SLAC – PUB – 4051
August 1986
(T)

CORRELATIONS IN 2-DIMENSIONAL CRITICAL THEORIES

MICHAEL P. MATTIS* †

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

We study the correlation functions of 2D statistical models at a critical point, using the techniques based on conformal invariance developed by Belavin, Polyakov and Zamolodchikov and Friedan, Qiu and Shenker. These functions are known to obey systems of linear partial-differential equations. We show that in many cases, determinable from properties of the operator product expansion, these systems reduce to first-order equations, soluble by inspection. The method is used to calculate 4- and 5-point functions in the Ising and tricritical Ising models. Finally, we propose a connection between the number of independent solutions of the differential equations and the existence of nontrivial symmetries such as the Kramers-Wannier duality.

Submitted for Publication

⋆ Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.
† Address after Sept. 1, 1986: Enrico Fermi Institute, Univ. of Chicago, Chicago IL 60637.

1. Introduction

The last two years have seen enormous progress in our understanding of critical 2D statistical systems, based on the work of Belavin, Polyakov and Zamolodchikov[1] (BPZ). The starting point is the recognition that a system undergoing a second-order phase transition is invariant, not only under global changes of scale ("dilatations"), but under local ones as well: in other words, the system is conformally invariant. This is an especially rigid constraint in 2D, where the conformal algebra is infinite-dimensional. Specifically, if we think of the 2D space (x, t) as the complex plane, then the conformal transformations consist of all analytic mappings $z \to f(z)$, and the powerful machinery of complex analysis can be brought into play.

It turns out that conformally invariant 2D theories can be classified by a parameter $c \geq 0$, defined below. We shall focus exclusively on the range $0 < c < 1$, which is of particular relevance to statistical mechanics. Theories in this range exhibit some remarkable properties, which we can only briefly touch upon:

1) To begin with, as Friedan, Qiu and Shenker have shown,[2] only the discrete series

$$c = 1 - \frac{6}{m(m+1)}, \quad m \geq 3 \tag{1a}$$

produces a unitary theory. For each of these values, the corresponding "minimal model" possesses only a finite number of "primary fields" $\Phi_{pp'qq'}(z, \bar{z})$, which can often be written as a product[1]

$$\Phi_{pp'qq'}(z, \bar{z}) = \psi_{p,q}(z) \times \xi_{p',q'}(\bar{z}), \quad 1 \leq p, p' \leq m-1, \quad 1 \leq q, q' \leq m. \tag{1b}$$

(Primary fields are fields that transform like tensors under conformal transformations; see Eq. (4) below.) The scaling dimension and spin of each such field is completely determined. Pleasingly, the first four theories in this set, with $m = 3, 4, 5, 6$, correspond to known spin systems: the Ising, tricritical Ising, 3-state Potts and tricritical 3-state Potts models, respectively.[1,2,3] Physical realizations of the remaining members of the series have also been constructed.[4] We should point out for later use that the labeling scheme of (1*b*) is actually quadruply redundant, since

$$\psi_{p,q} \equiv \psi_{m-p,m+1-q} \ , \tag{2}$$

and likewise for ξ. (This is a special case of Eq. (37) below.)

2) These fields form a closed set under the operation of taking operator products.[1] There is a simple prescription, known as the "fusion rules," for determining which fields occur in the operator product of any string of primary fields (see Section 4). Correlation functions will be nonzero only if the identity operator $\Phi_{1,1,1,1}$ appears in the operator product of the fields. In general, the coefficients in the operator product expansion can be determined by the requirement of associativity.

3) Finally, any correlation function involving the field $\psi_{p,q}(z)$ (and likewise $\xi_{p,q}(\bar{z})$) must satisfy two linear homogeneous partial differential equations, of order pq and $(m-p)\cdot(m+1-q)$.[1] (This will be reviewed in the following Section.)

