SLAC–PUB–4072
August 1986
(T)

COMPACTIFICATION OF SUPERSTRINGS AND CHAIN OF ORIENTED STRINGS IN INTERACTIONS*

Robert Olivares Morales†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

Superstring theories command the study of their various possible compactifications, and their consequence physics. Thus, the role of topology is likely to be far more central, in particular in ten-dimensional physics. Topological invariants on a chain of oriented strings in interaction are discussed. Attempts to link superstrings with the reality of the physical world in four dimensions are discussed.

Submitted to *Physical Review Letters*

⋆ Work supported in part by the Department of Energy, contract DE-AC03-76SF00515, and by the University of Paris VI-CNRS-Laboratorie de Physique Theorique and Henri Poincaré Institute.

† SLAC Visiting Scientist from Post-Doctoral Fellowship.

1. INTRODUCTION

Superstring theories[1] are natural candidates for unified theories containing gravity. They are supersymmetric: Local supersymmetry broken (gravitino mass $\neq 0$). Effective low-energy theory (supersymmetric gauge theory: $m_{\tilde{q}}, m_{\tilde{g}} \neq 0$). They are anomaly free, only if the Yang–Mills symmetry group associated with them are SO(32) and $E_8 \times E_8$.[2,3] They exist in ten-dimensions (ten-dimensional Einstein–Yang–Mills supergravity theory) and the fundamental equations of motion[4]

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R \;=\; & -\gamma \left[2\partial_M \phi \, \partial_N \phi - g_{MN} \left(\nabla \phi \right)^2 \right] \\ & + \beta \, e^{-\phi} \left(4 F^{ij}_{MP} \, F^{ijP}_{N} - g_{MN} \, F^2 \right) \\ & - \alpha \, e^{-2\phi} \left(6 H_{MPQ} \, H^{PQ}_{N} - g_{MN} \, H^2 \right) \quad , \end{aligned} \tag{1}$$

$$\nabla_M \nabla^M \phi - \frac{\beta}{2\gamma} e^{-\phi} F^2 + \frac{\alpha}{\gamma} e^{-2\phi} H^2 = 0 \quad , \tag{2}$$

$$\nabla_M \left(e^{-2\phi} H^{MNK} \right) = 0 \quad , \tag{3}$$

$$D_N \left(e^{-\phi} F^{NM} \right) + F_{NL} H^{NLM} = 0 \quad , \tag{4}$$

corresponding to the modified Chapline–Manton action,[5]

$$\begin{aligned} S = \int_{M^{10}} dx \, \sqrt{-g} \Big[& -\frac{1}{2} R - \alpha e^{-2\phi} H_{MNP} \, H^{MNP} \\ & - \beta e^{-\phi} F^{ij}_{MN} F^{ijMN} + \gamma \nabla_M \phi \nabla^M \phi \Big] \end{aligned} \tag{5}$$

do not involve any arbitrary parameter, where capital letters are in the range of 0 to 9, ∇ is the covariant derivative defined with the Christoffel symbol Γ^A_{BC}, R is known as the Ricci form as it is just the complete structure times the Ricci tensor, F is the field strength, g is a metric tensor, and $\propto$ is a fermion zero mode of the Dirac Operator on the compact manifold. This action is the boson part, including the extra Chern–Simons term discovered by Green and Schwarz[2] in the definition of H_{MNP} the field strength for the antisymmetric tensor field. Extra terms include the supersymmetrizations of the anomaly cancelling terms which may turn out to be nonpolinomial, together with all possible higher derivatives, order α' terms such as R^2, *etc.* In this paper these terms will be neglected. We notice that Candelas *et al.*[6] have discussed the compactification of (5) requiring:

1. A manifold, *i.e.*, a direct product $M^4 \times K^6$, among M^4, the Minkowski space, and some compact six-dimensional manifold (for example, a Calabi–Yau Space K^6, *i.e.*, a Ricci–flat Kähler manifold with SU(3) holonomy group).
2. Four-dimensional low-energy physics with a suitable number of standard fermion generations.
3. An unbroken $N = 1$ supersymmetry in $D = 4$.

