2, 3, 4 and that Eq. (4.12) reproduces Eq. (4.11) for other values of m, n as well. Namely, the amplitude must be given by $G_1 + G_3$ for $m = 1$, $n = 0$, by $G_2 + G_3$ for $m = 1$, $n = 4$, and by $G_1 + G_2$ for $m = 5$, $n = 0$. Thus, we can finally write down a covariant expression for the four-point correlator,

$$\begin{aligned}
&\left\langle V_{\alpha_1}(z_1)V^{\alpha_2}(z_2)V_{\alpha_3}(z_3)V^{\alpha_4}(z_4)\right\rangle \\
&= K''' \sum_\nu \delta_\nu \Bigg\{ - \left[\vartheta_\nu\left(\frac{z_1+z_3-z_2-z_4}{2}\right)\right]^2 \left[\vartheta_\nu\left(\frac{z_1+z_4-z_2-z_3}{2}\right)\right]^2 \\
&\qquad \frac{1}{\vartheta_1(z_2-z_1)\vartheta_1(z_3-z_4)}\delta_{\alpha_1}{}^{\alpha_2}\delta_{\alpha_3}{}^{\alpha_4} \\
&\quad + \left[\vartheta_\nu\left(\frac{z_1+z_3-z_2-z_4}{2}\right)\right]^2 \left[\vartheta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)\right]^2 \\
&\qquad \frac{1}{\vartheta_1(z_4-z_1)\vartheta_1(z_3-z_2)}\delta_{\alpha_1}{}^{\alpha_4}\delta_{\alpha_3}{}^{\alpha_2} \\
&\quad + \left[\vartheta_\nu\left(\frac{z_1+z_2-z_3-z_4}{2}\right)\right]^2 \left[\vartheta_\nu\left(\frac{z_1+z_4-z_2-z_3}{2}\right)\right]^2 \\
&\qquad \frac{1}{\vartheta_1(z_3-z_1)\vartheta_1(z_2-z_4)}(\gamma^\mu)_{\alpha_1\alpha_3}(\gamma^\mu)^{\alpha_2\alpha_4} \Bigg\}.
\end{aligned} \tag{4.13}$$

It is now clear that the above expression vanishes identically. This can be seen by using the Riemann theta identity (Eq. R5 of Ref.[14]),

$$\sum_\nu \delta_\nu \vartheta_\nu(x)\vartheta_\nu(y)\vartheta_\nu(u)\vartheta_\nu(v) = 2\vartheta_1(x_1)\vartheta_1(y_1)\vartheta_1(u_1)\vartheta_1(v_1) \tag{4.14}$$

where

$$\begin{aligned}
x_1 &= \frac{1}{2}(x+y+u+v) \\
y_1 &= \frac{1}{2}(x-y+u-v) \\
u_1 &= \frac{1}{2}(x+y-u-v) \\
v_1 &= \frac{1}{2}(x-y-u+v)
\end{aligned} \tag{4.15}$$

A similar result can also be proved for $\langle V_{\frac{1}{2}} V_{-\frac{1}{2}} \rangle$. This is as expected from the non-renormalization theorems[18]. Thus our analysis provides an explicit verification of the results of Ref.[18], and at the same time gives a general prescription for calculating the $2n$-point fermion amplitude in the covariant formulation, which are expected to be non-vanishing for sufficiently large n.

V. Conclusions

In this paper we have calculated the general n-point correlation function for $SO(2)$ spin operators on a torus. Using this result, one can calculate the general n-point correlation function of $SO(2N)$ spin fields for arbitrary n. We have also shown how our results can be used to calculate various one-loop amplitudes in string theories for external states whose vertex operators are given in terms of spin fields.

Our analysis also seems to generalize to higher genus Riemann surfaces. The main problem is essentially to write down the Greens function involving the fermion fields in the presence of spin fields. The Greens function will have certain singularities determined from the operator product expansion which are independent of which Riemann surface we are working on. Moreover, it must satisfy certain periodicity properties as a function of its arguments. These depend on which Riemann surface we are working on and on the spin structure of the fermions. Such a function can be written down in terms of the prime forms [14] by generalizing the results of Sonoda[19]. Once the Greens function is written down, the derivation of differential equations for the correlation function should be straightforward and follows the procedure illustrated in Sec. II.

We also expect that the non-renormalization theorems on higher genus surfaces will be consequences of identities involving generalized ϑ-functions analogous to the Riemann theta identities[20]. The results will be reported elsewhere[21].

