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# SPACE CHARGE EFFECTS — TUNE SHIFTS AND RESONANCES*

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# 1. INTRODUCTION

In an accelerator or storage ring there exist electromagnetic fields of various types and of different origins. Some of them are put there on purpose to make an accelerator, but some of them are parasitic and whose effects have to be minimized or corrected. For examples, the dipole and quadrupole fields are created to guide and confine the beam, and the longitudinal electrical fields of cavities are used for acceleration. On the other hand, the space charge force and beam-beam force, in general they are unwarranted, are present due to the particular situation of the beam condition.

When a charged particle travels around an accelerator, it also sees the neighboring particles. If the electromagnetic field created by those neighboring particles are strong enough, the actual motion of the particle will be modified. Historically those forces produced by particles travelling in the same direction are called space charge force and those produced by the particles travelling in the opposite direction are called beam-beam force.

In principle those forces can create both coherent and incoherent effects, in this lecture we will only consider the effect on single particle motion in the transverse degree of freedom. In other words, the effect on betatron oscillations of the beam.

In Chapter 2 we introduce the concept of space charge and the tune shift resulting from it, which usually limits the maximum intensity attainable for proton synchrotron at injection. In Chapter 3 a theoretical model is introduced, making use of the fact that the space charge induced tune shift is dependent on the amplitude of betatron oscillation, to show that the resonances are self-limiting in the presence of space charge forces. Beam-beam interactions and resonances are introduced and treated in weak-strong approximation to show the similarities and differences from that of space charge effects in Chapter 4.

# 2. SPACE CHARGE FORCE AND DETUNING

## 2.1 Space Charge Force

To find the space charge force of other particles in the same beam, let us consider a DC beam of circular cross section with uniform charge density $\rho$ of radius $a$ as shown in Fig. 1. The electromagnetic fields satisfy the Maxwell's equations:

$$\epsilon_0 \, \nabla \cdot (E) = \rho \tag{2.1}$$

$$\frac{1}{\mu_0} \, \nabla \times B = J = \rho v \quad . \tag{2.2}$$

Neglecting the curvature of the path, the solutions are found to be:

$$E_r = \frac{\rho}{2\epsilon_0} \, r \ , \quad B_\phi = \frac{\rho}{2\epsilon_0} \, \frac{v}{c^2} \, r \ , \quad r < a \quad . \tag{2.3}$$

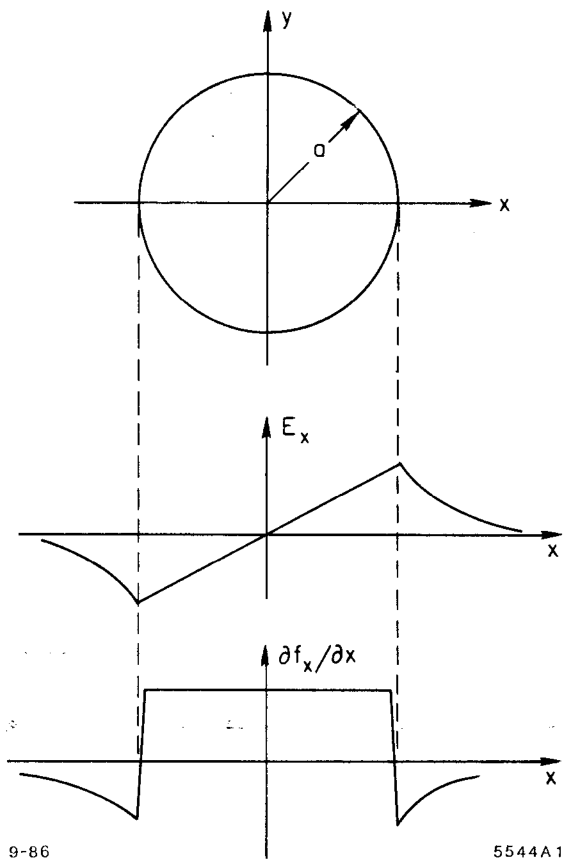


Fig. 1. Space Charge force of a uniform cylindrical beam.

Therefore a particle at the center of the beam experiences additional space charge force of

$$F_r = e(E + v \times \beta) = \frac{e\rho}{2\epsilon_0}\,(1-\beta^2)\,r \tag{2.4}$$

and the corresponding defocusing force is,

$$k(s) = \frac{1}{m\gamma}\,\frac{dF_r}{dr} = \frac{N r_0}{R\pi\, a^2\, \beta^2\, \gamma^3} \tag{2.5}$$

where $N$ is the total number of particles in the ring,

$$N = 2\pi R(\pi\, a^2)\,\rho \tag{2.6}$$

and $r_0$ is the classical radius of proton or electron,

$$r_0 = \frac{1}{4\pi\epsilon_0}\,\frac{e^2}{mc^2} = \begin{cases} 2.8 \times 10^{-15}\ \mathrm{m} & \text{for } e \\ 1.5 \times 10^{-18}\ \mathrm{m} & \text{for } p \end{cases} \quad . \tag{2.7}$$

In the following section, the space charge induced defocusing force of Eq. (2.5) will be used to estimate the tune shift.

### 2.2 INCOHERENT TUNE SHIFT

It has been shown that a particle executes betatron oscillation with respect to the equilibrium orbit in an accelerator obeys the equation of motion[1,2]:

$$\frac{dy}{ds} + K(s)\,y = 0 \quad . \tag{2.8}$$

The solution of the equation is usually expressed as

$$y(s) = a\beta^{1/2}\cos(\nu\phi(s) + \delta) \tag{2.9}$$

where

$$\phi(s) = \int \frac{ds}{\nu\beta} \tag{2.10}$$

represents the phase advance along the beam path and $\nu$ is the tune of the machine.

In the presence of small gradient perturbation $k(s)$, the tune will be changed by[1]

$$d\nu = \frac{1}{4\pi} \int k(s)\ \beta(s)\ ds \quad . \tag{2.11}$$

From Eqs. (2.5) and (2.11) the expected incoherent tune shift is

$$\begin{aligned} d\nu &= \int \frac{1}{4\pi}\ \beta(s)\ k(s)\ ds \\ &= \frac{NR\ r_0}{\pi\nu}\ \frac{1}{\beta^2\gamma^3}\ \frac{1}{a^2} \quad . \end{aligned} \tag{2.12}$$

It is important to recognize the fact that the space charge tune shift given in Eq. (2.12) is proportional to the ring radius and inversely proportional to the tune, cubic of energy and beam area. Often time people are interested in knowing the total number of particles acceptable by a machine for a given tune shift $d\nu$,

$$N = \frac{2\pi\nu\ d\nu}{R\ r_0}\ \beta^2\ \gamma^3\ a^2 \quad . \tag{2.13}$$

The above derivation only takes into account the defocusing force produced by the beam itself, in the presence of vacuum tube the EM field seen by the particle have to be modified. For example, for a uniform beam of elliptical cross section inside an elliptical vacuum chamber in a parallel iron wall as shown in Fig. 2, the final expression of the number of particle acceptable is[3]

$$N = \frac{\nu\ d\nu\ \beta^2\ \gamma^3}{R\ r_0}\ \pi b(a+b)\ BF \tag{2.14}$$

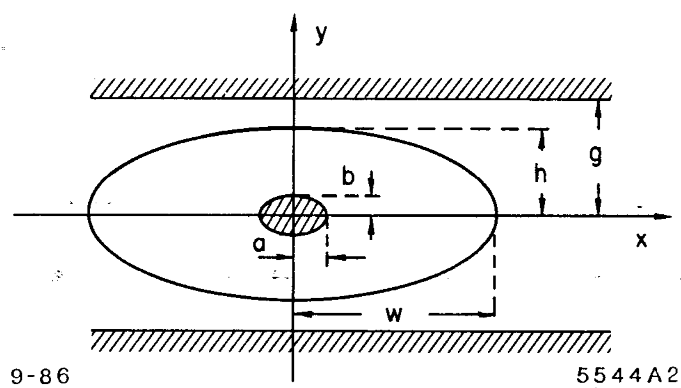


Fig. 2. Cross sectional view of an elliptical beam inside an elliptical chamber and parallel iron face.

with

$$F = \left[1 + b(a+b)\left(\frac{E_1(1+B\beta^2\gamma^2)}{h^2} + \frac{E_2\, C_m\, B\beta^2\gamma^2}{g^2}\right)\right]^{-1} \qquad (2.15)$$

where $a$ and $b$ are half horizontal and vertical beam size, $h$ and $g$ are half height of chamber and iron gap respectively, $B$ is the bunching factor, $C_m$ is the fraction of the circumference occupied by the magnet, and $E_1$ and $E_2$ are geometric coefficients. $E_2$ is close to 0.206 and $E_1$ is 0.172 for a chamber with width to height ratio of two.[3] Usually the vertical dimension $g$ is smaller than the horizontal dimension, the tune shift given by Eq. (2.15) is for vertical tune shift which is commonly called the "Laslett tune shift."

It is important to note that due to the energy dependence term, for an accelerator with large dynamic range the tune shift at low energy is caused predominantly by the direct space charge force. While at high energy the image charge effect becomes important. Expression Eq. (2.14) is also valid for a beam with Gaussian charge distribution, only then $a$ and $b$ should be interpretated as $\sqrt{2}\,\sigma_x$ and $\sqrt{2}\,\sigma_y$ respectively.[4]

### 2.3 Examples from AGS and PSB

Usually the injection into a proton synchrotron is by multi-turn injection from a linac of much higher rf frequency. For all practical purposes the injected beam is continuous and uniformally fill up the circumference of the synchrotron. After injection, the rf voltage is turned on to accelerate the beam. At that time the beam is gradually bunched and experiences the largest space charge tune shift which usually sets the limitation of the attainable intensity of a proton synchrotron. For example listed in Table 1 are the parameters and the estimated tune shifts of the AGS at BNL and PSB at CERN during injection.

Traditionally, at the design stage, the machine parameters are chosen in such a way that the tune shifts stay below 0.25 unit in both planes. Because people believed that, under that condition the tune would not cross low order resonance lines and the beam would be stable. But after the accelerators have been built, under the pressure of experimental program requesting for more beams, the actual intensity and tune shifts are grossly exceeded. For example, from Table 1 it is clear that for few msec the beam has a tune shift exceeds half unit for both AGS and PSB. It certainly crosses half integer and third integer resonance lines. To keep the beam, elaborate stopband correction system and beam distribution shaping methods have to be implemented. The measurements of AGS tunes during capture are shown in Fig. 3[5] and that of PSB are shown in Fig. 4.[6]

Table 1. Tune Shift of the AGS, AGSB and PSB

| | AGS[5] | AGSB[7] | PSB[6] |
|---|---|---|---|
| $E_{\text{inj}}$ (MeV) | 200 | 200 | 50 |
| $\beta^2\gamma^3$ | 0.572 | 0.572 | 0.115 |
| $R$ (m) | 128.5 | 32.125 | 25.0 |
| $\nu_y$ | 8.75 | 4.83 | 5.45 |
| $a/b$ (cm) | 4.2/2.85 | 4.2/2.85 | |
| $F/B$ | 1.31/0.5 | 1.31/0.5 | 1.3/0.58 |
| $N \times 10^{12}$ | 22 | 15 | 8.5 |
| $d\nu_y$ | 0.58 | 0.3 | 0.55 |

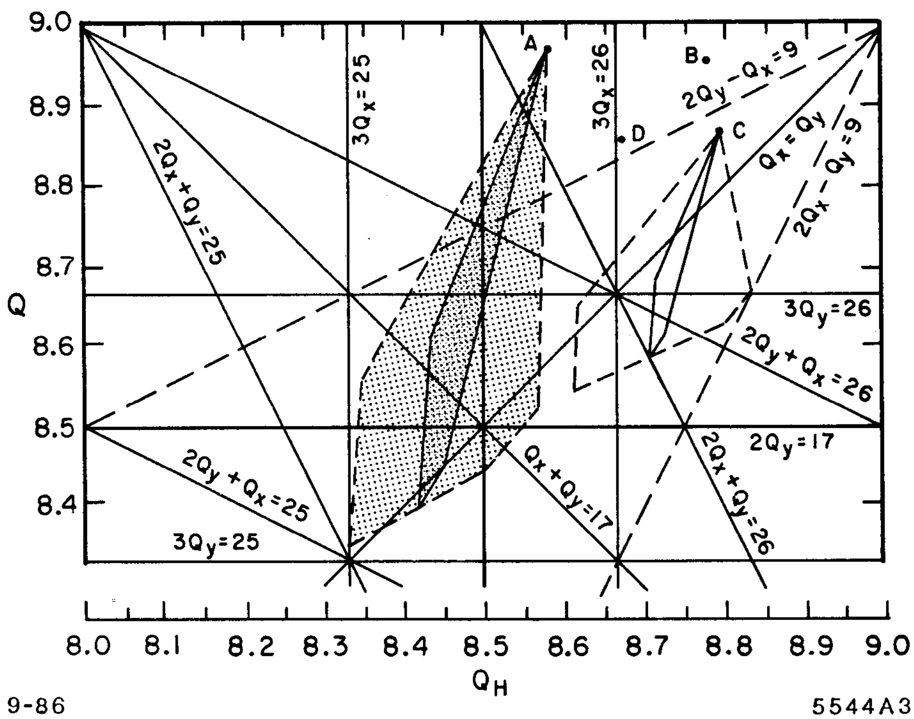


Fig. 3. Space charge tune shift of the AGS.