The fact that correlation functions in the minimal models satisfy such equations implies that, in principle, they are calculable! Feigin and Fuchs[5] have shown that they can be expressed as multiple contour integrals of exponentials of free Bose fields. With the help of this representation, Dotsenko and Fateev,[6] in an ambitious series of papers, have derived general expressions for the correlators in terms of multiple integrals of algebraic functions, weighted by products of ratios of Γ-functions. This approach, which involves sophisticated use of complex analysis, can be used to solve for the coefficients of the operator product expansion. One subtlety is that the integrals involved actually diverge when c assumes one of the discrete values given above, and can only be given meaning via an analytic continuation in c.

In this paper we adopt a much more straightforward approach to the correlation functions of conformally invariant 2D theories. We point out that the systems of partial differential equations that the correlators obey can often be greatly simplified. In fact, in many cases (determinable from the fusion rules) they are equivalent to first-order ordinary differential equations, which are trivially soluble. We shall illustrate this "reduction algorithm" by explicitly calculating some 4- and 5-point functions in the Ising and tricritical Ising models. (2- and 3-point functions are trivial at the critical point.) We also speculate on the connection between the number of independent solutions of these equations and the existence of nontrivial symmetries in the theory such as the famous Kramers-Wannier duality.

2. The PDE's of BPZ

Let us first review some basic notions of conformal invariance, following BPZ. At a critical point, the physical fields of a theory must scale in a well-defined way under dilatations:

$$\Phi(z,\bar{z}) \to \lambda^{\Delta_{\mathrm{phys}}} \cdot \Phi(\lambda z, \lambda \bar{z}). \tag{3}$$

There are many ways that one might imagine generalizing this to the case of local scale transformations $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$. "Primary fields" are operators that transform in a particularly simple fashion:

$$\Phi(z,\bar{z}) \to \left(\frac{dw}{dz}\right)^{\Delta}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}}\Phi(w,\bar{w}). \tag{4}$$

Here Δ and $\bar{\Delta}$ are independent numbers; in the case of the minimal models of Eq. (1), they are specified fractions:

$$\Delta_{p,q} = \Delta_{m-p,m+1-q} = \frac{(p(m+1)-qm)^2-1}{4m(m+1)}. \tag{5}$$

The physical scaling dimension Δ_{phys} and spin s of Φ are given by $\Delta+\bar{\Delta}$ and $\Delta-\bar{\Delta}$, respectively. (The requirement that correlators of physical fields be single-valued restricts s to integral or half-integral values.)

It is convenient to think of $\Phi(z,\bar{z})$ as a product $\psi(z)\xi(\bar{z})$, where $\psi(z) \rightarrow (dw/dz)^{\Delta}\psi(w)$, and likewise for ξ, in which case correlators of the $\Phi(z,\bar{z})$'s factor simply into correlators of the $\psi(z)$'s and correlators of the $\xi(\bar{z})$'s. We can focus on the analytic correlators $G =< \psi_1(z_1) \cdots \psi_n(z_n) >$, since the analysis for the ξ's is completely parallel. We shall also frequently exploit the equivalence between statistical systems and Euclidean quantum field theories by writing G as a time-ordered vacuum expectation value:

$$G =< 0|\psi_1(z_1) \cdots \psi_n(z_n)|0 > . \tag{6}$$

We are interested in the conformal properties of the G's. Consider the infinitesimal analytic transformations $z \rightarrow z+\epsilon z^{n+1}$, $n \in Z$, which we can represent by an abstract generator L_n. The infinitesimal version of (4) then reads

$$[L_n, \psi(z)] = z^{n+1}\psi'(z) + (n+1)\Delta z^n \psi. \tag{7}$$

The L_n's can be shown to satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n-1)(n+1)\delta_{n+m,0} \ , \tag{8}$$

where the central charge c is the parameter mentioned earlier.