The same, Gross *et al.*,[7] in heterotic superstring theory found at the lowest nontrivial order in a^1, promising solutions of the effective field equations for massless fields, where the internal six dimensions compactify on a space with a Ricci–flat Kähler metric.[6] These lowest-order solutions have the important property of preserving an $N = 1$ supersymmetry in the effective four-dimensional theory. Corrections to the effective fields equations at higher orders in α', which will lead to modifications of the Ricci–flat Kähler solutions, have been also investigated.[8]

Superstring theories command the study of their various possible compactifications, and their consequence physics. Thus, the fundamental and interesting question would be: How to compactify superstrings in order to obtain a four-dimensional low-energy theory of our physical world from a ten-dimensional superstring? One must find an acceptable compactifying solution to the equations of motion (1–4) of the ten-dimensional effective theory. The low-energy theory of our physical world is, of course, four-dimensional and, therefore, some of the ten-dimensional must be compactified.

There are various options for compactifying, which depend on the compactification conditions in order to have consistent theories for any even dimensions. Among these options, we have:

1. Compactification on manifolds such as the Calabi–Yau spaces,[6] *i.e.*, Ricci–flat Kähler manifolds with SU(3) holonomy group which break the $E_8 \otimes E_8$ and SO(32) gauge groups[2] down to many different subgroups in the ten-dimensional Einstein–Yang–Mills supergravity theory. The (non–Ricci–flat) compact coset spaces, *i.e.*, the six-dimensional compact coset manifolds K with torsion for K = SU(3)/U(1) $\times$ U(1), G_2/SU(3),[9] which will provide a perturbative solution of the classical string field equations in the compactification of the heterotic string theory. For K = Sp(4)/SU(2) $\times$ U(1),[10,11] and only that, the compactification on K gives three standard generations, that transform under the phenomenologically interesting gauge group SU(5) $\times$ SU(3)$_F$ $\times$ U(1)$_F$, the last two factors being flavor symmetries.

   The coset space compactification seems also to be interesting for type II superstring theories, which do not have gauge fields in ten dimensions.

In this case, some of the four-dimensional gauge fields can arise from the isometries of the compact coset manifold.

2. Compactification on orbifolds,[12,13] *i.e.*, manifolds with singularities that correspond to those obtained by dividing a smooth manifold by the nonfree action of a discrete group. The resulting smooth manifold is a rather interesting one called the K3 surface, because of being the only four-dimensional manifold of SU(2) holonomy. (This K3 surface is simply T^4 divided by reflections with blown-up singularities.)

3. Compactification on identification,[6] or nonidentification,[14,15] of the spin connection ω with the gauge connection A.

Here, we will discuss the first choice. We would like to note that the expression of the action (5) leads for the groups $E_8 \otimes E_8$ and SO(32) to a superstring theory free of anomalies,[2,3] and that for the equations of motion (1–4) corresponding to this action, the Bianchi identities must be satisfied:[18]

$$\partial_{[S}H_{RMN]} \sim Tr(F_{[SR}F_{MN]} - R_{[SR}R_{MN]}) \quad , \tag{6}$$

Table 1 gives recent results for the superstring theory of the groups $E_8 \otimes E_8$ and SO(32) that are anomaly free after compactifying and remain consistent in every even dimension. For the $E_8 \otimes E_8$ theory there is a compactification for which one E_8 group is broken to E_6 in four dimensions with zero mode, transforming as $m(27) + (\bar{2}7)$ chiral multiplets, in addition to the 78 gauge multiplet with a number of generations Ng : Ng $= \frac{\Delta}{2}|X|$, where X is the Euler characteristic of K.