Therefore the experiences show that the amount of tune shift surviable is not a hard number. The AGS booster under construction will be the first machine starting with a design space charge tune shift larger than 0.25.[7] Judging from the experiences from AGS and PSB, it should not be too tough a challenge, as long as proper stopband corrections and beam distribution shaping methods are incorporated. In the next chapter a theoretical attempt will be made to show that the amplitude growth due to resonance is self-limiting in the presence of space charge force.

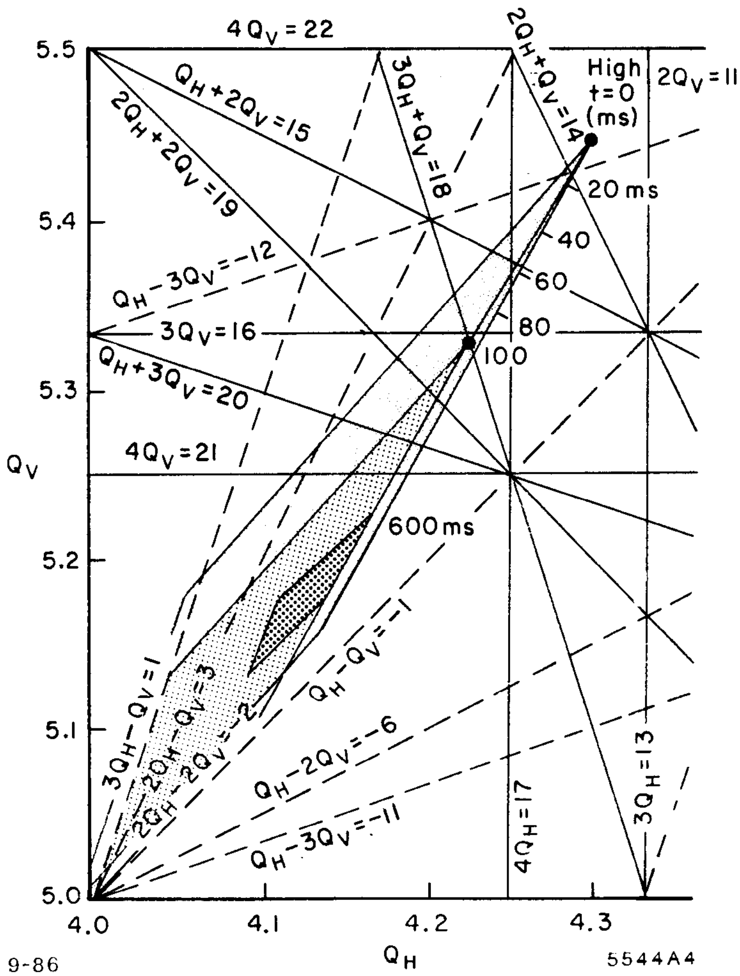


Fig. 4. Space charge tune shift of the PSB.

## 3. RESONANCES IN THE PRESENCE OF THE SPACE CHARGE FORCE

We have shown that high transverse density particle beam at low energy in a synchrotron are strongly influenced by the self-field space charge forces. Because of the relativistic cancellation of electric and magnetic contributions to this force, space charge effects decrease as $1/\gamma^3$ and they are therefore important only at low energy.

The dominant impact of the space charge force is to introduce a characteristic detuning of particles in the beam. This means that the frequency of betatron oscillation becomes a function of amplitude and that resonant conditions can only be sustained over an amplitude region of phase space which decreases for increasing space charge. Thus a magnetic field resonance will be restricted to a bounded phase space amplitude range and results in an amplitude modulation rather than instability. In a DC beam, neglecting scattering and noise effects, the tune of any given particle involves only small amplitude modulation around the resonance amplitudes and the betatron oscillations remain stable.

When radiofrequency accelerating fields are applied to a beam, this situation is drastically altered. After rf turn-on, the beam becomes bunched, with the current distribution following the bunch structure. The space charge force thus develops a current modulation which peaks at the bunch center and vanishes at the ends. At regions of small or vanishing space charge, nonlinear resonance stabilization tends to disappear, thus leaving particles susceptible to resonance blow-up.

In the following, we will study the problem of resonance behavior in the presence of space charge by treating it as one-dimensional resonance problem. Since the self-consistent requirements are ignored, the results describe only the onset or early behavior of the beam. The presentation in this chapter follows closely that in Ref. 8.

For example, let us consider a charged particle beam with Gaussian distribution in both transverse dimension and of elliptical cross section. The scalar potential generated by such a beam is[9]

$$U(x,y) = -\frac{e\lambda}{4\pi\epsilon_0}\int_0^\infty \frac{1 - e\left(-\frac{x^2}{2\sigma_x^2+t}+\frac{y^2}{2\sigma_y^2+t}\right)}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}}\,dt \quad . \tag{3.1}$$

With the potential known, the equation of motion can be found by incorporating the additional force into the betatron oscillation of Eq. (2.8).

## 3.1 Equations of Motion

In the presence of the space charge force the equation of motion of a particle becomes

$$\frac{dy}{ds} + K(s)y + \frac{\lambda R_p}{\beta^2\gamma^3}\,\frac{dU}{dy} = 0 \quad . \tag{3.2}$$

Keeping only the $y$-dependent term by integrating over $x$, the equation can be simplified to be[8]

$$\frac{dy}{ds} + K(s)y - \frac{\lambda R_p}{\beta^2\gamma^3\sigma^2}\,yH(Y^2) = 0 \tag{3.3}$$

where

$$Y = \frac{y}{\sqrt{2}\,\sigma} \quad , \quad \sigma^2 = \beta\epsilon_{\rm rms} \tag{3.4}$$

and

$$H(Y^2) = \int_0^1 e(-tY^2)\, dt = \frac{1 - e(-Y^2)}{Y^2} \quad . \tag{3.5}$$

To study the effects of magnetic imperfections in the vertical plane, we need an expansion of the radial magnetic field, $B_x$, around $y = o$. We can write,

$$\frac{B_x(s,y)}{(B\rho)} = -\sum_{p=0}^{\infty} d_p(s) y^p \ , \tag{3.6}$$

where, $(B\rho)$ is the particle magnetic rigidity, and $d_p(s)$ is the distribution of field errors. It can be shown straightforwardly that the relation between the appropriate error field derivative and the error distribution $d_p(s)$ for each value of $p$ is given by

$$d_p(s) = (-1)^{1/2(p+1)} \frac{B^{(p)}}{p!(B\rho)} , \quad p = 1, 3, \ldots \tag{3.7}$$

and

$$d_p(s) = (-1)^{1/2(p+2)} \frac{B^{(p)}}{p!(B\rho)} , \quad p = 2, 4, \ldots \tag{3.8}$$

where

$$B^{(p)} = \left. \frac{\partial^p B_y}{\partial_x^p} \right|_{x=y=0} \quad . \tag{3.9}$$

The resonance forcing term modifies the linear equation, with the equation becoming,

$$y'' + K(s)y + d_{(p-1)}(s) y^{p-1} = 0 \, , \tag{3.10}$$

where we have kept a single resonance term of order $p - 1$ ($p = 1$, dipole; $p = 2$, quadrupole; $p = 3$, sextupole; and so on).

For a resonance in the presence of space charge, we add to the linear equation both resonance and space charge terms, leading to,

$$y'' + Ky - \frac{\lambda r_p}{\beta^2 \gamma^3 \sigma^2} y H(Y^2) + d_{p-1}(s) y^{p-1} = 0 \ . \tag{3.11}$$

We define the space charge strength parameter $\xi$,

$$\xi = \frac{N r_p}{4\pi \ \epsilon_{\text{rms}} \beta^2 \gamma^3} \ , \tag{3.12}$$

where $N$ is the number of particles in the beam, and $\epsilon_{\text{rms}}$ is the rms emittance, defined by

$$\epsilon_{\rm rms} = \frac{\sigma^2}{\beta(s)} , \tag{3.13}$$

with $\beta(s)$ is the beta-function or Twiss parameter.[1]

To solve the linear part of the equation we introduce the normalized coordinates $(\eta, \dot{\eta})$, by

$$\begin{bmatrix} \eta \\ \dot{\eta}/\nu \end{bmatrix} = \begin{bmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} , \tag{3.14}$$

with

$$\alpha = -\frac{1}{2}\frac{d\beta}{ds} , \tag{3.15}$$

and differentiation is with respect to the betatron phase, $\phi$, given by

$$\phi = \frac{1}{\nu} \int_0^s \frac{ds}{\beta(s)} . \tag{3.16}$$

In the smooth approximation, the phase $\phi$ is simply the azimuth $\theta$, i.e. $\phi \rightarrow \theta = s/R$, where $R$ is the average ring radius. The equation of motion in terms of $\eta$, $\dot{\eta}$ and $\phi$, is

$$\ddot{\eta} + \nu^2\eta - \frac{2\nu^2\xi\beta(s)}{R}\,\eta H(Y^2) + \nu^2(\beta(s))^{1/2(p+2)}\; d_{p-1}(s)\eta^{p-1} = 0 , \tag{3.17}$$

where

$$Y^2 = \frac{\eta^2}{2\epsilon_{\rm rms}} . \tag{3.18}$$

We further introduce amplitude and phase variables, or action and angle variables, $(I, \psi)$, related to $(\eta, \dot{\eta})$ by

$$\eta = \sqrt{I}\cos\psi , \tag{3.19}$$

$$\dot{\eta} = -\nu\sqrt{I}\sin\psi . \tag{3.20}$$

Inverting and differentiating we obtain the equations of motion for $I$ and $\psi$ in the form,

$$\dot{I} = -\frac{2\sqrt{I}}{\nu}\,\sin\psi(\ddot{\eta} + \nu^2\eta) , \tag{3.21}$$

and

$$\dot{\psi} = \nu - \frac{\cos\psi}{\nu\sqrt{I}}(\ddot{\eta} + \nu^2\eta) . \tag{3.22}$$

When the tune of the machine is close to a rational number, we can define the resonant phase variable,

$$\tau = \psi - \frac{m}{p}\,\phi + \text{constant phase}\ ,$$

having the property of a slow time variation:

$$\dot{\tau} = \dot{\psi} - \frac{\dot{m}}{p} \approx 0\ , \qquad \text{with } \dot{\psi} \approx \nu\ .$$

In other words, we require that the tune for betatron oscillations $\nu$ be close to the order $p$ resonance value, $m/p$. Here, $m$ is the ring azimuthal harmonic (in the variable $\phi$) of the resonant force term appropriate to the resonance of order $p$.