Conformal invariance requires that G be invariant under the subset of analytic transformations that preserve the "in" and "out" vacua, *i.e.*, that are regular as $t \rightarrow \pm\infty$. If one adopts the "radial time ordering" prescription discussed in Refs. 1 and 2, one finds:

$$L_k|0>= 0 \quad : \quad k \geq -1 \tag{9a}$$

and

$$< 0|L_k = 0 \quad : \quad k \leq 1. \tag{9b}$$

G will thus be invariant under the subalgebra $\{L_{-1}, L_0, L_1\}$, which annihilate both $|0>$ and $<0|$. One therefore has

$$0 = <0|[L_k, \phi_1(z_1)] \cdots \phi_n(z_n)|0> + \cdots + <0|\phi_1(z_1) \cdots [L_k, \phi_n(z_n)]|0>, \; k = -1, 0, 1 \tag{10}$$

or equivalently

$$\sum_{i=1}^{n} \partial_i <0|\psi_1(z_1) \cdots \psi_n(z_n)|0> = 0; \tag{11a}$$

$$\sum_{i=1}^{n} (z_i \partial_i + \Delta_i) <0|\psi_1(z_1) \cdots \psi_n(z_n)|0> = 0; \tag{11b}$$

$$\sum_{i=1}^{n} (z_i^2 \partial_i + 2 z_i \Delta_i) <0|\psi_1(z_1) \cdots \psi_n(z_n)|0> = 0. \tag{11c}$$

These conditions imply, respectively, invariance of G under translations, dilatations, and "special conformal transformations."

Of course, (11) holds for *any* conformal theory, not just the minimal models defined by (1) and (5). We shall see shortly that the correlators in the minimal models satisfy additional PDE's.

In addition to primary fields, a conformally invariant theory will contain an infinite number of "descendant" or "secondary" fields, which transform in a more complicated way than (4). Taken together, the primary and secondary fields form a complete set of operators in a theory. We can construct the secondary fields by the following prescription. First, we note the existence of a 1-to-1 correspondence between primary fields $\psi(z)$ and states $|\psi> = \psi(0)|0>$. Using (7) and (9), one finds:

$$\begin{aligned} L_0|\psi> &= \Delta|\psi>, \\ L_k|\psi> &= 0, \quad k \geq 1 \end{aligned} \tag{12a}$$

and likewise

$$\begin{aligned} <\psi|L_0 &= <\psi|\Delta, \\ <\psi|L_{-k} &= 0, \quad k \geq 1. \end{aligned} \tag{12b}$$

Thus the L_k's can be viewed as "annihilation operators." The secondary states are obtained simply by acting on $|\psi>$ with an arbitrary string of "creation operators" $L_{-k_1}L_{-k_2}\cdots L_{-k_n}$, $k_i \geq 1$.

BPZ have shown that correlators involving secondary fields can be expressed in terms of correlators of the corresponding primary fields by means of linear partial-differential operators. To see this, consider the quantity

$$< 0|\psi_1(z_1)\cdots\psi_n(z_n)L_{-k}\psi(0)|0> \tag{13}$$

containing one secondary and n primary fields. Commuting L_{-k} to the left and using (7) and (9b), one can rewrite this as*

$$\hat{\mathcal{L}}_{-k} < 0|\psi_1(z_1)\cdots\psi_n(z_n)\psi(0)|0>, \tag{14}$$

where

$$\hat{\mathcal{L}}_{-k} = \sum_{i=1}^{n}\Big(-z_i^{-k+1}\frac{\partial}{\partial z_i} - (1-k)\Delta_i z_i^{-k}\Big). \tag{15}$$

Note that the $\hat{\mathcal{L}}$'s satisfy the Virasoro algebra (8). Similarly one obtains

$$\begin{aligned} &< 0|\psi_1(z_1)\cdots\psi_n(z_n)L_{-k_1}\cdots L_{-k_l}\psi(0)|0> \\ &\qquad = \hat{\mathcal{L}}_{-k_1}\cdots\hat{\mathcal{L}}_{-k_l} < 0|\psi_1(z_1)\cdots\psi_n(z_n)\psi(0)|0> . \end{aligned} \tag{16}$$