2. COMPACTIFICATION

The basic idea of superstring compactification is to find a ground state solution which is a direct product of Minkowski space and K^6, some compact six-imensional manifold. Consider the following superstring equation in two dimensions[6,19]

$$R_{ab} = 0 \quad , \tag{7}$$

where R_{ab} is the Ricci tensor of a two–dimensional metric (a flat metric) obtained explicitly that this is order $\frac{1}{v^2}$ in the field equations, and v is the radius of the compactified manifold. Let us try to solve this equation on the manifolds shown by Fig. 1. The topology classifies these manifolds by their number of handles, *i.e.*, a property which does not change when one deforms the manifold without breaking it. A relation discoveries by Gauss shows that his number h is connected to the integral of the intrinsic curvature of all metric defined on the manifold. For example, in two dimensions for compact manifolds the number of handles h is given by[19]

$$h = 1 - \int d^2x \, \frac{\sqrt{g}R}{4\pi} \quad . \tag{8}$$

If R is zero, there is one handle, so this is possible only for the torus. What is the total curvature of a torus? The first thing to notice is that any torus has the same total curvature as any other, *i.e.*, any torus has zero total curvature. More generally, if M is a closed orientable manifold (closed string) of handles h with a Riemann metric, then

$$\iint_M K \, dA = 4\pi \, (1 - h) \quad . \tag{9}$$

Namely, if we have a surface that is topologically the same as a sphere with n handles attached, then the total curvature K is given by

$$K(\text{sphere with } n \text{ handles}) = 4\pi\,(1-m) \quad . \tag{10}$$

Thus, for the case of the torus,

$$K = 4\pi\,(1-1) = 0 \quad . \tag{11}$$

The importance of knowing about manifolds with handles is this: Any closed surface (for example, the strings and superstring shown by Figs. 2, 3, 4 and 5) in the three-dimensional space is topologically equivalent to a sphere with a bunch of handles attached. Thus, the surface in Fig. 5 is equivalent to a sphere with six handles. Thus, we may see from Eq. (9) that the total curvature of any closed surface in three-dimensional space is an integer multiple of 4π. Furthermore, as the quantity $V-E+F$ (V vertices, E edges, and F faces) is the Euler characteristic χ of a surface. Using this notation we rewrite our curvature formula as

$$K = 2\pi(V-E+F) = 2\pi\chi \quad , \tag{12}$$

which is valid for any closed surface, where K and χ are both topological invariants.

Remarks:

1. In Eq. (9), $(1-h)\,4\pi$ is the algebraic area of $f(s_0)$ ($f: s_0 \rightarrow \Sigma$ is a spherical map) on Σ, *i.e.,* the sum of the areas covered positively minus the sum of the areas covered negatively. Since 4π is the area of Σ, $1-h$ represents the algebraic proportion of Σ that is covered by $f(s_0)$ (s_0 is the parameter surface).

2. If $h = 0$, then $\iint_M K dA = 4\pi$. If P is the set of points where $K > 0$, then $\iint_M K dA \geq 4\pi$. Any field on such a surface has at least one singularity, and if it has, at most, a finite number of singularities, then at least one singularity has a positive index.

3. If $h = 1$, then $\iint_M K dA = 0$. This is the only case where it is possible to define a Riemannian metric such that $K = 0$, and the only case in which it is possible to define a field of line elements without singularities.

4. If h is very large, then K is negative on most of the surface.

Figure 6 illustrates some closed surfaces with K curvature, and h handles. Since we have compact manifolds with other than one handle, there is said to be a topological obstruction (first Chern class) to solving (7). The superstring ground state equation on a compact manifold just states that the first Chern class must vanish (Calabi–Yau manifolds). Equation (7) (differential equation) is a topological equation. It is solved by the choice M^4 x K^6 with the corresponding Ricci–flat metric.