Applying the method of phase averaging, we can reduce the equations for the amplitude and phase variables to the resonance equations. The relevant phase variables are $\psi$, the phase space phase and $\phi$ the ring azimuth phase. The explicit dependence on $\phi$ represents the azimuthal harmonic content of the perturbing force in the original equations and is the necessary ingredient for resonance behavior. The phase averaging procedure involves two steps. First, the space charge term has no explicit $\phi$ dependence except for the $\beta$-function variation, whose symmetry tends to suppress resonance excitation. Therefore the space charge term simply oscillates rapidly in $\psi$ and $\phi$ about some mean value. Averaging over $\psi$ and $\phi$ as a consequence replaces this term by its "long term" average. On the other hand, the resonance force term, arising for example from magnet imperfections, has no significant long term average independent of phase. However, the resonance condition that the betatron oscillation frequency be close to $m/p$ leads to a term which contains the phases combined in the form $(\psi - m\phi/p)$. Near the resonant tune, this phase varies slowly in time and does not average to zero. Thus, the second part of the averaging procedure is to retain the slow phase term in the equations for $\dot{I}$ and $\dot{\psi}$.

Averaging the space charge term over $\psi$ and $\phi$, we find in the case of no resonance force term,

$$\dot{I} = 0\ , \tag{3.23}$$

and

$$\dot{\psi} = \nu - \xi F(\alpha)\ , \tag{3.24}$$

where

$$\alpha = I/2\epsilon_{\text{rms}}\ , \tag{3.25}$$

and

$$F(\alpha) = \frac{1}{\pi} \int_0^{2\pi} du \ \cos^2 u \int_0^1 dt \ e^{-t\alpha \cos^2 u} \ . \tag{3.26}$$

Here, use has been made that the average $\beta \approx R/\nu$ and $\xi > 0$.

The function of $F(\alpha)$ can be expressed in terms of the Bessel Functions,

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{z \cos \theta} \cos n\theta \ d\theta \ , \tag{3.27}$$

by

$$F(\alpha) = \frac{2}{\alpha} \left(1 - e^{-\alpha/2} I_o \left(\frac{\alpha}{2}\right)\right) \ . \tag{3.28}$$

The phase equation in the presence of space charge thus has the property that the oscillating tune, $\dot{\psi}$, becomes a function of amplitude. Thus, a resonant condition on the tune becomes amplitude dependent and resonances become restricted to certain amplitude regions in the phase space.

To average the resonant terms, we must look for the term containing the slow phase, $\psi - (m/p)\phi$. Consider the resonant term in the equation for $\dot{\psi}$:

$$\frac{\cos \psi}{\nu \sqrt{I}} \ \nu^2 \beta^{1/2(p+2)} d_{p-1}(\phi) \eta^{p-1} \ , \tag{3.29}$$

or, with $\eta = I^{1/2} \ \cos \psi$, we have

$$\nu \beta^{1/2(p+2)} d_{p-1} \ (\phi) \ I^{1/2(p-2)} \cos^p \psi \ . \tag{3.30}$$

Since the only term in $\psi$ which will contribute to the slow phase terms is $\cos \ p\psi$, we make the replacement

$$\cos^p \psi \to \frac{1}{2^{p-1}} \ \cos \ p\psi \ . \tag{3.31}$$

Also, we expand $d_{p-1}(\phi)$ in a Fourier series in $\phi$ and retain only the $m^{\text{th}}$ harmonic, which is the only term contributing to the slow phase. That is,

$$d_{p-1}(\phi) \to e_{pm} \cos m_\phi + f_{pm} \sin m_\phi \ , \tag{3.32}$$

where

$$e_{pm} = \frac{1}{2\pi} \int_0^{2\pi} d_{p-1}(\phi) \cos m_\phi d_\phi \ , \tag{3.33}$$

and

$$f_{pm} = \frac{1}{2\pi} \int_0^{2\pi} d_{p-1}(\phi) \sin m_\phi d_\phi \ . \tag{3.34}$$

Making the substitutions for $\cos^p \psi$ and $d_{p-1}(\phi)$ and retaining only the slow phase term, we have

$$\nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \{e_{pm} \cos(p\psi - m\phi) - f_{pm} \sin(p\psi - m\phi)\} \ . \tag{3.35}$$

If we now define

$$\Gamma_p e^{i\gamma_p} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ d_{p-1}(\phi) e^{im_\phi} \ , \tag{3.36}$$

we obtain for this term in the phase equation,

$$\nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \Gamma_p \cos(p\psi - m\phi + \gamma_p) \ . \tag{3.37}$$

The equation for $\dot{\psi}$ is therefore

$$\dot{\psi} = \nu - \xi F(\alpha) + \nu\beta^{1/2(p+2)} I^{1/2(p-2)} \frac{1}{2^p} \Gamma_p \cos(p\psi - m\phi + \gamma_p) \ . \tag{3.38}$$

In a similar way, the equation for the amplitude $I$, in the presence of resonance, can be written,

$$\dot{I} = 2\nu\beta^{1/2(p+2)} \ I^{1/2p} \ \frac{1}{2^p} \Gamma_p \sin(p\psi - m\phi + \gamma_p) \ . \tag{3.39}$$

Now the corresponding slow phase becomes, $\tau = \psi - m\phi/p + \gamma_p/p$, with the property that $\dot{\tau}$ is small, i.e. that $\tau$ is slowly varying when the betatron tune is close to resonance. Using the variable $\alpha = I/2\epsilon_{\text{rms}}$, we have equations for $\alpha$ and $\tau$:

$$\dot{\tau} = \nu - \frac{m}{p} - \xi F(\alpha) + \frac{\nu\beta^{1/2(p+2)}}{2^{1/2(p+2)}} \ \epsilon_{\text{rms}}^{1/2(p-2)} \alpha^{1/2(p-2)} \ \Gamma_p \cos p\tau \ , \tag{3.40}$$

and

$$\dot{\alpha} = \frac{1}{2^{1/2p}} \ \nu\beta^{1/2(p-2)} \ \epsilon_{\text{rms}}^{1/2(p-2)} \alpha^{1/2p} \ \Gamma_p \sin p\tau \ . \tag{3.41}$$

Define the stopband width with $\Delta_e$ by

$$\Delta_e = \frac{1}{2^{1/2(p+2)}} \ \nu\beta^{1/2(p+2)} \ \epsilon_{\text{rms}}^{1/2(p-2)} \ \Gamma_p \ . \tag{3.42}$$

Thus, the $\tau$ and $\alpha$ equations become,

$$\dot{\tau} = \nu - \frac{m}{p} - \xi F(\alpha) + \Delta_e\ \alpha^{1/2(p-2)} \cos\ p\tau\ , \tag{3.43}$$

and

$$\dot{\alpha} = 2\ \Delta_e\ \alpha^{1/2p} \sin p\tau\ . \tag{3.44}$$

### 3.2 Amplitude Dependence of the Tune

Using the $\dot{\alpha}$ and $\dot{\tau}$ equations, we can construct an invariant $C$ by requiring that

$$\dot{\tau} = \frac{\partial C}{\partial \alpha} \tag{3.45}$$

and

$$\dot{\alpha} = -\frac{\partial C}{\partial \tau}\ . \tag{3.46}$$

In this case, since $C$ is explicitly independent of $\phi$, $C$ is an invariant in the sense that

$$\dot{C} = \frac{\partial C}{\partial \alpha}\ \dot{\alpha} + \frac{\partial C}{\partial \tau}\ \dot{\tau} = 0\ . \tag{3.47}$$

It is seen in a straightforward way that $C$ is given by,

$$C = \Delta_L \alpha - \xi U(\alpha) + \frac{2}{p}\ \Delta_e \alpha^{p/2} \cos p\tau\ , \tag{3.48}$$

where we have written

$$\Delta_L = \nu - \frac{m}{p}\ , \tag{3.49}$$

and

$$U(\alpha) = \int_0^\alpha F(\alpha) d\alpha\ . \tag{3.50}$$

The resonance invariant defines a set of curves in the phase space $(\alpha, \tau)$ which represent the phase space structure for particle motion near the resonance. There are three terms in the invariant, each represented by a parameter: $\Delta_L$ is the distance of the linear unperturbed tune from resonance; $\xi$ is the strength of the space charge force; and $\Delta_e$ is the resonance stopband width, proportional to the $m^{\text{th}}$ harmonic of the magnetic error field exciting the resonance. It should be noted that in the presence of the space charge force, the resonance tune condition is amplitude dependent. Furthermore $\Delta_L$ need not be small, since the space charge term also contains a linear tune shift which is not included in $\Delta_L$. We therefore have defined $\nu$ as the linear unperturbed tune as determined by the external focusing structure. Under space charge conditions, the particle linear tune is depressed by $\xi$, i.e.

$$\nu_{\text{particle}} = \nu - \xi(\text{small amplitudes}) \ .$$

Also, the tune amplitude dependence is given by

$$\nu_{\text{particle}}(\alpha) = \nu - \xi F(\alpha) \ . \qquad (3.51)$$

Equation (3.51) represent the fact that the tune of the betatron oscillation in the presence of space charge is amplitude-dependent which is shown in Fig. 5. From Fig. 5 it is clear that the detuning is most complete for small amplitude, while at large amplitude there is little effect.

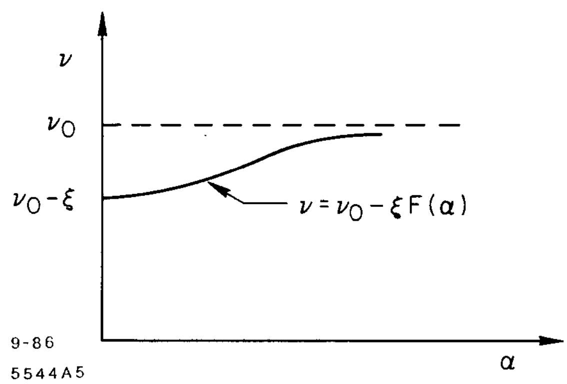


Fig. 5. The amplitude dependence of the space charge detuning. $\nu_0$ is the tune provided by external focusing and $\nu$ is the tune results from space charge.

### 3.3 Particle Behavior Under Resonance and Space Charge Conditions

When particle trajectories in the phase space are isolated points, these are called fixed points. They can be defined by $\dot{\alpha} = \dot{\tau} = 0$. For a resonance of order $p$, the fixed points come in sets of $p$ points. A fixed point is stable if nearby trajectories are elliptical around it, and is unstable if nearby trajectories are hyperbolic and move towards and away from it. For a different approach to the same subject, see Ref. 10. If the phase space structure is defined by the set of invariant curves,

$$C = \Delta_L \alpha - \xi U(\alpha) + \frac{2}{p} \Delta_e \alpha^{p/2} \cos p\tau \ , \tag{3.52}$$

then the fixed points are obtained from

$$\frac{\partial C}{\partial \alpha} = \frac{\partial C}{\partial \tau} = 0 \ . \tag{3.53}$$

To determine the nature of the fixed point, we must expand the function $C$ to second order in deviations from the fixed point. Let $\alpha_F$ and $\tau_F$ be the value of $\alpha$ and $\tau$ at a fixed point. Let $\delta_\alpha = \alpha - \alpha_F$, $\delta\tau = \tau - \tau_F$ be small deviations from the fixed point. Then, a small deviation in $C$ can be expressed by

$$\delta_C = \frac{\partial^2 C}{\partial \alpha^2} (\delta\alpha)^2 + \frac{\partial^2 C}{\partial \tau^2} (\delta\tau)^2 + \frac{\partial^2 C}{\partial\alpha\partial\tau} (\delta\alpha)\ (\delta\tau) \ , \tag{3.54}$$

where derivatives are evaluated at the fixed point. To test whether the fixed point is elliptic or hyperbolic, we rotate the coordinates through the angle $w$ by the relation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \cos w & \sin w \\ -\sin w & \cos w \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\tau \end{bmatrix} \tag{3.55}$$

Choosing the angle $w$ to be,

$$\cot 2w = \frac{\dfrac{\partial^2 C}{\partial\alpha^2} - \dfrac{\partial^2 C}{\partial\tau^2}}{\partial^2 C / \partial\alpha\partial\tau} \ , \tag{3.56}$$

we remove the $pq$ cross term. After some algebraic manipulation, we can show that the 2nd order equation is elliptic or hyperbolic according to the rule,

$$4 \frac{\partial^2 C}{\partial\alpha^2} \left( \frac{\partial^2 C}{\partial\alpha\partial\tau} \right)^2 \tag{3.57}$$

$> 0$ elliptic (stable fixed point, s.f.p.)