An interesting peculiarity of the Virasoro algebra is the existence of primary states $|\psi>$ that are annihilated, not only by lowering operators L_k, $k \geq 1$, but also by a special combination of *raising* operators L_{-k}.[7] In other words, one can construct primary states that have associated with them a vanishing secondary state. *All primary states* $|\psi_{p,q}>$ *and* $|\xi_{p,q}>$ *of the minimal models have this property!*[1] (It is thanks to this property that these theories are unitary.[2]) As

* We are assuming here that $\psi(0)$ can be pulled out of the time-ordering implicit in all of these correlators. This is always justified in the case of radial quantization, since $z=0$ is equivalent to $t=-\infty$.

an illustration, let us consider the Ising model ($m = 3$). One can check using (8) and (12) that the secondary state

$$(L_{-1}{}^2 - \frac{4}{3}L_{-2})|\psi_{2,1}> \tag{17}$$

vanishes; that is, it is orthogonal to every state in the theory.† Consequently, every correlator of primary fields containing $\psi_{2,1}$ must satisfy the linear homogeneous second-order PDE

$$(\hat{\mathcal{L}}^2_{-1} - \frac{4}{3}\hat{\mathcal{L}}_{-2}) < 0|\psi_1(z_1)\cdots\psi_n(z_n)\psi_{2,1}(0)|0 >= 0. \tag{18}$$

The general result[1,7] is that the primary fields $\psi_{p,q}$ of the minimal models are associated with partial differential operators of order $p \cdot q$ and (thanks to (2)) $(m-p)\cdot(m+1-q)$. Thus an n-point function of such fields will need to satisfy a system of $2n$ PDE's, in addition to (11). We will see in the next Section that these systems are frequently equivalent to first-order ODE's, which can be solved by inspection.

3. Correlation functions made easy

The formalism of the previous Section can immediately be put to use. It is easy to show that (11) completely determines the form of the 2- and 3-point functions in conformally invariant theories.[1] One finds:*

$$< 0|\psi_1(z_1)\psi_2(z_2)|0 >\propto (z_1 - z_2)^{-2\Delta_1}\delta_{\Delta_1,\Delta_2} \tag{19a}$$

and

$$\begin{aligned} &< 0|\psi_1(z_1)\psi_2(z_2)\psi_3(z_3)|0 >\propto \\ &(z_1 - z_2)^{\Delta_3-\Delta_1-\Delta_2}(z_2 - z_3)^{\Delta_1-\Delta_2-\Delta_3}(z_3 - z_1)^{\Delta_2-\Delta_3-\Delta_1}. \end{aligned} \tag{19b}$$

However, the n-point functions with $n \geq 4$ are not fixed by (11); they can be expressed in terms of an arbitrary function g of $n-3$ independent "anharmonic

---

† $\psi_{2,1}$ is the "hidden fermion" of the Ising model.[8]

* Note that the constants of proportionality in (19) vanish if the identity operator does not appear in the operator product of the ψ's; see Section 4 below.

quotients"[1]

$$\chi_{kl}^{ij} = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_k)(z_j - z_l)}. \tag{20}$$

In particular, the 4-point function must be of the form

$$\begin{aligned} < 0|\psi_1(z_1)\psi_2(z_2)\psi_3(z_3)\psi_4(z_4)|0 >= (z_1 - z_4)^{-2\Delta_1}(z_2 - z_4)^{\Delta_1-\Delta_2+\Delta_3-\Delta_4} \\ \times (z_3 - z_4)^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}(z_2 - z_3)^{-\Delta_1-\Delta_2-\Delta_3+\Delta_4} \cdot g(\chi_{34}^{12}). \end{aligned} \tag{21}$$