3. CONNECTION BETWEEN LINKS AND STRING THEORY

String are piecewise smooth maps of an interval or of a circle S^1 into a manifold M (a space-time). In closed string theories, points in space-time are replaced by loops in space-time. A close string $\alpha : S^1 \rightarrow M$ is often called loop. When points are replaced by loops in string theory, one obtains Figure 7 where all graphs of loops are described by the basic interaction (a)[3] out of which all Riemann surfaces are constructed [20].

By an n–link L $= (\alpha_1, \ldots \alpha_n)$ will be meant an ordered collection $(\alpha_1, \ldots \alpha_n)$ of maps $\alpha_i : S^1 \rightarrow M$ where the image are to be disjoint $(\alpha_1(S^1) \cup \alpha_2(S^1) \cup \ldots \cup \alpha_n(S^1) = |L|)$. For each link L let $G(L)$ denote the fundamental group of the completment $M - |L|$ (the link group).

Definition:

Let $|L| = \alpha_1 \cup \alpha_2$ be a link of two components. Let $\alpha_1 \cap \alpha_2$ the set of crossings of α_1 with α_2. Then, the linking number N(L) for a given diagram is defined as:

$$\overline{N}(|L|) = N(\alpha_1 \cup \alpha_2) = \frac{1}{2} \sum_{P \in \alpha, \Lambda_{\alpha_2}} \in (p) \quad , \tag{13}$$

where p runs over all crossings in the diagram.

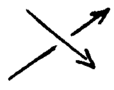
Example:

$\in = +1$

$N\Big[\alpha_1$  $\Big] = \frac{1}{2}(1+1) = +1 \quad .$

$\in = +1$

This will conform with the usual right-hand rule. In order to do this we associate a sign $\in$ to each crossing,

$\in$ [] $= +1$, $\in$ [] $= -1$.

Remarks:

1. Each closed loop $\alpha_1, \ldots, \alpha_n$ generates the fundamental group $G(L)$.
2. N is a topological invariant of links. The first invariants of knots were obtained by studying the complement of the knot in $\Re^3$ and its associated knot group (or link group).
3. All knots and knot groups belong to the classical case $(S^1 \, C \, S^3 \, C \, S^4)$.
4. An oriented knot (or link) is an imbedded circle S^1 in $\Re^3$.
5. One attempt to detect chirality of a knot N (or oriented link) is the signed crossover number $\in$. A knot N is chiral if $N \neq \overline{N}$, where $\overline{N}$ denotes the mirror image of N, obtained by reversing the orientation of 3-space (or 4-space).

In real life one works with pieces of string. If one wants to distinguish between different knot types in this context, then one must require that the endpoints of the string are essentially kept fixed. An special case of the general problem is to determine when a knot is unknotted. For example, if we are given a chain of string, how can one tell if there is a knot present?

It will be assumed that $M (= \Re^{d+1}, \Re^{d-1,1}$, where $\Re^d, S^d, T^\tau, G$, *etc.*) is an orientable manifold; fixed orientations have been chosen for M and S^1, and that the space of maps $\alpha : [0, \pi] \rightarrow \Re^{d-1,1}$, where $\Re^{d-1,1}$ is a supermanifold, may be written as $\alpha = (\alpha_{B1}\alpha_F)$, where α_B and α_F refer to bosonic and fermionic coordinates.

We wish to describe a chain of oriented strings (considered as a oriented closed superstring) in terms of the Kähler geometry of the superloop space, the space of maps from the circle S^1 to the manifold M with periodic boundary conditions in the fermionic coordinates. A chain of oriented strings (physically represented as a series of graphs of Fynman for strings) is a chain where all the circles are oriented. An oriented circle (or loop) is a circle where the two senses (right and left) of running up have been distinguished. A group of invariance is associated to a chain by the operations of linking up (*i.e.,* the way of engendering a chain). A chain of oriented strings is represented by Fig. 8. In these figures, when loops are central, N–loops form a Riemann surface and $\alpha : N \rightarrow \Re^{d-1,1}$ is a map.