$< 0$ hyperbolic (unstable fixed point, u.f.p.).

The phase space structure nearby a stable fixed points are rings of closed curves and that near an unstable fixed points are divergent lines. Figure 6 shows the general features of them in a phase space.

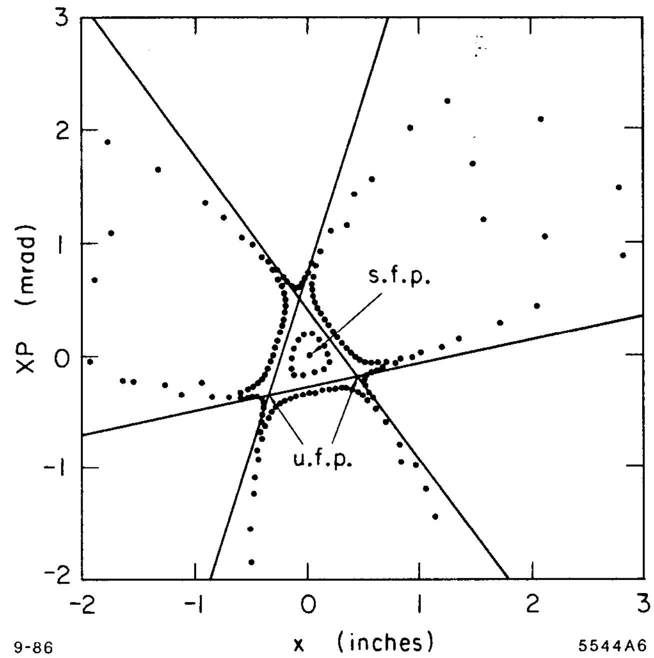


**Fig. 6. Phase space structure of stable fixed point (elliptic) and unstable fixed point (hyperbolic). This is taken from the simulation of AGS third-order resonance extraction.**

The fixed points $(\alpha_F, \tau_F)$ are found from

$$\frac{\partial C}{\partial \alpha} = \frac{\partial C}{\partial \tau} = 0 \ , \qquad \text{or} \tag{3.58}$$

$$\begin{cases} \sin p\tau_F = 0 \ , & (3.59) \\ \Delta_L - \xi F(\alpha_F) \pm \Delta_e \ \alpha_F^{p/2-1} = 0 \ , & (3.60) \end{cases}$$

which does not include the fixed point at $\alpha_F = 0$. To determine the nature of these fixed points, we find the second derivatives,

$$\frac{\partial^2 C}{\partial \alpha^2} = -\xi F'(\alpha) + \left(\frac{p}{2} - 1\right) \ \Delta_e \ \alpha^{p/2-2} \cos p\tau \ , \tag{3.61}$$

$$\frac{\partial^2 C}{\partial \tau^2} = - \ 2p \ \Delta_e \ \alpha^{p/2} \cos p\tau \ , \tag{3.62}$$

and

$$\frac{\partial^2 C}{\partial \alpha \partial \tau} = - \ p \ \Delta_e \ \alpha^{p/2-1} \sin p\tau \ . \tag{3.63}$$

At the fixed point, $\sin p\tau_F = 0$, and

$$\frac{\partial^2 C}{\partial \alpha^2} = -\xi F'(\alpha_F) \pm \left(\frac{P}{2} - 1\right) \Delta_e \ \alpha_F^{p/2-2} \ , \tag{3.64}$$

$$\frac{\partial^2 C}{\partial \tau^2} = \pm \ 2p \ \Delta_e \ \alpha_F^{p/2} \ , \tag{3.65}$$

and

$$\frac{\partial^2 C}{\partial \alpha \partial \tau} = 0 \ . \tag{3.66}$$

Thus, the fixed points at $\alpha_F$ are stable or unstable according as,

$$\pm\xi F'(\alpha_F) - \left(\frac{p}{2} - 1\right) \Delta_e \ \alpha_F^{p/2-2} \quad \begin{array}{ccc} > & 0 & \text{(s.f.p)} \\ < & 0 & \text{(u.f.p)} \end{array} \ . \tag{3.67}$$

or, alternatively

$$\xi F'(\alpha_F) \alpha_F (\Delta_L - \xi F(\alpha_F)) \tag{3.68}$$

$$+ \left(\frac{p}{2} - 1\right) (\Delta_L - \xi F(\alpha_F))^2 \quad \begin{array}{ccc} < & 0 & \text{(s.f.p)} \\ > & 0 & \text{(u.f.p)} \end{array} \ . \tag{3.69}$$

In the case $\xi = 0$, the condition becomes

$$\left(\frac{p}{2} - 1\right) \Delta_L^2 \quad \begin{array}{ccc} < & 0 & \text{(s.f.p)} \\ > & 0 & \text{(u.f.p)} \end{array} \ . \tag{3.70}$$

Therefore, for dipole resonances, $p = 1$, we have stable fixed points only; for quadrupole resonances, $p = 2$, we have no fixed points for $\alpha_F \neq 0$; for sextupole or higher order resonances, $p \geq 3$, we have unstable fixed points only. If $\xi$ is not zero, we can see from the above conditions that if $p \geq 2$ the top sign gives an unstable fixed point since $\xi > 0$ and $F' < 0$. Thus the set of $p$ unstable fixed points are obtained from,

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e \ \alpha_F^{p/2-1} = 0 \ . \tag{3.71}$$

In other words, $\cos p\tau_F = +1$ is for the unstable fixed points.

The separatrix is the phase space trajectory which passes through a set of unstable fixed points. The unstable fixed points are given by

$$\cos p\tau_F = +1 \; ; \quad \Delta_L - \xi F(\alpha_F) + \Delta_e \; \alpha_F^{p/2-1} = 0 \; . \tag{3.72}$$

To evaluate the separatrix, we evaluate $C_F$, the constant, at the fixed points:

$$C_F = \Delta_L \; \alpha_F - \xi U(\alpha_F) + \frac{2}{p} \; \Delta_e \; \alpha_F^{p/2} \; . \tag{3.73}$$

Thus, the separatrix equation is,

$$\Delta_L(\alpha - \alpha_F) - \xi(U(\alpha) - U(\alpha_F)) + \frac{2}{p} \; \Delta_e \; (\alpha^{p/2} \cos p\tau - \alpha_F^{p/2}) = 0 \; , \tag{3.74}$$

or to second order in $\delta\alpha = \alpha - \alpha_F$,

$$r^2 \left\{ 1 + \left(\frac{1}{2}\right) \left(\frac{p}{2}\right) \left(\frac{p}{2} - 1\right) h \cos p\tau \right\} - \left(\frac{P}{2}\right) rh(1 - \cos p\tau) - h(1 - \cos p\tau) = 0 \; , \tag{3.75}$$

where $r = \delta\alpha / \alpha_F$, and

$$h = - \; \frac{4 \; \Delta_e}{\xi \alpha_F F'(\alpha_F) p} \; \alpha_F^{p/2-1} \; . \tag{3.76}$$

For small $h$, we can solve this equation for $r$ and take the leading term, which goes like $\sqrt{h}$, to obtain,

$$r \simeq \pm \; \sqrt{h(1 - \cos \; p\tau)} \; .$$

Thus, for small $h$, the separatrix is a string of $p$ islands around the origin, with a width,

$$w = \pm \; \sqrt{2h} \quad .$$

The illustrations of the phase space structure will be given in the next two sections for specific $p$.

## 3.4 Resonance Phase Space Structure With No Space Charge

In order to see the phase space structure with no space charge, the only thing we have to do is to set $\xi = 0$ in Eq. (3.48), then we can write the set of invariant curves defining the resonance phase space structure as,

$$C = \Delta_L\, \alpha + \frac{2}{p}\, \Delta_e\, \alpha^{p/2} \cos p\tau \ . \tag{3.77}$$

The fixed points for $\alpha_F \neq 0$ can be found from,

$$\Delta_L \pm \Delta_e\, \alpha_F^{p/2-1} = 0 \ . \tag{3.78}$$

If these exist, they will be stable or unstable according to,

$$\left(\frac{p}{2} - 1\right) \Delta_L^2 \quad \begin{matrix} < & 0 & \text{(s.f.p)} \\ > & 0 & \text{(u.f.p)} \end{matrix} \ . \tag{3.79}$$

Consider the case where a particle is on resonance; that is, $\Delta_L = 0$. Then, if $p \geq 2$, the only fixed point is at $\alpha_F = 0$. The phase space structure is then controlled by an unstable fixed point at the origin, with all the trajectories being of the unstable form (that is, trajectories with unlimited amplitude). The set of phase space trajectories are given by

$$\frac{2}{p}\, \Delta_e \alpha^{p/2} \cos p\tau = C \ , \tag{3.80}$$

and the trajectories passing through the unstable fixed point at $\alpha_F = 0$ are given by

$$\alpha^{p/2} \cos p\tau = 0 \ , \quad p \geq 2 \ . \tag{3.81}$$

This is a set of $p$ straight lines passing through the origin of the $(\sqrt{\alpha}, \tau)$ phase plane. Defining the lines for the angle $\tau$ in the range $-\pi/2 < \tau \leq \pi/2$ , we can express the equations of the lines by,

$$\tau = \pm \frac{q\pi}{2p} \ , \quad q = 1, 3, \ldots (p-1) \quad \text{(for even } p) \ ; \tag{3.82}$$

and

$$\tau = \begin{cases} \pm \frac{q\pi}{2p} \ , & q = 1, 3, \ldots (p-2) \\ \pi/2 \ , & q = p \end{cases} \quad \text{(for odd } p) \ . \tag{3.83}$$

For example, for a dipole resonance, $p = 1$, the phase space trajectories are given by

$$C = \Delta_L\ \alpha + 2\ \Delta_e \sqrt{\alpha}\ \cos\tau\ , \tag{3.84}$$

with a stable fixed point given by,

$$\Delta_L \pm \frac{\Delta e}{\sqrt{\alpha_F}} = 0\ . \tag{3.85}$$

Thus we see that for $\Delta_L \neq 0$, there is a stable fixed point at

$$\sqrt{\alpha_F} = \left|\frac{\Delta_e}{\Delta_L}\right|\ , \quad \text{with}\ \tau = \begin{cases} 0, & \text{if} \quad \Delta_L < 0 \\ \pi, & \text{if} \quad \Delta_L > 0 \end{cases}\ . \tag{3.86}$$

The trajectories are circles around this fixed point. At the resonance, $\Delta_L = 0$, and the phase space structure degenerates into the straight lines

$$\sqrt{\alpha}\ \cos\tau = \ \text{constant}\ . \tag{3.87}$$

Notice that for the dipole resonance, the fixed point at $\sqrt{\alpha} = 0$ vanishes because of the resonance. All other resonance orders retain the $\sqrt{\alpha} = 0$ fixed point. This is easily seen by recognizing that to get the fixed points, $\partial C/\partial\sqrt{\alpha}$ must be set to zero, rather than $\partial C/\partial\alpha$. Of course, for the $\sqrt{\alpha} \neq 0$ fixed points, using the latter is appropriate.

The phase space structure close to resonance condition for $p = 1, 2$ and 3 with no space charge are illustrated in Fig. 7.