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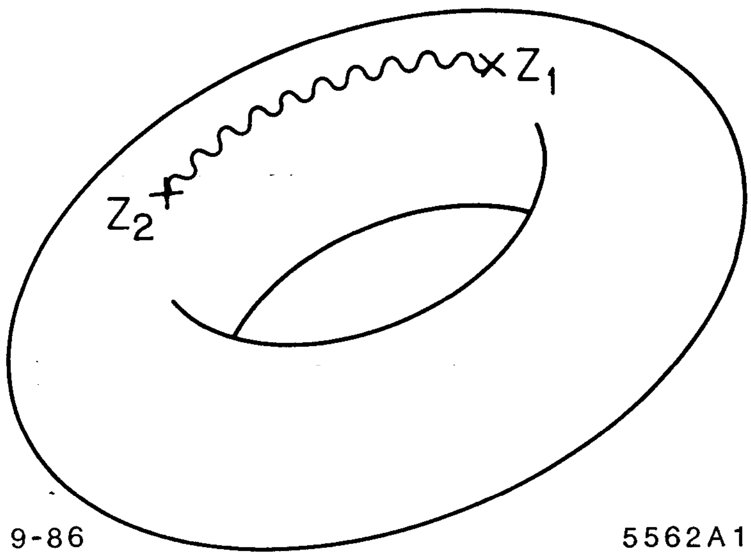

Fig. 1

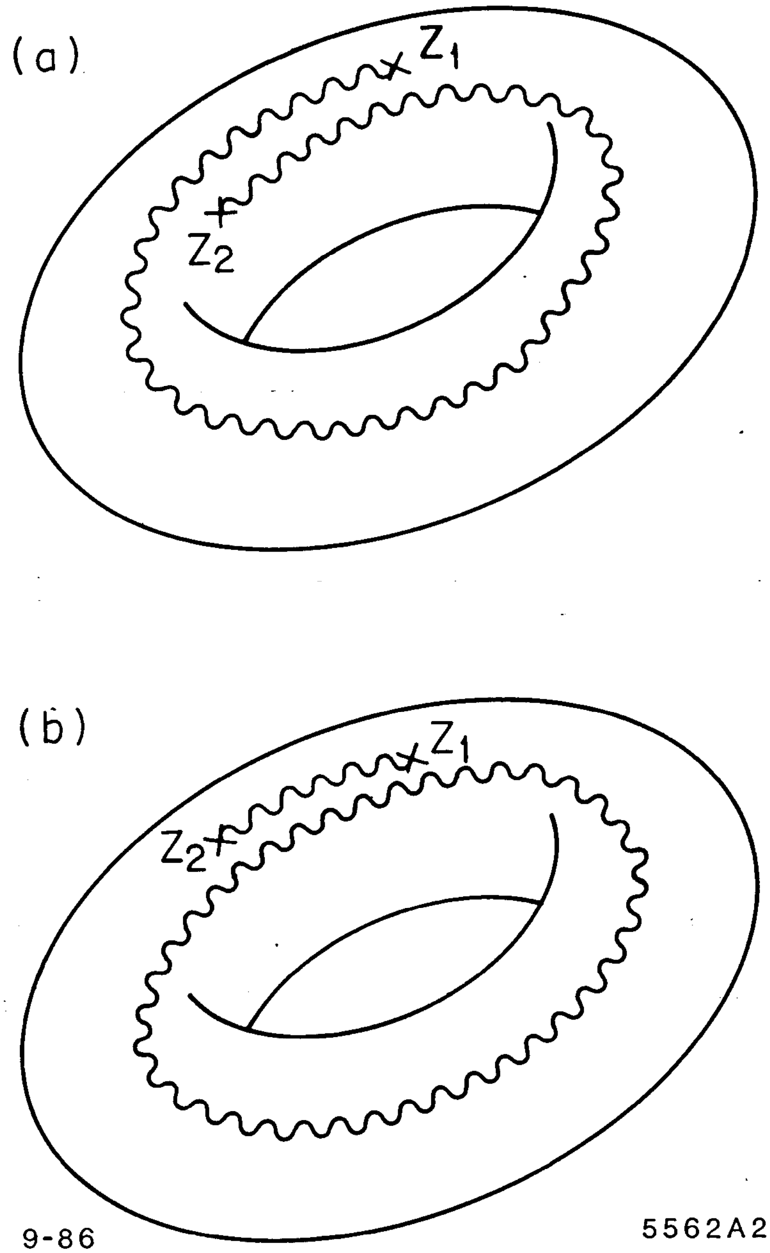

Fig. 2