In a chain of oriented strings (or topologically equivalent a string to five rounds) we can study its invariants and its index of Milnor [21].
a) Its invariants are:

$$\begin{aligned} K(abcde) &= K(baced) \\ &= K(decba) \\ &= K(edcab) \\ &= K(abcde)\,, \end{aligned} \tag{14}$$

where $K(= K_n)$ is the chain's function of the n–oriented string (n is the number of rounds).

b) Its index of Milnor $m(i_1 \ldots i_{N-\in}, n_1 n)$ are:

$$
\begin{aligned}
M_{abcde} &= +1 \qquad & M_{aedcb} &= -1 \quad , \\
M_{adecb} &= +1 \qquad & M_{abced} &= -1 \quad , \\
M_{abedc} &= +1 \qquad & M_{acdeb} &= -1 \quad , \\
M_{acedb} &= +1 \qquad & M_{abdec} &= -1 \quad , \\
M_{abdce} &= \ldots \qquad & M_{acdbe} &= 0 \quad ,
\end{aligned}
\tag{15}
$$

according to the following formulas:

$$
\begin{aligned}
M_{xyztu} &= M_{yztux}, \\
M_{xyztu} &= -M_{xutzy}.
\end{aligned}
\tag{16}
$$

4. CONCLUSION AND DISCUSSION

Thus, superstring theories commands the study of their various possible compactifications, and their consequence physics. For example, in a general Kaluza–Klein compactification on a compact manifold K (Calabi–Yau spaces with a constant holomorphic 3-form) this involves overlap integrals of zero-mode wave functions on K. In general, direct evaluation of these integrals can be quite difficult. Further, the requirement that the fundamental group be large enough to allow an acceptable E_6-breaking pattern will reduce the number of considerably acceptable manifolds, in particular on three-dimensional manifolds.

Topological invariants such as Betti numbers, Euler characteristic on surfaces, linking number, *etc.*, could represent a fundamental role in superstring theories.[21]

Some questions are interesting after this study:

1. Why the unperturbed Calabi–Yau manifold has vanishing Ricci tensor?
2. Since the string theory is valid for 1–loop diagrams, a surface of genus one and for n-loop diagrams, a surface with n handles, is this theory valid for a chain of oriented strings in interactions?
3. What is the cohomology of the direct product $M^4 \times K^6$,

$$H^n(M^4 \times K^6) = \oplus H^p(M^4) \otimes H^q(K^6) \qquad \text{for every } n \quad ,$$

$$p + q = n \quad ,$$

according to the Künneth formula [22]? In $O(16) \times O(16)$ heterotic string theory, we have

$$H^{10}(S^4 \times K) \cong H^0(S^4) \times H^{10}(K) + H^4(S^4) \times H^6(K) \cong H^6(K) \quad .$$

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Table 1. Summary of the results of the anomaly free $E_8 \otimes E_8$ and SO(32) superstring theory after compactifying remain consistent in every even dimension.

Manifold Dimension	Present Anomalies	Compactification Condition: $\int_M \left(\mathrm{tr}\, R_0^2 - \frac{1}{30}\mathrm{Tr}\, F_0^2\right) = 0$	Theory Consequences
2 [16,17]	Gravitational and Yang–Mills and Mixed Anomalies	Yes $M = 8$	Anomaly Free
4 [16,18]	No Gravitational Anomalies	Yes $M = 6$	Anomaly Free
6 [2,3,6,16]	Yang–Mills Anomalies	Yes $M = 4$	Anomaly Free
8 [16]	No Gravitational Anomalies	No	Anomaly Free
10 [2,3,17]	Gravitational and Mixed and Yang–Mills Anomalies	$H = dB + W_{3L} - \frac{1}{30}W_{3y}$	Anomaly Free
6 Calabi–Yau [6]	No Gravitational and Yang–Mills and Mixed Anomalies	Yes $M = 8$	Anomaly Free

ACKNOWLEDGMENTS

I would like to thank R. Blankenbecler for his hospitality at SLAC, and for the support as participant to the 1986 SLAC Summer Institute, and to all members of the Theory Group, in particular Michael Peskin for accepting this paper to be published at SLAC, and scientific secretaries.