### 3.5 Resonance Phase Space Structure with Space Charge

If we include space charge, we have for the $p$ unstable fixed points,

$$\Delta_L - \xi F(\alpha_F) + \Delta_e\ \alpha_F^{p/2-1} = 0\ . \tag{3.88}$$

On the other hand, the $p$ stable fixed points can be found from the expression,

$$\Delta_L - \xi F(\alpha_F) - \Delta_e\ \alpha_F^{p/2-1} = 0\ . \tag{3.89}$$

If a particle is oscillating in the phase space not close to the fixed points, i.e. not near the resonance, then the behavior is simply

$$\dot{\alpha} = 0\ , \tag{3.90}$$

and

$$\dot{\psi} = \nu - \xi_F(\alpha)\ , \tag{3.91}$$

where $\nu$ is the externally applied betatron tune and $\xi$ is the space charge strength. $\dot{\psi}$ is simply a perturbed tune for the amplitude $\alpha$. Calling this

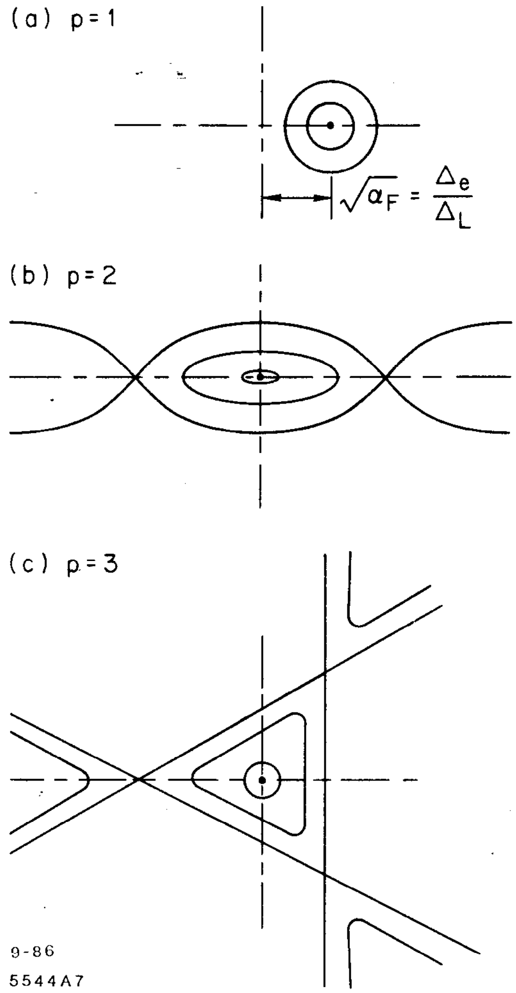


Fig. 7. The phase space structure close to resonance of order $p = 1, 2$ and 3 with no space charge force present. If right at resonance, for $p = 1$ the s.f.p. moves to infinity and for $p = 2$ and 3, the s.f.p. turn into u.f.p. with no stable region.

$\nu_p(\alpha)$, we have,

$$\nu_p(\alpha) = \nu - \xi F(\alpha) \ . \tag{3.92}$$

At amplitudes near the fixed points, i.e. near the resonance, the phase space structure is generally a string of $p$ islands around the origin. The fixed points define the amplitude at which these islands are located. The amplitude variation can be found from the approximate expression for the separatrix,

$$\frac{\Delta\alpha}{\alpha_F} = \pm \sqrt{h(1 - \cos p\tau)} \ . \tag{3.93}$$

This means that the amplitude modulation around the fixed points, or the extent of the resonance in amplitude, is given by

$$\frac{(\Delta\alpha)_{\max}}{\alpha_F} = \pm\sqrt{2h} \ , \tag{3.94}$$

where

$$h = -\frac{4\Delta_e \alpha_F^{p/2-1}}{p\xi\alpha_F F'(\alpha_F)} \; . \tag{3.95}$$

*For large space charge strength, $h \ll 1$, and we see that the space charge has stabilized the resonance. Without space charge the resonances fill the entire phase space with unstable trajectories. Adding space charge reduces the impact to a small phase space amplitude region.*

Consider the calculation for the case where the unstable fixed points are at $\alpha_F = 1$, corresponding to the beam edge (i.e. $\sqrt{2}\,\sigma_{\rm rms}$). We have,

$$\Delta_L - \xi\ F(1) + \Delta_e = 0 \; , \tag{3.96}$$

or

$$\Delta_L = \xi\ F(1) - \Delta_e \; . \tag{3.97}$$

Since it is assumed that $\xi \gg \Delta_e$, both the stable and unstable fixed points are very near $\alpha = 1$, and the tune required for this condition is

$$\Delta_L = \xi\ F(1) \; . \tag{3.98}$$

In this case the maximum amplitude modulation around $\alpha_F = 1$ is

$$(\Delta\alpha)_{\max} = \pm\sqrt{\frac{8\Delta_e}{p\xi|F'(1)|}} \; . \tag{3.99}$$

A good approximation to $F(\alpha)$ is

$$F(\alpha) = \frac{1}{1+\alpha/2} \; , \tag{3.100}$$

which satisfies $F(0) = 1$, and $F(\infty) \to 2/\alpha$. Using this form, $F(1) = 2/3$ and $F'(1) = 2/9$, where $F'(\alpha) = -\frac{1}{2(1+\alpha/2)^2}$. We therefore have:

$$\Delta_L = \frac{2}{3}\,\xi \; , \quad \text{and} \quad (\Delta\alpha)_{\max} = \pm\sqrt{\frac{36\Delta_e}{p\xi}} \; . \tag{3.101}$$

#### 3.5.1 $p \geq 3$ Case

As $\Delta_L$ approaches the resonance, $\Delta_L = \xi$, the fixed points shift toward the origin and the islands shrink in width. For $p \geq 3$, and $\Delta_L = \xi$, the fixed points are determined by

$$\xi(1 - F(\alpha_F)) \pm \Delta_e \; \alpha_F^{p/2-1} = 0 \; , \tag{3.102}$$

or

$$\alpha_F = 0 \; . \tag{3.103}$$

At $\alpha_F = 0$, we have,

$$(\Delta\alpha)_{\max} = \pm\alpha_F\sqrt{2h} \rightarrow 0 \; . \tag{3.104}$$

The phase space structure is sketched in Fig. 8(c).

#### 3.5.2 $p = 2$ Case

For the quadrupole resonance, $p = 2$, the fixed points are determined by,

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e = 0 \; , \tag{3.105}$$

and the set of invariant curves are given by,

$$C = \Delta_L \alpha - \xi U(\alpha) + \Delta_e \; \alpha \cos 2\tau \; . \tag{3.106}$$

To see the nature of the fixed points, write the fixed point equation in the form,

$$\Delta_L - \xi + \xi(1 - F(\alpha_F)) \pm \Delta_e = 0 \; , \tag{3.107}$$

noting that $1 - F(\alpha_F) \geq 0$ for $\alpha_F \geq 0$. Then we have,

- For $\Delta_L - \xi > \Delta_e$, no fixed points. (The origin is a stable fixed point.)
- For $\Delta_L - \xi < -\Delta_e$, 2 sets of fixed point pairs, one stable, the other unstable (islands strung around the origin). (The origin is a stable fixed point.)
- For $-\Delta_e < \Delta_L - \xi < \Delta_e$, one set of stable fixed points. (The origin is an unstable fixed point.)

Thus, the no-space-charge-stopband still exists in a sense. If the linear particle tune (small amplitude) is above the stopband there are no fixed points. That is, we have distorted circles of varying tune in phase space. Below the stopband, islands develop around the origin. We have calculated the width of

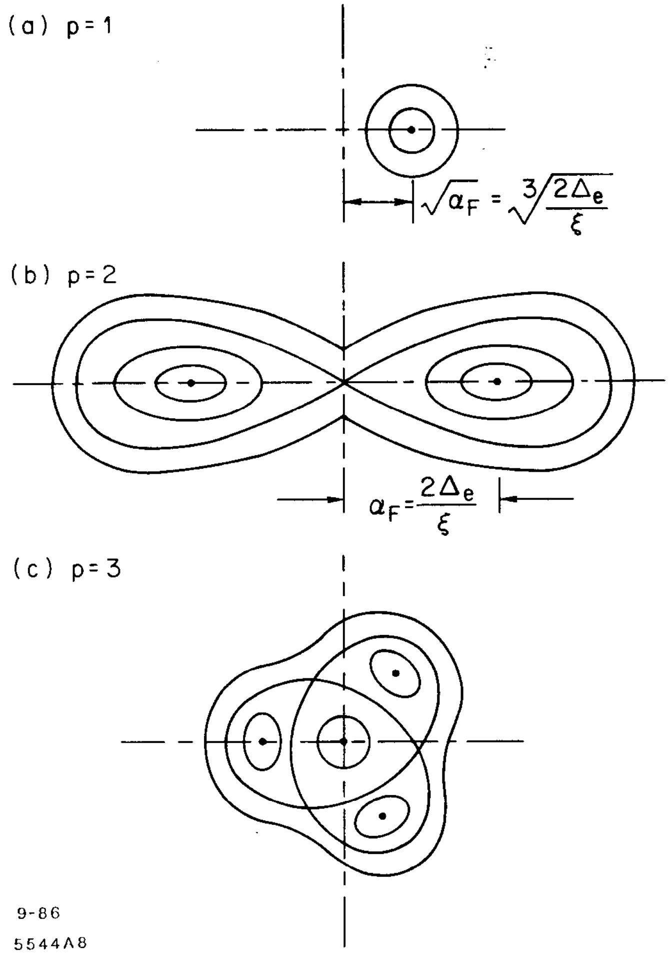


Fig. 8. The phase space structure close to resonance of order $p = 1, 2$ and 3 in the presence of space charge force. If $\xi$ is not zero, stable solutions always exist even at resonance.

these islands before, using an expansion about the unstable fixed points for $\alpha_F \neq 0$. We obtained a width given by

$$(\Delta\alpha)_{\max} = \pm\sqrt{\frac{4\Delta_e\alpha_F}{3|F'(\alpha_F)|}} \quad . \tag{3.108}$$

Again, if the tune is chosen so that the fixed point is at the beam edge, $\alpha_F = 1$, we have

$$(\Delta\alpha)_{\max} = \pm\sqrt{\frac{18\Delta_e}{\xi}} \quad . \tag{3.109}$$

This is a new effect introduced by the space charge force. Inside the stopband, the space charge force adds a stable fixed point to the already existing unstable fixed point at the origin, leading to a "figure-8" type separatrix. Since the separatrix passes through the origin, it is found by taking the constant $C = 0$, that is

$$\Delta_L \alpha - \xi U(\alpha) + \Delta_e \alpha \, \cos 2\tau = 0 \ , \tag{3.110}$$

To determine the general nature of the phase space structure, we use the approximate expression for $F(\alpha)$,

$$F(\alpha) = \frac{1}{1 + \alpha/2} \ . \tag{3.111}$$

Solving for the fixed points, we obtain

$$\frac{1}{2} \, \alpha_F = \frac{\xi}{\Delta_L \pm \Delta_e} - 1 \ . \tag{3.112}$$

The stable fixed point when the tune is inside the stopband is found using the "–" sign. At the center of the stopband, $\Delta_L = \xi$, and we have for the stable fixed point,

$$\frac{1}{2} \, \alpha_F = \frac{\Delta_e}{\xi - \Delta_e} \approx \frac{\Delta_e}{\xi} \ , \tag{3.113}$$

and for the separatrix

$$\xi(\alpha - U(\alpha)) + \Delta_e \, \alpha \cos \, 2\tau = 0 \ . \tag{3.114}$$

Expand about the unstable fixed point at $\alpha = 0$. Then,

$$U(\alpha) = \alpha - \frac{1}{4} \, \alpha^2 + \ldots \ . \tag{3.115}$$

Thus, we have for the separatrix,

$$\alpha = - \frac{4\Delta_e}{\xi} \cos 2\tau \ . \tag{3.116}$$

Thus, this is "vertical figure-8", with the stable fixed points along $\tau = \pi/2$ and $\tau = 3\pi/2$, with maximum amplitudes along this line given by

$$\alpha_{\max} = \frac{4\delta_e}{\xi} \ . \tag{3.117}$$

To the extent that $\alpha_{\max}$ is small, i.e. $\xi \gg \delta_e$, the space charge force has bounded the resonant trajectories to regions in amplitude around the origin of order $\Delta_e/\xi$. The phase space structure is sketched in Fig. 8(b).

