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A NEW FORMULATION FOR THE LATTICE FERMION DERIVATIVE – LOCALITY AND CHIRALITY WITHOUT SPECTRUM DOUBLING.*

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ABSTRACT

We present a formulation for the lattice fermion derivative which is both local and explicitly chiral. Provided that the continuum and infinite volume limits are taken properly the formulation avoids spectrum doubling, and gives a satisfactory weak coupling perturbation theory for the case of Yukawa-coupled scalars. For gauge theories we find that a sum over long string paths must be introduced in order to maintain the correct weak-coupling perturbation theory in the continuum limit.

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In this paper we present a *local* lattice theory of chiral fermions which is free of spectrum doubling. Our interest in this problem is motivated by a desire to develop methods which allow a non-perturbative study of truly chiral gauge theories (e.g., most grand-unified theories). We find that one can avoid the implications of the Nielsen-Ninomiya theorem[1] because, as with most no-go theorems, its proof requires an implicit assumption; namely that one should take the limit of infinite volume before taking the continuum limit. For a theory initially defined in any *finite volume*, it is trivial to avoid spectrum doubling if one takes the infinite volume and continuum limits carefully. We present a method which explicitly demonstrates these properties. Our formulation involves an additional parameter ϵ which vanishes in the continuum limit.

In an earlier paper[2] we addressed a class of local free fermion derivatives which have no spectrum doubling. However, we found that these derivatives lead to non-covariant current-current correlation functions in the free theory and to non-covariant propagators in the case of the theory of a fermion interacting with a scalar field through a Yukawa interaction. We concluded that these problems could only be avoided by using a long-range derivative of the SLAC type. The purpose of this letter is to demonstrate that this

In this paper we present a local chiral fermion theory which not only has an undoubled spectrum but which also produces covariant Green's functions provided that the continuum limit is properly taken.

Unfortunately, when this derivative is rendered gauge invariant by the usual technique of introducing only the shortest string-paths, the resulting Hamiltonian does not correctly reproduce continuum weak-coupling perturbation theory for QED. The origin of this problem lies not in the derivative but in the way the coupling to the gauge field is introduced. In the simplest straight-line gauging one obtains an anomalously large coupling to the high-lying spurious fermion modes. This coupling becomes so large that even the modes go to infinite energy their contribution to the photon propagator survives in the continuum limit. This problem can be avoided, but only at the expense of introducing a correctly weighted sum over string paths in order to suppress the coupling of spurious modes to transverse photons. The coupling to transverse photons is then reintroduced in the form of an additional nearest neighbor interaction.

Obviously summing over arbitrarily long strings makes the Hamiltonian considerably more complicated, even though the fermionic derivative is local. To date we have not been able to find an alternative to this solution. We wish to emphasize that this weighted sum over strings reduces, at strong coupling, to the usual straight line prescription; thus, for practical calculations on a finite lattice at finite g^2, one can introduce a cut-off on the string length and probably not seriously affect the results for physical quantities which scale early, e.g mass ratios. Furthermore, the sensitivity to such a cut-off can readily be checked, so it would appear that this formulation will be useful for practical computations.

Although our discussion will focus on the Hamiltonian formulation of the theory, this derivative, unlike the long-range SLAC derivative, can be used in a Euclidean action formalism, at least in $A_0 = 0$ gauge. The theory obtained in this way is not identical to the Hamiltonian theory, except in the limit of zero lattice spacing, where both versions yield the usual continuum perturbation theory. The Euclidean version in fact converges to the continuum theory a little more rapidly as the $a \to 0$ limit is taken.

The Hamiltonian

In d-spatial dimensions consider a free fermion Hamiltonian of the form

$$\mathcal{H} = \frac{i}{2\epsilon a} \sum_{\vec{j},\hat{i}} \left\{ \bar{\psi}(\vec{j})\gamma_i \psi(\vec{j}+\hat{i}) - \frac{1-\epsilon}{2d}\, \bar{\psi}(j)\gamma_i \psi(j+2\hat{i}) \right.$$

$$\left. - \sum_{\hat{n}\neq\hat{i}} \frac{(1-\epsilon)}{\sqrt{2d}}\, \bar{\psi}(j)\, \frac{(\gamma_i+\gamma_n)}{\sqrt{2}}\, \psi(\vec{j}+\hat{\imath}+\hat{n}) \right\} + h.c. \tag{1}$$

$$+ m \sum_{\vec{j}} \overline{\psi}(j)\psi(j)$$

The sign convention $\gamma_{-\hat{n}} = -\gamma_{\hat{n}}$ is used throughout this paper and the sum over $\hat{n}$ runs over $2d$ unit vectors.