If we restrict our attention to the minimal models, then g, too, can be calculated. Let us work through one example in detail. Consider the 4-point function $< \epsilon\epsilon\epsilon\epsilon >$ in the Ising model.‡ (We shall focus on the Ising model in this Section; the tricritical Ising model is dealt with in the Appendix.) Here, ϵ represents the energy density fluctuation field, with spin 0 and scaling dimension $\Delta_{\text{phys}} = 1$; in the notation of Eq. (1b), it can be factored as

$$\epsilon(z, \bar{z}) = \psi_{2,1}(z) \times \xi_{2,1}(\bar{z}). \tag{22}$$

The 4-point function of the $\psi_{2,1}$'s must satisfy (18); using (21), we obtain

$$g''(\chi) + \frac{1}{8}\left[\frac{6}{\chi} - \frac{3}{\chi - 1} + \frac{7}{\chi + 1}\right]g'(\chi) - \frac{5}{4\chi^2} \cdot g(\chi) = 0, \quad \chi \equiv \chi_{34}^{12}. \tag{23}$$

We can simplify our task further by noting that $\psi_{2,1}$ is the same field as $\psi_{1,3}$ ($cf.$ (2)); this implies the existence of a third-order equation as well. The required null state of the Virasoro algebra is easy to work out, and one finds:

$$\left(\hat{\mathcal{L}}_{-1}^3 - 3\hat{\mathcal{L}}_{-1}\hat{\mathcal{L}}_{-2} + \frac{15}{4}\hat{\mathcal{L}}_{-3}\right) < \psi_{2,1}\,\psi_{2,1}\,\psi_{2,1}\,\psi_{2,1} >= 0. \tag{24}$$

One can reduce (24) *ab initio* to a second-order equation, by applying $\hat{\mathcal{L}}_{-1}$ to

‡ We shall frequently use $< \psi_1 \cdots \psi_n >$ as an abbreviation for $< 0|\psi_1(z_1) \cdots \psi_n(z_n)|0 >$.

(18) and subtracting; this gives

$$\left(-\frac{5}{3}\hat{\mathcal{L}}_{-1}\hat{\mathcal{L}}_{-2}+\frac{15}{4}\hat{\mathcal{L}}_{-3}\right) < \psi_{2,1}\,\psi_{2,1}\,\psi_{2,1}\,\psi_{2,1} >= 0, \tag{25}$$

which becomes

$$g''(\chi)+\frac{2}{3}\left[\frac{2}{\chi}-\frac{1}{\chi-1}\right]g'(\chi)-\frac{2}{3\chi^2}\cdot g(\chi)=0. \tag{26}$$

Finally, subtracting (26) from (23) yields the *first*-order equation

$$g'(\chi)+\left[\frac{1}{\chi}-\frac{2\chi-1}{\chi^2-\chi+1}\right]g(\chi)=0, \tag{27}$$

which can be solved by inspection:

$$g=\frac{\chi^2-\chi+1}{\chi}. \tag{28}$$

Of course, the identical calculation goes through for the 4-point function of the $\xi_{2,1}$'s. All in all, we find:

$$\begin{aligned} &< \epsilon(z_1,\bar{z}_1)\epsilon(z_2,\bar{z}_2)\epsilon(z_3,\bar{z}_3)\epsilon(z_4,\bar{z}_4) > \\ &= \left|(z_1-z_4)^{-1}(z_2-z_3)^{-1}\frac{\chi^2-\chi+1}{\chi}\right|^2 \\ &= \left|\frac{(z_1^2z_2z_3+\text{perms.})-(z_1^2z_2^2+\text{perms.})-6z_1z_2z_3z_4}{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)(z_3-z_4)}\right|^2. \end{aligned} \tag{29}$$

Note that (29) is properly symmetric in the z_i's. It also satisfies cluster decomposition: as one separates one pair of z_i's from the other, the correlator collapses into the product of two propagators.