### 3.5.3 $p = 1$ Case

For the dipole resonance, $p = 1$, the fixed points are determined by

$$\Delta_L - \xi F(\alpha_F) \pm \Delta_e/\sqrt{\alpha_F} = 0 \ . \tag{3.118}$$

It should be noted that in the dipole case there is no added fixed point at the origin. This is a reflection of the fact that dipole resonances directly affect the closed orbit. The set of resonant trajectories are given by,

$$C = \Delta_L\ \alpha - \xi U(\alpha) + 2\Delta_e\ \alpha^{1/2} \cos\tau \ . \tag{3.119}$$

To see the general nature of the fixed point structure, write the fixed point equation in the form,

$$\Delta_L - \xi + \xi(1 - F(\alpha_F)) \pm \frac{\Delta e}{\sqrt{\alpha_F}} = 0 \ . \tag{3.120}$$

The point $\alpha_F = 0$ is never a fixed point as long as $\Delta_e \neq 0$. On resonance, $\Delta_L - \xi = 0$, and the fixed point which was at infinite amplitude when there was no space charge has moved to a finite amplitude which can be found from the equation,

$$\sqrt{\alpha_F}\ (1 - F(\alpha_F)) = \frac{\Delta_e}{\xi} \ , \tag{3.121}$$

where only the bottom sign in the fixed point equation gives a fixed point, which is a stable fixed point. An approximate solution can be found by noting that for $\Delta_e/\xi$ small, the solution must have $\alpha_F < 1$. So, expand about $\alpha_F = 0$, giving the equation

$$|F'(0)|\ \alpha_F^{3/2} = \frac{\Delta_e}{\xi} \ , \tag{3.122}$$

or

$$\sqrt{\alpha_F} = \left(\frac{\Delta_e}{\xi|F'(0)|}\right)^{1/3} = \left(\frac{2\Delta_e}{\xi}\right)^{1/3} \ . \tag{3.123}$$

Now, if $\Delta_L - \xi > 0$, this fixed point must move closer to the origin since a solution requires that the term $(-\Delta_e/\sqrt{\alpha_F})$ to increase in magnitude. The phase space structure is illustrated in Fig. 8(a).

We can estimate the fixed point structure as a function of $\Delta_L - \xi$ by using the approximate form of $F$ or by expanding $F$ to lowest order in $\alpha$ since all the fixed points will be close to $\alpha = 0$ if $h \ll 1$. Writing,

$$1 - F = \frac{\alpha}{2+\alpha} \approx \frac{\alpha}{2} \quad \text{if} \quad \alpha \ll 1 \ , \tag{3.124}$$

and defining

$$h = \frac{\Delta_e}{\xi} \ , \tag{3.125}$$

$$\epsilon = \frac{\Delta_L - \xi}{\xi} \ , \tag{3.126}$$

$$\chi = \sqrt{\alpha} \ , \tag{3.127}$$

we have for the fixed point equation,

$$\chi^3 + 2\epsilon\chi \pm 2h = 0 \ . \tag{3.128}$$

Consider the fixed points as a function of $\epsilon = (\Delta_L - \xi)/\xi$. On resonance, $\epsilon = 0$, and we have the solution,

$$\chi = (2h)^{1/3} \ . \tag{3.129}$$

To complete the analysis of the dipole resonance, the cubic equation for the fixed points has been solved off resonance ($\epsilon \lesseqgtr 0$) as well, with the conclusion that the space charge force does indeed preserve a stable region of phase space near the origin. In fact, the worst situation is on resonance, where the fixed point moves from infinity (no space charge stabilization) to a distance $(2h)^{1/3}$ from the origin, i.e. a distance on the order of $(\Delta e/\xi)^{1/3}$.

Recently I. Hofman and K. Beckent wrote a simulation program to look into the resonance behavior of particles under space charge force with self-consistent charge distribution. They found out the basic fact we showed here that the space charge force tend to stabilize the resonance. Their simulation shows that if the dipole resonance is crossed by adjusting external tune $\nu_0$, the bunch experiences a position offset. However, if the crossing is due to space charge detuning, the centroid of the bunch never experiences any movement as shown in Fig. 9.[11]

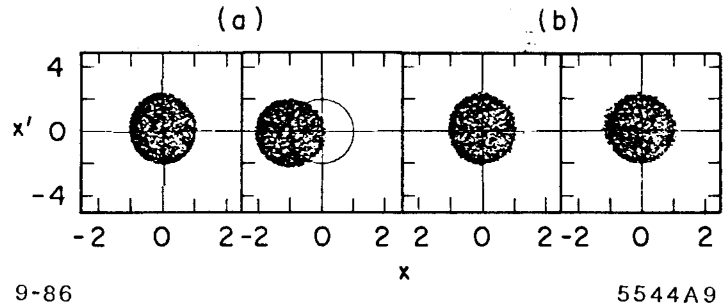


Fig. 9. Phase space projections for dipole error before and after integer crossing. (a) crossing of $\nu_0$; (b) crossing of $\nu$ ($\nu_0$ fixed).

### 3.6 Comparison Between Uniform and Bunched Beam

From the above analysis, we can draw some general observations about beam behavior under space charge condition. A uniform current beam has the characteristic that all particles are mediated by the same space charge strength $\xi$. Particles of different amplitudes will have different tunes, but the force is the same for all particles. A second point is that, except for scattering and noise effects, the linear tune remains constant for each particle, although the beam will have a spread in linear tune due to the chromaticity.

If the linear tune of a particle is fixed, and if the space charge strength is fixed, to determine stability we need only examine the "fixed" phase space structure. In the case of large current, or large $\xi$, we have seen that the resonances of all orders have only a minor impact on the phase space structure. Thus, even on resonance, the beam will be stabilized by the strong resonance detuning effect of the space charge force.

When a beam is bunched by an applied radiofrequency field, two effects manifest themselves, which change this picture of space charge induced stability. The most important impact is that the space charge strength $\xi$ is a function of the local current density, which is a constant in the non-bunched case. At both ends of the bunch, the local current drops to zero. Thus, the detuning effect of the space charge force disappears for these particles and if they are on a resonance, the phase space structure will be strongly affected as we have previously seen. A second important effect of bunching is the synchrotron motion of particles around the bunch. In other words, particles rotate around the center of the bunch and in particular, those of large synchrotron amplitude move from front to center to back and so on. Thus, a particle can move from a region where it is space charge stabilized to one where it is resonance vulnerable. The problem is to control the tune such that uncorrected resonances are only located around the bunch center where space charge stabilization is effective; while the tunes corresponding to the bunch ends are kept free of such resonances. This becomes increasingly difficult as $\xi$ increases since the bunch tune spread from center to ends is on the order of $\xi$.

Application of this model to low energy beam capture in a synchrotron leads to the conclusion that in the bunching phase if the beam in tune space must be located on an uncorrected resonance, then it is stable if the locally dense portion of the beam is directly on the resonance and the locally dilute portion of the beam is free of this resonance. That observation is consistent with the practices at AGS and PSB as shown in Figs. 3 and 4.

To compare the detailed prediction presented above with the experimental observation, the theory has to be extended to include coupling from the other transverse dimension. Which we plan to work out in the future; in the meantime, there is some effort spent in this direction using computer simulation by G. Parzen[12] at Brookhaven.

## 4. BEAM–BEAM FORCE AND DETUNING

In a colliding beam storage ring, a major source of nonlinear resonance excitation resides in the beam-beam collisions as the stored beams repeatedly cross each other. During such collisions, particles in each beam see the electromagnetic field generated by the counter rotating beam. The beam-beam collisions therefore perturb the particle motion, causing

1. the transverse beam size to blow up and loss of luminosity,
2. the beam lifetime to be reduced,
3. and rapid beam loss as beam intensity increases beyond a more or less distinct threshold.

Considerable efforts, both experimental and theoretical, have brought some insight into the beam-beam instability problem and often led to improvements in luminosity. But the nature of the instability and its associated underlying mechanism(s) are not yet fully understood and remain an outstanding problem for the designers of colliding beam storage rings.

Nonlinear resonance behavior is the underlying process of beam-beam interaction, this chapter will identify the role of nonlinear resonances in the beam-beam problem. To reduce the scope of this effect, we will consider only head-on collisions of bunched beams. For the sign convention of the beam-beam force, we assume the two colliding beams to have opposite charges. We will emphasize on the detuning effect and its comparison to similar effects introduced by space charge force. For more complete coverage, the articles by Kheifets[13] and Chao et al.[14] are recommended.

In this section we will develop the single resonance analysis that is applicable to the strong-weak case of the beam-beam interaction. In this case, the strong beam is unperturbed by the beam-beam interaction; motions of the weak beam particles are then analysed in the presence of the nonlinear electromagnetic force produced by the strong beam at the collision points.

## 4.1 Beam-Beam Force and Resonances

In the strong-weak picture, the motion of a test particle in the weak beam is governed by the Hamiltonian[14]

$$\begin{aligned} H &= H_0 + H_1 \\ &= \frac{1}{2}\left(p_x^2 + K_x x^2\right) + \frac{1}{2}\left(p_y^2 + K_y y^2\right) + U(x,y)\,\delta_p(s) \end{aligned} \tag{4.1}$$

where $U(x,y)$ is the equivalent potential produced by the strong beam, $x$ and $y$ are the horizontal and vertical coordinates that describe the test particle motion. The $\delta$-function represents the periodic collisions with a period $2\pi R/S$ where $R$ is the average radius of the storage ring and $S$ is the number of collision points around the ring. The unperturbed Hamiltonian $H_0$ represents the usual two dimensional betatron motions with focussing structures described by $K_x(s)$ and $K_y(s)$.

The equations of motion described by the Hamiltonian (4.1) are

$$\frac{d^2 z}{ds^2} + K_z(s) z = -\frac{\partial U}{\partial z}\,\delta_p(s)\ , \quad z = x, y \tag{4.2}$$

The potential $U$ depends on the distribution of the strong beam at the collision points. Assuming the strong beam has an upright bi-gaussian distribution, the potential can be written as[9]

$$U(x,y) = \frac{N r_e}{\gamma} \int_0^\infty dt\, \frac{1 - \exp\left[-\dfrac{x^2}{2\sigma_x^2 + t} - \dfrac{y^2}{2\sigma_y^2 + t}\right]}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \tag{4.3}$$

where $r_e = e^2/mc^2$ is the classical radius of the particle, $\gamma$ is the relativistic factor, $N$ is the number of particles per bunch and $\sigma_{x,y}$ are the rms beam dimensions of the strong beam at the collision points. Throughout this section on strong-weak single resonance treatment, we will assume the gaussian potential given by Eq. (4.3). Equation (4.3) is identical to Eq. (3.1) for space charge force, the only difference is the sign of the force and the spatial distribution along the ring.

Equations (4.1) and (4.2) can be solved in various stages of approximations and sophistications. The simplest treatment is to consider only the linear effects by Taylor expanding $U(x,y)$ and keeping only terms quadratic in $x$ and $y$. The problem is then solved exactly in the same way as ordinary gradient perturbations. The linear beam-beam perturbations give rise to betatron tune shifts $\xi_{x,y}$ which are given by

$$\xi_z = \frac{N r_e \beta_z^*}{2\pi\gamma\sigma_z(\sigma_x+\sigma_y)} \,, \qquad z = x, y \tag{4.4}$$

where $\beta_{x,y}^*$ are the betatron functions at the collision point. This effect was first pointed out by F. Amman and D. Ritson.[15]

In the linear approximation, the $x$- and the $y$-motions are decoupled. The motion in each dimension is completely determined by two parameters, i.e. the betatron tune per revolution $\nu$ and the beam-beam strength parameter $\xi$. The simplest resonance effect manifest itself when $\nu$ is sufficiently close to a half integer, the particle motion becomes unstable due to the gradient perturbation of the beam-beam force.

When the complete potential $U$ is taken into account, the particle motion is affected by the beam-beam perturbation whenever a nonlinear resonance condition is approximately satisfied

$$2n\nu_x + 2m\nu_y + k \approx 0 \tag{4.5}$$

where $n$, $m$ and $k$ are integers. The even coefficients in front of $\nu_x$ and $\nu_y$ is due to the polarity of the beam-beam force. Resonances with odd coefficients are not excited except for non-head-on collisions, which we do not consider in this lecture. Extension of single resonance analysis to include the non-head-on cases is straightforward with proper modifications on the beam-beam force.

To treat the general strong-weak problem, it is a matter of taste whether to start with the Hamiltonian (4.1) or with the equations of motion (4.2), each gives the same answer. The Hamiltonian approach will be adopted here. In the following we will assume that $\xi_x = \xi_y = \xi$.