The spectrum of Eq. 1 is given by

$$E(\vec{k}) = \left\{ \sum_i D_i^2(\vec{k}) + m^2 \right\}^{1/2} \tag{2}$$

with

$$D_i(\vec{k}) = \frac{1}{\epsilon a} \sin k_i a \left\{ 1 - \frac{1-\epsilon}{2d} \sum_{\hat{n}} \cos k_{\hat{n}} a \right\} \tag{3}$$

This spectrum has the property that when all $k_i \to 0$

$$D_i(\vec{k}) \to k_i \tag{4}$$

However, the usual spurious states that appear where any one or more components of k_i are near π/a behave as

$$\begin{aligned} D_i(\vec{k}) &\propto \frac{k_i}{\epsilon} \qquad k_i \to 0 \quad k_{j\neq i} \to \pi/a \\ D_i(\vec{k}) &\propto \frac{\pi/a - k_j}{\epsilon} \qquad k_i \to \pi/a \end{aligned} \tag{5}$$

Thus, for finite volume, in the limit $\epsilon \to 0$, there is only one region of the spectrum corresponding to low lying states and $2^d - 1$ additional minima which we refer to as the ***spurious regions.***

The values of the k_i are given by

$$k_i = \frac{2\pi\, m_i}{(2N+1)a} \qquad (2N+1)a = L \tag{6}$$

for a lattice with sites $-N \leq j_i \leq N$. Since the lowest states in any spurious region have energies of order $\pi/\epsilon L$, if, in the limit of $a \rightarrow 0$ (L fixed), we require that $\epsilon \rightarrow 0$ then these spurious states disappear from the spectrum. This limit must be taken with $\epsilon/a^2 \rightarrow \infty$ in order that Eq. 4 will apply to an infinite range of physical momenta, k.

The next question one must ask is whether the contribution of these states to fermion loop integrals vanishes in the same limit. The simplest interacting fermion theory is one in which the fermions have Yukawa couplings to some scalar field. The fermion contribution to the scalar self energy is the most divergent fermion loop diagram in such a theory and therefore, the most sensitive to the presence of the spurious regions of the spectrum. Making the usual subtractions necessary for the continuum perturbation theory one readily calculates that the contributions of the spurious regions to the scalar self energy are, in the limit in which we take $a \rightarrow 0$ with L held fixed, at worst of order

$$\epsilon\, q^2\, \ell n \left(q^2 L^2\right) \tag{7}$$

For a Euclidean Lagrangian version this result becomes $\epsilon^2 q^2 \ell n(q^2 L^2)$. These contributions vanish in the limit $\epsilon \rightarrow 0$. A similar result is obtained for current-current correlation functions in the free field theory, with the currents defined as in Ref. 2.

The simplest way to render the Hamiltonian in Eq. 1 gauge invariant is to introduce gauge-link operators so that

$$\mathcal{H}_f = \frac{i}{2\epsilon a}\Bigg\{ \sum_{\vec{j},\hat{i}} \bar{\psi}(\vec{j})\gamma_i U(\vec{j},\hat{i})\psi(j+\hat{i}) - \frac{1-\epsilon}{2d}\,\bar{\psi}(\vec{j})\gamma_i U(\vec{j},\hat{\imath},\hat{\imath})\psi(\vec{j}+2\hat{i})$$
$$- \frac{1-\epsilon}{\sqrt{2}} \sum_{\hat{n}\neq\hat{i}} \bar{\psi}(\vec{j}) \left(\frac{\gamma_i+\gamma_n}{d\sqrt{2}}\right) \left[\frac{U(\vec{j},\hat{\imath},\hat{n})}{2} + \frac{U(\vec{j},\hat{n},\hat{\imath})}{2}\right] \psi(\vec{j}+\hat{n}+\hat{i}) \Bigg\} \tag{8}$$

where $U(\vec{j},-\hat{i}) = U^{\dagger}(\vec{j},\hat{i}) = e^{igA(\vec{j},\hat{i})}$, and $U(\vec{j},\hat{\imath},\hat{n}) = U(\vec{j},\hat{i})U(\vec{j}+\hat{\imath},\hat{n})$. This gives a coupling to fermions of the form

$$gA_\mu(q)\bar{\psi}(k+q)\gamma_\mu\psi(k)\Gamma_\mu(k,q) + \text{terms explicitly of order } a^2 \tag{9}$$

and

$$\Gamma_i(k,q) = a\frac{[D_i(k+q) - D_i(k)]}{sin\frac{1}{2}q_i a} + \mathcal{O}(a)\ . \tag{10}$$

With this choice of gauge invariant derivative the fermion-photon coupling is proportional to the derivative of the fermion propagator; hence, it is of order $1/\epsilon$ for fermions in the spurious regions. This means that the *photon* self energy receives contributions of order

$$\pi_{ii} \to \frac{1}{\epsilon}\, q_i^2\ \ell n\left(q^2L^2\right) \tag{11a}$$

or for a Euclidean Lagrangian version of the theory

$$\pi_{\mu\mu} \propto q_\mu^2\ \ell n\ \left(q^2L^2\right) \tag{11b}$$

which do not vanish in the $\epsilon \to 0$ limit.

The way out of this disaster is to recognize that the form (8) is not dictated by gauge invariance. An alternative formulation, which we have previously suggested in the context of the SLAC long range derivative, is to introduce a weighted average over all possible string paths between the fermion operators in the Hamiltonian.[3] In QED one can choose the weighting of a path to be

$$\omega_{\mathcal{P}} = -g^2 \sum_{\ell \in \mathcal{P}} E^2(\ell) \tag{12}$$

where l stands for a link belonging to the path $\mathcal{P}$. We have shown[2] that the coupling to transverse photons (plaquette variables) induced by this average over strings vanishes like e^{-1/g^2} and hence is zero to all orders in perturbation theory.