It would be interesting to see how this reduction of order manifests itself in the integral representations of Refs. 5 and 6.

The 4-point function $<\epsilon\epsilon\sigma\sigma>$ succumbs to an equally straightforward calculation. Here σ is the spin density, which has spin 0 and dimension $\Delta_{\text{phys}} = \frac{1}{8}$. It can thus be thought of as

$$\sigma(z,\bar{z}) = \psi_{1,2}(z) \times \xi_{1,2}(\bar{z}). \tag{30}$$

The correlator $<\epsilon\epsilon\sigma\sigma>$ likewise satisfies (18) and (24), but now with different expressions for the $\hat{\mathcal{L}}$'s determined by the new values of the Δ_i's. One now finds:

$$g = \frac{(\chi-2)\sqrt{\chi-1}}{\chi} \tag{31}$$

and hence

$$< \epsilon(z_1,\bar{z}_1)\epsilon(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4) >= \left|\frac{(z_1-z_2)^{-1}(z_3-z_4)^{-\frac{1}{8}}[(z_1z_2+z_3z_4)-\frac{1}{2}(z_1+z_2)(z_3+z_4)]}{\sqrt{(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4)}}\right|^2. \tag{32}$$

These two examples illustrate a general "reduction algorithm" geared toward finding the solutions of a system of differential equations of various orders. The algorithm entails repeatedly differentiating the lower-order equation(s) and subtracting from the higher-order one(s), until equations of still lower order are obtained, *etc.* Indeed, we might attempt to reduce the order of (27) still further, to a "zeroth order" equation, by differentiating (27), subtracting from (26), then subtracting the resulting first-order equation from (27); this procedure yields an equation of the form

$$f(\chi)\cdot g(\chi) = 0, \tag{33}$$

where $f(\chi)$ is an ordinary function. This means that $g(\chi) \equiv 0$ unless f vanishes identically! One can check that for both $<\epsilon\epsilon\epsilon\epsilon>$ and $<\epsilon\epsilon\sigma\sigma>$, f indeed vanishes, so that the reduction algorithm terminates in a first-order equation.

However, one can carry out the reduction algorithm for the more general 4-point function $< \psi_{2,1}\, \psi_a\, \psi_b\, \psi_c >$, where $\psi_{a,b,c}$ denote primary fields of arbitrary scaling dimension $\Delta_{a,b,c}$. In this case f vanishes only if the Δ's assume the values $\{\frac{1}{2}\frac{1}{2}\frac{1}{2}\}$, $\{\frac{1}{2}00\}$, $\{\frac{1}{2}\frac{1}{16}\frac{1}{16}\}$ or $\{0\frac{1}{16}\frac{1}{16}\}$. Note that $0, \frac{1}{2}, \frac{1}{16}$ are precisely the dimensions $\Delta_{p,q}$ of the Ising model fields. In this way, the reduction algorithm provides yet another check on the self-consistency, closure and uniqueness of the minimal models.

The reduction algorithm can also be applied to higher-point functions. Consider the 5-point function $< \epsilon\epsilon\epsilon\sigma\sigma >$. In order to satisfy (11), we must have

$$\begin{aligned} &< \psi_{2,1}\, \psi_{2,1}\, \psi_{2,1}\, \psi_{1,2}\, \psi_{1,2} >= \\ &(z_1 - z_4)^{-\frac{3}{2}}(z_1 - z_5)^{\frac{1}{2}}(z_2 - z_5)^{-1}(z_3 - z_5)^{-1}(z_4 - z_5)^{\frac{11}{8}} \cdot g(\chi_+, \chi_-), \end{aligned} \tag{34}$$

where $\chi_+ = \chi_{45}^{12}$ and $\chi_- = \chi_{45}^{13}$. In this case, the most efficient strategy for calculating g is to ignore the five higher-order equations associated with the ψ's, and to use, instead, the five second-order equations, all of which turn out to be independent. These can be combined to yield the two first-order equations