The first step is to make a canonical transformation on the Hamiltonian $H$ to remove the time-dependence from $H_0$, thus defining an equivalent harmonic oscillator with frequencies $\nu_x$ and $\nu_y$. The transformation has the generating function

$$F_1(x,y,\phi_x,\phi_y) = -\frac{1}{2}\sum_{z=x,y}\frac{z^2}{\beta_z(s)}\left[\tan\varphi_z - \frac{\beta_z'(s)}{2}\right]$$

with

$$\varphi_z = \phi_z + \int_0^s ds' \left[\frac{1}{\beta_z(s')} - \frac{\nu_z}{R}\right] \tag{4.6}$$

where $\beta_x(s)$ and $\beta_y(s)$ are the betatron functions defined by Courant and Snyder.[1]

The relation between the old coordinates $(z, p_z)$ and new coordinates $(J_z, \phi_z)$ is

$$z = \sqrt{2J_z\beta_z}\,\cos\phi_z$$

$$p_z = -\sqrt{\frac{2J_z}{\beta_z}}\left(\sin\phi_z - \frac{\beta_z'}{2}\cos\phi_z\right)\ , \quad z = x, y \tag{4.7}$$

We then normalize the action variables by

$$\alpha_z = \frac{J_z}{\epsilon_x + \epsilon_y}\ , \quad z = x, y \tag{4.8}$$

where $\epsilon_{x,y} = \sigma_{x,y}^2/\beta_{x,y}^*$ are the natural emittances of the strong beam. We also will change the time variable s to the azimuthal angle $\theta = s/R$.

Assuming equal linear tune shifts in the $x$ and $y$ planes, the new Hamiltonian becomes

$$H(\phi_x, \phi_y, \alpha_x, \alpha_y, \theta) = \nu_x\alpha_x + \nu_y\alpha_y + \frac{Nr_e}{2\pi\gamma}\,\frac{S}{\epsilon_x + \epsilon_y}\sum_{k=-\infty}^{\infty} e^{ikS\theta}$$
$$\times\int_0^\infty dt\,\frac{1 - \exp\left[-\alpha_x\,\frac{a+1}{at+1}\cos^2\phi_x - \alpha_y\,\frac{a+1}{a+t}\cos^2\phi_y\right]}{\sqrt{\left(\frac{1}{a}+t\right)(a+t)}} \tag{4.9}$$

where $a = \sigma_y/\sigma_x$ is the aspect ratio of the strong beam distribution. Note that the periodic delta-function in s has been replaced by infinite series of sinusoidal terms in $\theta$.

So far the manipulation on the Hamiltonian has been only mathematical. The physics comes in the next step — the "smooth approximation". To do that, we assume there is one and only one dominating nonlinear resonance that determines the motion of the weak beam particles.

Let the resonance be that of Eq. (4.5). Note that the Hamiltonian (4.9) contains complicated dependences on $\theta$, $\phi_x$ and $\phi_y$. In the smooth approximation, we need to remove the "fast oscillating" terms and extract only the "slowly varying" terms in the Hamiltonian. To do so, a triple Fourier expansion in $\theta$, $\phi_x$ and $\phi_y$ is performed on (4.9). Keeping only the slowly varying terms, we obtain a new Hamiltonian

$$H(\phi_x, \phi_y, \alpha_x, \alpha_y, \theta) \approx \nu_x\alpha_x + \nu_y\alpha_y + S\xi\,[H_{00}(\alpha_x, \alpha_y) \\ + 2(-1)^{n+m} H_{nm}(\alpha_x, \alpha_y)\cos(k\theta + 2n\phi_x + 2m\phi_y)] \tag{4.10}$$

There are two beam-beam terms in (4.10). The first term is independent of $\theta$, $\phi_x$ and $\phi_y$. The second term contains the Fourier component in the slow variable $kS\theta + 2n\phi_x + 2m\phi_y$. The functions $H_{00}(\alpha_x, \alpha_y)$ and $H_{nm}(\alpha_x, \alpha_y)$ are the Fourier components of the beam-beam perturbation

$$H_{nm} = \int_0^\infty dt\, \frac{P_{nm} - \exp\left[-\frac{\alpha_x}{2}\frac{1+a}{1+at} - \frac{\alpha_y}{2}\frac{a+1}{a+t}\right]}{\sqrt{\left(\frac{1}{a}+t\right)(a+t)}} \\ \times I_n\left(\frac{\alpha_x}{2}\frac{1+a}{1+at}\right) I_m\left(\frac{\alpha_y}{2}\frac{1+a}{a+t}\right) \tag{4.11}$$

where $P_{nm} = 1$ if $n = m = 0$ and 0 otherwise, $I_n$ and $I_m$ are Bessel functions.

There are two invariants for the smoothed Hamiltonian (4.10). The first one is

$$C = -m\alpha_x + n\alpha_y \tag{4.12}$$

For a one-dimensional resonance ($n = 0$ or $m = 0$), it trivially means that the other dimension is not affected, and is thus redundant in the treatment. For a two-dimensional resonance, it expresses the exchange of energy between the two coupled dimensions under the constraint of (4.12) and renders the problem effectively one-dimensional.

We now perform another canonical transformation using the generating function

$$F_2(\phi_x, \phi_y, K, C, \theta) = -\frac{1}{4mn}\left[K(2n\phi_x + 2m\phi_y + k\theta) + C(2n\phi_x - 2m\phi_y)\right] \tag{4.13}$$

The dynamical variables for the effective coupled dimension are

$$\begin{cases} K = -\,(m\alpha_x + n\alpha_y) \\ \psi = -\dfrac{1}{4mn}\left[2n\phi_x + 2m\phi_y + k\theta\right] \end{cases} \tag{4.14}$$

where $\psi$ is the slow phase. The corresponding Hamiltonian is[14]

$$H(K, \psi) = -\frac{K\delta\nu}{2mn} + S\xi\left[H_{00}(K) + 2(-1)^{m+n} H_{nm}(K)\cos(4mn\psi)\right] \tag{4.15}$$

where $\delta\nu = 2n\nu_x + 2m\nu_y + k$ specifies the distance from the exact location of the resonance. Note that this Hamiltonian is independent of the time variable

$\theta$, meaning it is the second constant of the motion. The functions $H_{00}$ and $H_{nm}$ are the characteristic functions of the beam-beam problem in the single resonance picture.

### 4.2 BEAM-BEAM DETUNING

The phase space structure for the motion described by the Hamiltonian (4.10) or (4.15) depends on the behavior of the functions $H_{00}$ and $H_{nm}$. In the absence of all resonances, the detuning term gives the effective tune shifts as functions of the oscillation amplitudes $\alpha_{x,y}$, i.e.

$$\Delta\nu_{x,y}(\alpha_x, \alpha_y) = S\xi \, \frac{\partial H_{00}(\alpha_x, \alpha_y)}{\partial \alpha_{x,y}} \quad . \tag{4.16}$$

The tune shifts at vanishing amplitudes are simply given by $\xi$ per beam-beam crossing in both dimensions. For larger amplitudes the tune shifts become smaller. The detuning mechanism is schematically shown in Fig. 10.

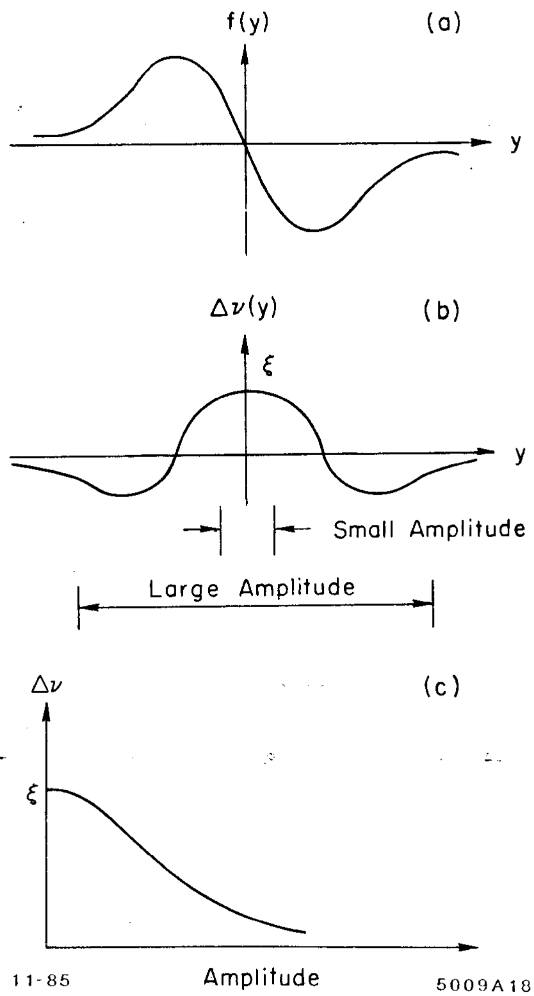


Fig. 10. Schematic illustration of the beam-beam tune shift mechanism. (a) shows the beam-beam force. (b) is the slope of this force. The tune shift is obtained by averaging $\partial f/\partial y$ over the range reached by a given amplitude. (b) also shows two such ranges, one for a small amplitude particle and one for a large amplitude particle. The result after averaging gives the beam-beam detuning curve which looks like (c).

In addition to a net shift (4.16), the instantaneous rate of change in the phase variable $\psi$ contains a slowly varying term proportional to $H'_{nm}(K)$, according to the Hamiltonian (4.15). The width of the resonance (in $n\nu_x + m\nu_y$ unit) can be defined to be

$$W_{nm}(K) = 4mn\ H'_{nm}(K) \quad . \tag{4.17}$$

It should be pointed out that (4.17) is the simplest possible definition of resonance width. More sophisticated definitions taking more carefully into account the phase space structure also exist, but will not be considered in the following analysis.

Due to the detuning mechanism, the beam-beam force introduces a spread in the weak beam tunes. The working point specified by the unperturbed tunes becomes a working area in the tune space. Figure 11 shows this behavior for three different values of the aspect ratio $a$. In the presence of nonlinear resonances, the working area should avoid resonance lines, according to the single resonance model. The working area therefore needs to fit into a "resonance free" region in the tune space, as shown in Fig. 11(d). For a flat beam with small aspect ratio, an inspection of the shape of the working area in Fig. 11 shows that it is better to choose the unperturbed working point to lie on the lower right side of the destructive resonances than to the upper left side. In particular, when applied to the diagonal $2\nu_x - 2\nu_y = n$ resonance, this means the unperturbed working point should be below the resonance line.

However, having a resonance line trespassing the working area does not necessarily mean instability of particle motion. The stability depends on the phase space structure which in turn depends on the behavior of both the tune shift (4.16) and the resonance width (4.17). Figure 12 shows three typical situations, each gives rise to a qualitatively different phase space structure, and therefore different stability behavior. For instability, the resonance width must dominate the tune shift. Otherwise a particle temporarily finds itself in resonance will grow in amplitudes, but a larger amplitude means a large tune shift which automatically brings it out of the resonance. The beam-beam interaction, it turns out, is one in which the tune shift dominates and therefore does not cause instability.