The coupling to transverse photons can be reintroduced by adding to $\mathcal{H}_f$ a term

$$\frac{l}{2a} \sum_{\vec{j},\hat{i}} \bar{\psi}(j)\gamma_i \psi(\vec{j}+\hat{i}) \left[U(j,\hat{\mu}) - \frac{\sum_{paths} \prod_{\ell \in \mathcal{P}} U(\ell)\omega_{\mathcal{P}}}{\sum_{paths} \omega_{\mathcal{P}}} \right] . \tag{13}$$

This gives a coupling of the form

$$\Gamma_i = \cos(k_i + q_i/2)a \tag{14}$$

for the fermions. With this choice of coupling, the result (7) applies also for the photon-self-energy. Hence, this Hamiltonian reproduces weak-coupling perturbation theory when the limits $a \to 0$ $\epsilon \to 0$ and $\epsilon/a^2 \to \infty$ are taken.

For a non-abelian gauge theory a weighting such as (11) is very difficult to implement, but it is also not necessary. If, instead, one chooses

$$\omega_{\mathcal{P}} = e^{\epsilon L_{\mathcal{P}}/a} \tag{15}$$

where $L_{\mathcal{P}}$ is the total length of the path P one finds[2] that the coupling to transverse photons appears first at order ϵ^2 and hence that contributions such as (11) are removed4The usual coupling is again reintroduced by the addition of a term of the form (13). Weak coupling perturbation theory is then correctly reproduced in the limit $a \to 0$ and $\epsilon \to 0$ with $\epsilon/a^2 \to \infty$.

Summary and Comments

We have introduced a simple local fermionic derivative which allows one to formulate lattice theories of chiral fermions in a totally straightforward manner. This derivative avoids both the restrictions of the Nielsen-Ninomiya theorem and the complexity of infinite range derivatives of the SLAC type. Although the derivative is itself local on the lattice, the transcription of the method to a gauge theory is not without complications. We found no way around the requirement that we introduce weighted sums over string configurations in order to control the strength of gauge couplings to the spurious modes.

This type of Hamiltonian is clearly more difficult to compute with than that obtained by straight line gauging, however it is not without certain nice features. First, and perhaps most important is that in the $m = 0$ limit it is explicitly chiral. Second, one can on the basis of a simple counting argument show that the effective theory at strong and weak coupling are qualitatively quite different. Amusingly, the expected roughening of the string joining charged particles now is explicit in the structure of the Hamiltonian itself. These points are discussed in more detail in the context of the SLAC long range derivative in our recent paper.[2] Obviously, as we suggested there, one way to use this formulation for practical calculations is to establish a cutoff on the maximum string length and study the behavior of computed quantities on this cutoff. In fact, one would expect this method work better in this case because the fermion derivative itself is explicitly local. In addition, unlike the case of the SLAC derivative, there is the hope that one can use this formalism to carry out $A_0 = 0$ gauge Euclidean computations.

The a-dependence of the parameter ϵ is not uniquely defined by the requirements that the spectrum and perturbative two-point functions are correctly reproduced in the $a \to 0$ limit. In principle, for an asymptotically free theory, the continuum theory is reproduced along some trajectory in the (ϵ, g^2) plane. The scaling of various physical quantities can be used to isolate this trajectory in the same way as is done when using the Symanzik improved lattice actions.[5] In practice we expect that for calculations at finite g^2 the behavior of the low-lying states will be quite insensitive to the value of ϵ, provided only that ϵ is sufficiently small.

Finally, we would like to conclude by pointing out, as we did in Ref. 2, that the fact that we have not found a way to avoid the sum over string trajectories is not to be construed as a proof that it cannot be done. We encourage the reader to try to devise a way around this cumbersome feature of our formulation.

REFERENCES

1. H. B. Nielsen and M. Ninomiya, *Phys. Lett.* **105B**, 219 (1981)
2. Helen R. Quinn and Marvin Weinstein, SLAC-PUB 3905, to be published in Phys. Rev. D
3. In a Euclidean formulation of the theory it is the introduction of these sums over string paths which forces us to restrict ourselves to $A_0 = 0$ gauge. Without this restriction there would be terms in the action connecting states arbitrarily separated in time. In that case it would not be at all clear that such a theory would be unitary. Of course, in $A_0 = 0$ gauge the derivative in the time direction has at most range 2 and one can satisfactorily define a transfer matrix for this theory by considering the basic unit to be two layers thick.
4. In Ref.2 we suggested a weighting $\exp[-g^2(L_P/a)]$ which we showed suppresses coupling to transverse protons by an additional factor of g^4. For an asymptotically free theory this choice may also be satisfactory, but the choice (15) gives a well defined continuum limit for any theory.
5. K. Symanzik, *Nucl.Phys.* **B226**, 187 (1983), *Nucl.Phys.* **B226**, 205 (1983)