This work was supported in part by the Department of Energy, contract DE-AC03-76SF00515, and by the University of Paris VI–Laboratoire de Physique Théorique and Henri Poincaré Institute. Work done in part at Harvard University, Lyman Laboratory of Physics, Cambridge, MA 02138. From this University, I would like to thank Prof. Coleman and Alvarez–Gaumé during my stay as Visiting Scholar at Lyman Laboratory, Year 1985–1986.

FIGURE CAPTIONS

Fig. 1. Riemann surfaces for interaction strings.

Fig. 2. Strings in interaction of the Yang–Mills type.

Fig. 3. Strings in interaction of the Gravity type.

Fig. 4. Compact surface for superstrings of the 11 type, and for heterotic strings.

Fig. 5. Surface topologically equivalent to topological sphere with six handles.

Fig. 6. Closed surfaces with V vertices, E edges, F faces, χ Euler characteristic, K total curvature, and h handles.

FIGURE	V	E	F	χ	K	h
SPHERE[a]	1	1	2	2	4π	0
SPHERE[b]	6	12	8	2	4π	0
TORUS [c]	1	2	1	0	0	1
TORUS [d]	4	8	2	−2	-4π	2

Fig. 7. Diagrams where all points are replaced by loops. All graphs of loops are described by the basic cubic interaction (a)'.

Fig. 8. A chain of oriented strings (or topologically equivalent a string to five rounds) in interaction. Its Feynman graphs and its resultant strings.

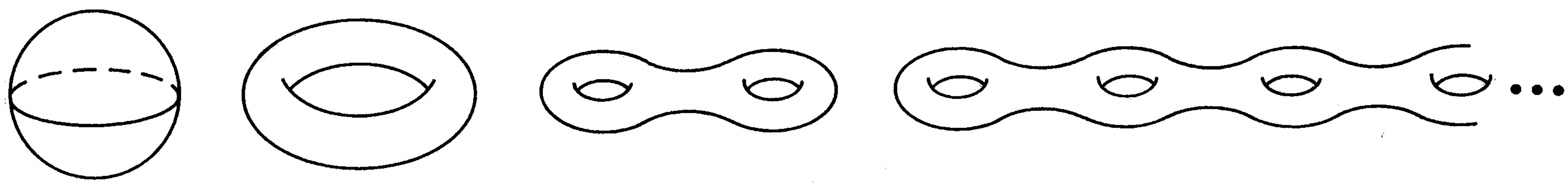

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Fig. 1

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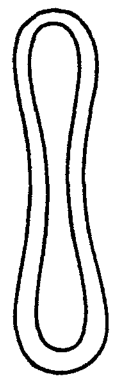

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Fig. 2

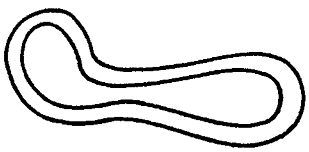

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Fig. 3

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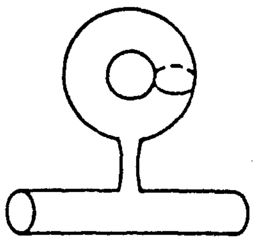