The tune shift and the resonance width as functions of amplitude for the beam-beam interaction looks like that sketched in Fig. 12(c). More quantitatively,[17] let us consider a round beam with $a = 1$ and consider a particle with no horizontal motion, i.e. $\alpha_x = 0$ and $\alpha_y = \alpha$. The tune shift

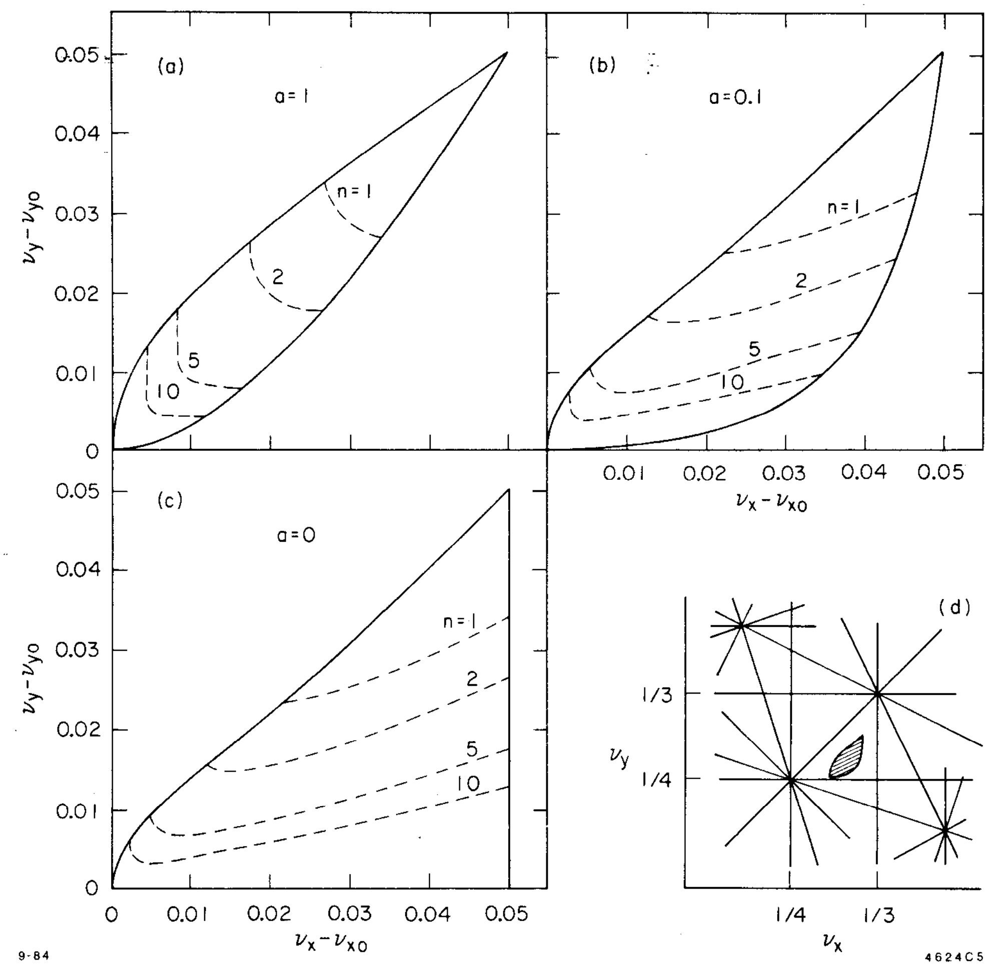


Fig. 11. Beam-beam tune spreads. We assume the two beams have opposite charges. $(\nu_{x0}, \nu_{y0})$ is the unperturbed working point. With beam-beam collisions, the working point extends into a working area. The dotted lines are the contours for particles with amplitudes satisfying $x^2/\sigma_x^2 + y^2/\sigma_y^2 = n^2$. We assume $\xi_x = \xi_y = 0.05$. Case (a) is when the aspect ratio is $a = 1$, i.e. a round beam. Case (b) is when $a = 0.1$, i.e. a flat beam. Case (c) gives the result in the limit $a = 0$. (d) shows fitting the working area (shaded region) into a resonance free region in the tune space.

and the width functions are given by

$$\begin{aligned} \Delta\nu(\alpha) &= S\xi \, \frac{1 - e^{-\alpha} I_0(\alpha)}{\alpha} \\ W_n(\alpha) &= 4S\xi \, \frac{e^{-\alpha} I_n(\alpha)}{\alpha} \end{aligned} \tag{4.18}$$

These functions are plotted in Fig. 13, taken from Ref. 16. It is seen that

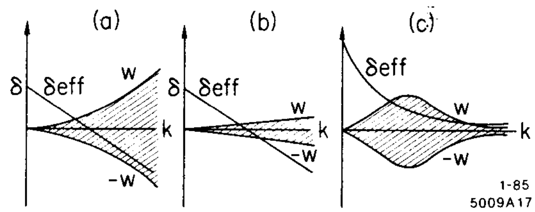


**Fig. 12.** A few typical detuning and width functions: (a) magnetic multipoles with weak detuning; (b) magnetic multipoles with strong detuning; and (c) beam-beam interaction.

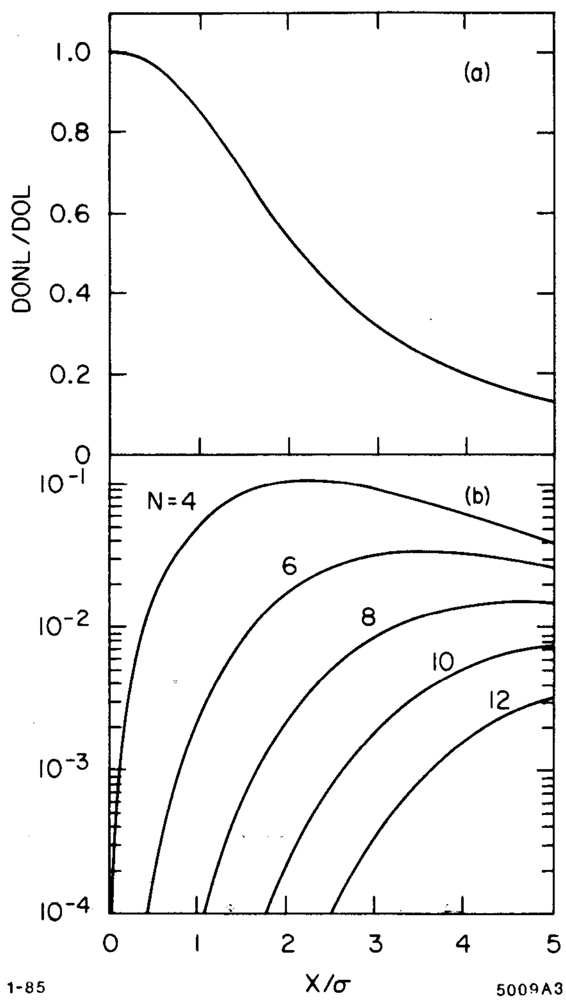


**Fig. 13.** Beam-beam detuning (a) and width (b) in the case of a round beam.

the resonance widths are typically much smaller than the tune shift and they decrease quickly with increasing resonance order $n$. Furthermore the asymptotic behavior of (4.18) for large $\alpha$ is $\Delta\nu \to 1/\alpha$, $W_n \to 1/\alpha^{3/2}$, meaning the tune shift always dominates the width for large amplitudes. The phase space structure is therefore closed and motion necessarily stable. In fact, only the lowest order resonances are capable of producing large islands in phase space and even then the islands are usually not large enough to cause beam loss. To explain the beam-beam instability, something more has to be added to the single resonance picture. Figure 14 demonstrate the situation by a computer simulation. A fourth order resonance is being studied. Case (c) has a phase space structure like Fig. 12(b) while case (d) has the structure like Fig. 12(c).

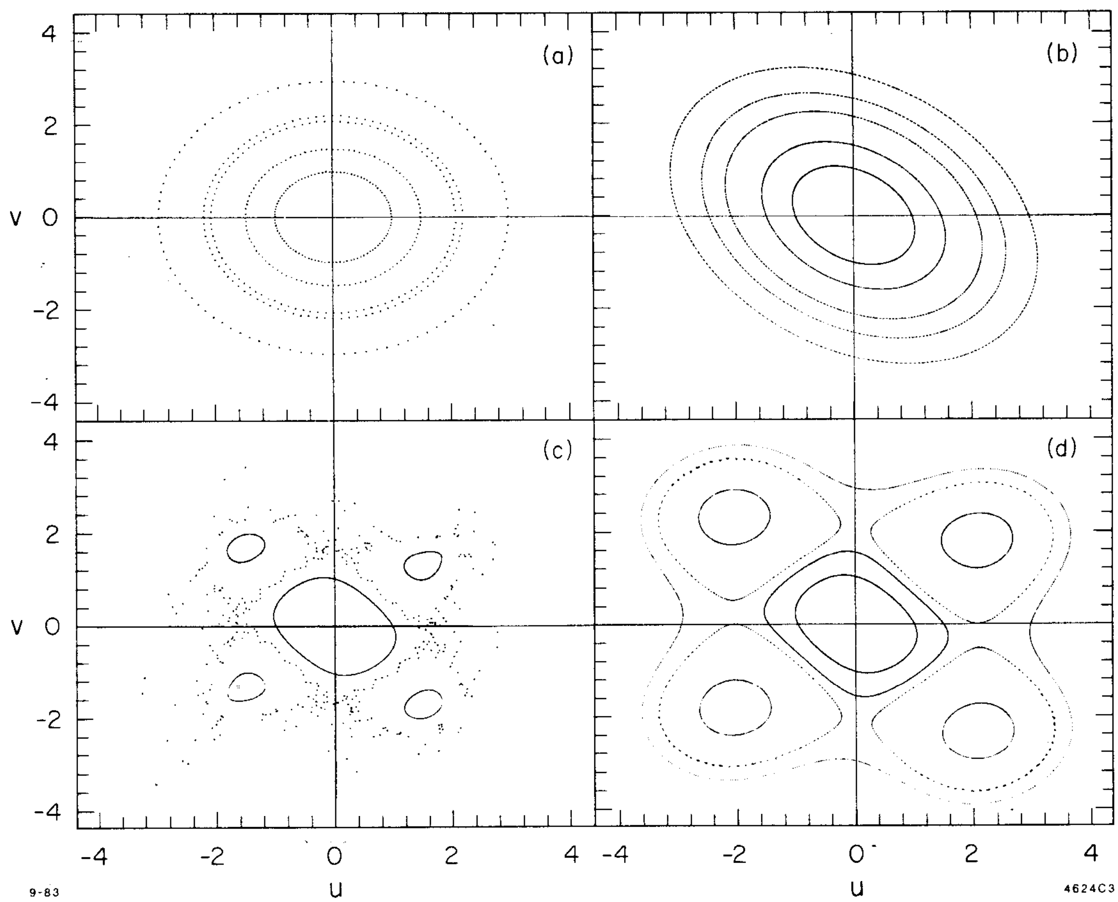


Fig. 14. Weak beam trajectories in the normalized phase space $(u, v)$, where $u = y/\sigma$, $v = \beta_0^* y'/\sigma$. We assume $\nu = 0.23$. (a) ignores the beam-beam force. (b) includes only the linear term of the beam-beam force. (c) inlcudes the linear and the octupole terms and (b) takes into account of the complete beam-beam force. In each diagram, trajectories of the same five sets of initial conditions are followed. Note the qualitative difference between (c) and (d).

It is interesting at this stage to introduce the actual observation of both space charge and beam-beam induced tune shifts in one machine, the $Sp\bar{p}S$.[18] In Fig. 15 $Q_0$ represents the machine tune provided by external focusing only. Point $Q_{cp}$ is the tune shift due to space charge for the proton beam and point $Q_{c\bar{p}}$ represents the tune shift for an anti-proton beam due to collision with proton. The former is a defocusing effect and the latter is a focusing effect. They extend into different direction in the tune diagram and make the control of working point more important for stable operation.

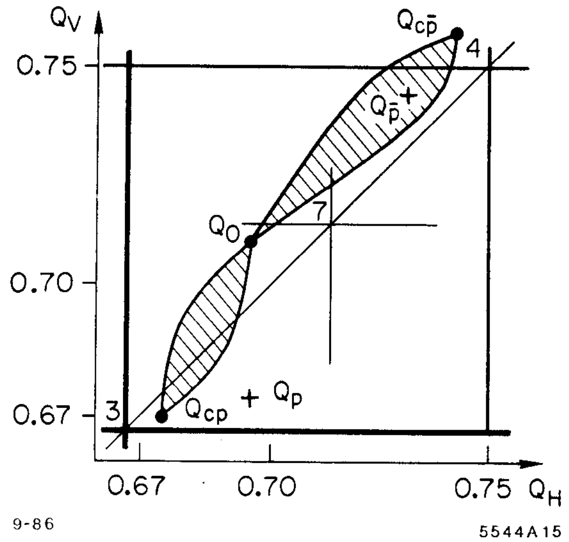


Fig. 15. Space-charge and beam-beam effects at injection (6$p$ bunches, 1$\bar{p}$ bunch) of $Sp\bar{p}S$.


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