$$\frac{\partial g}{\partial \chi_\pm} + \left[\frac{1}{\chi_\pm} + \frac{\frac{1}{2}}{\chi_\pm - 1} \pm \frac{1}{\chi_+ - \chi_-} - \frac{\chi_\mp^2 + 2\chi_+\chi_- - 4\chi_\pm + 2\chi_\mp}{\chi_+\chi_-^2 + \chi_-\chi_+^2 - 2\chi_+^2 - 2\chi_-^2 + 2\chi_+\chi_-}\right] g = 0, \tag{35}$$

which can be solved by inspection:

$$g = \frac{\chi_+\chi_-^2 + \chi_-\chi_+^2 - 2\chi_+^2 - 2\chi_-^2 + 2\chi_+\chi_-}{\chi_+\chi_-(\chi_+ - \chi_-)\sqrt{\chi_+ - 1}\,\sqrt{\chi_- - 1}}. \tag{36}$$

Apparently, the reduction algorithm ceases to be of practical use for n-point functions when n is large, since the number of mixed partial derivatives grows faster than the number of equations. The algorithm still works in principle if one utilizes the following generalization of (2):[1]

$$\psi_{p,q} \equiv \psi_{km+p,k(m+1)+q} \equiv \psi_{(k+1)m-p,(k+1)(m+1)-q}\,, \qquad k = 0, 1, 2, \dots . \tag{37}$$

This implies the existence of an infinite number of PDE's of order $(km+p)(k(m+$

$1) + q)$ and $((k+1)m - p)((k+1)(m+1) - q)$. In practice, however, the use of these equations for $k > 0$ seems hopeless.

Not all systems of equations satisfied by the correlators in the minimal models can be reduced to first (or zeroth) order equations. Consider, for example, the 4-point function $< \sigma\sigma\sigma\sigma >$. The field $\psi_{1,2} \equiv \psi_{2,2}$ is associated with both a second-order operator $\mathcal{D}_2(\chi)$ and a fourth-order operator $\mathcal{D}_4(\chi)$. However, $\mathcal{D}_4$ turns out *not* to yield an independent equation, since it is factorable as $\tilde{\mathcal{D}}_2 \circ \mathcal{D}_2$. There are thus two *bona fide* independent solutions for the 4-point functions of the $\psi_{1,2}$'s and $\xi_{1,2}$'s, hence four independent solutions for $< \sigma\sigma\sigma\sigma >$.[1,9] The same phenomenon occurs for $< \epsilon\sigma\sigma\sigma\sigma >$.

What is the physical meaning of this degeneracy? Recall the existence in the Ising model of the Kramers-Wannier duality, whereby the spin operator σ is mapped into the "disorder operator" μ, and *vice versa.* Of course, μ must have the same critical properties as σ; in particular, it is associated with the same differential operators. The four-fold degeneracy that we found is precisely what is needed in order to account for the four linearly independent correlators $< \sigma\sigma\sigma\sigma >$, $< \sigma\sigma\mu\mu >$, $< \sigma\mu\mu\sigma >$, and $< \sigma\mu\sigma\mu >$.*

More generally, it is tempting to conjecture that *the existence of a multiplicity of solutions to the BPZ equations in a given minimal model is always associated with the presence of multiple operators of a given scaling dimension, as is the case when the theory possesses a Kramers-Wannier duality.* Conversely, operators such as ϵ which are associated only with first-order equations would have to be self-dual under such symmetry transformations.

In the following Section, we shall give the general rule for determining the multiplicity of a general n-point function in the minimal models.

4. Fusion Rules Redux

---

* The latter three, although related by a permutation of the z_i's, correspond to different functions of χ and $\bar{\chi}$. Note also that the other four nonvanishing 4-point functions of σ's and μ's are equal to these by the duality.