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Fig. 4

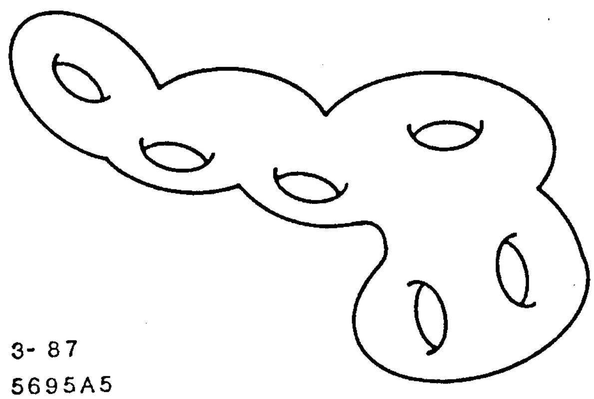

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Fig. 5

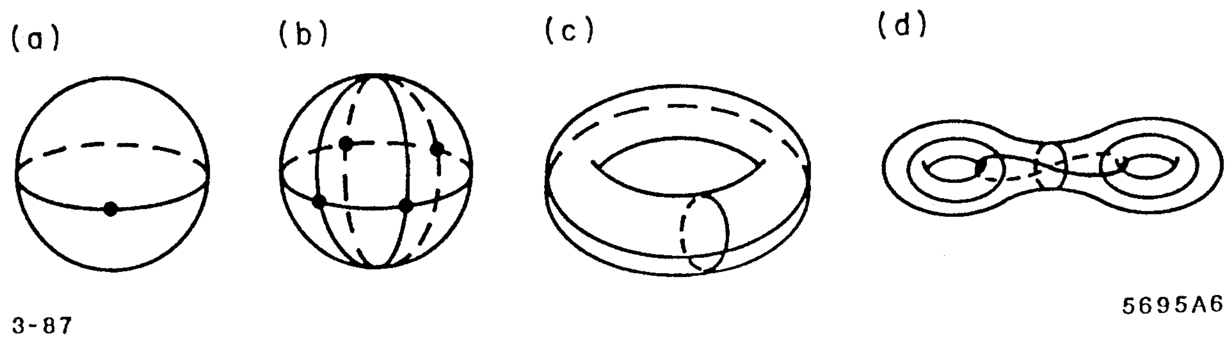

Fig. 6

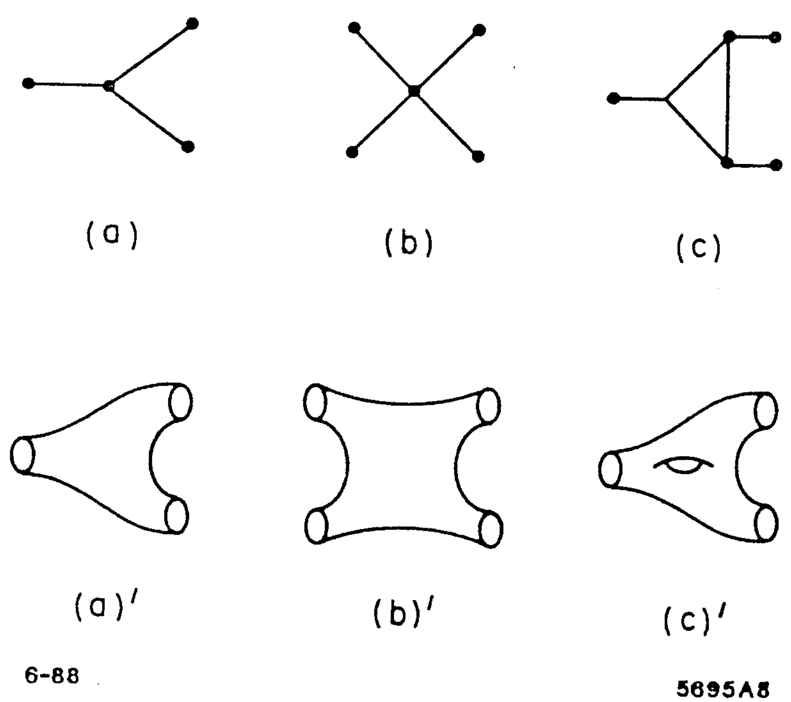

Fig. 7

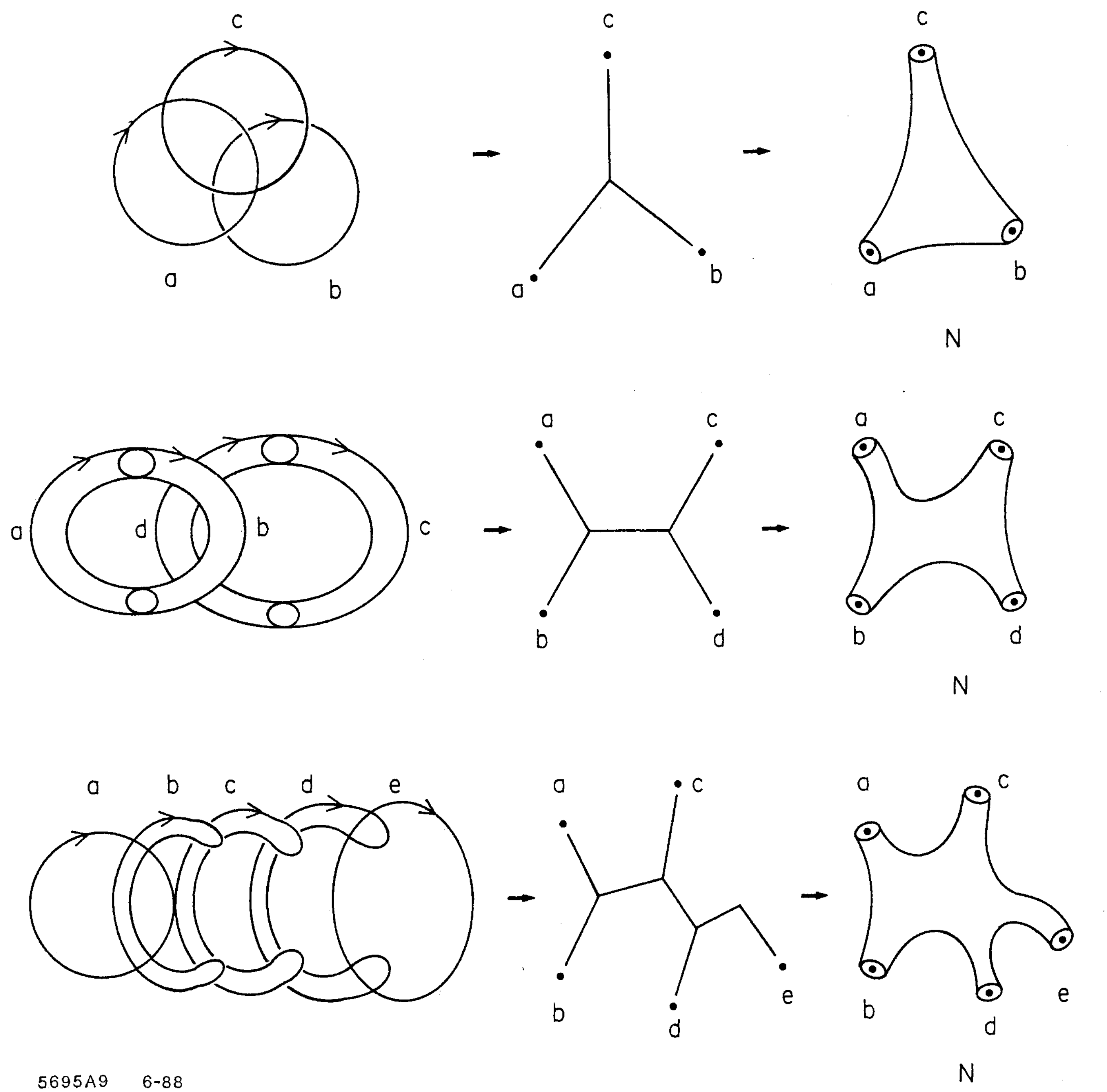

Fig. 8