As mentioned earlier, one of the striking features of the minimal models discovered by BPZ is closure under the operator product expansion. This property is expressed in the "fusion rules," which can be thought of as the "Clebsch-Gordan series" of the Virasoro algebra:

$$\psi_{p_1,q_1}(z_1)\psi_{p_2,q_2}(z_2) = \sum_{l=1-p_1}^{p_1-1} \sum_{k=1-q_1}^{q_1-1} \left[\psi_{p_2+l,q_2+k}\left(\frac{z_1+z_2}{2}\right)\right]. \tag{38}$$

Here $[\psi_{p,q}]$ stands for the primary field $\psi_{p,q}$ together with its associated secondary fields, summed against c-number functions of z_1 and z_2; the index k (l) runs over even or odd values, depending on whether q_1 (p_1) is odd or even, respectively. We can constrain the allowed fields in the right-hand side of (38) still further, by reexpressing the left-hand side in the following three equivalent ways: $\psi_{m-p_1,m+1-q_1}(z_1)\psi_{p_2,q_2}(z_2)$, $\psi_{p_2,q_2}(z_2)\psi_{p_1,q_1}(z_1)$, $\psi_{m-p_2,m+1-q_2}(z_2)\psi_{p_1,q_1}(z_1)$. The allowed fields will be restricted to those that appear on the right-hand side of all four versions of the fusion rules.

In this way, we easily obtain the fusion rules for the Ising $(m = 3)$ and tricritical Ising $(m = 4)$ models:

Ising Model

$$\psi_{1,1}\psi_{1,1} = [\psi_{1,1}], \quad \psi_{1,1}\psi_{1,2} = [\psi_{1,2}], \quad \psi_{1,1}\psi_{2,1} = [\psi_{2,1}],$$
$$\psi_{1,2}\psi_{1,2} = [\psi_{1,1}] + [\psi_{2,1}], \quad \psi_{1,2}\psi_{2,1} = [\psi_{1,2}], \quad \psi_{2,1}\psi_{2,1} = [\psi_{1,1}]$$

Tricritical Ising Model

$$\psi_{1,1}\psi_{1,1} = [\psi_{1,1}], \quad \psi_{1,1}\psi_{3,1} = [\psi_{3,1}], \quad \psi_{1,1}\psi_{1,2} = [\psi_{1,2}], \quad \psi_{1,1}\psi_{1,3} = [\psi_{1,3}],$$
$$\psi_{1,1}\psi_{2,2} = [\psi_{2,2}], \quad \psi_{1,1}\psi_{2,1} = [\psi_{2,1}], \quad \psi_{3,1}\psi_{3,1} = [\psi_{1,1}], \quad \psi_{3,1}\psi_{1,2} = [\psi_{1,3}],$$
$$\psi_{3,1}\psi_{1,3} = [\psi_{1,2}], \quad \psi_{3,1}\psi_{2,2} = [\psi_{2,2}], \quad \psi_{3,1}\psi_{2,1} = [\psi_{2,1}],$$
$$\psi_{1,2}\psi_{1,2} = [\psi_{1,1}] + [\psi_{1,3}], \quad \psi_{1,2}\psi_{1,3} = [\psi_{1,2}] + [\psi_{3,1}], \quad \psi_{1,2}\psi_{2,2} = [\psi_{2,1}] + [\psi_{2,2}],$$

$$\psi_{1,2}\psi_{2,1} = \left[\psi_{2,2}\right], \quad \psi_{1,3}\psi_{1,3} = \left[\psi_{1,1}\right] + \left[\psi_{1,3}\right], \quad \psi_{1,3}\psi_{2,2} = \left[\psi_{2,2}\right] + \left[\psi_{2,1}\right],$$
$$\psi_{1,3}\psi_{2,1} = \left[\psi_{2,2}\right], \quad \psi_{2,2}\psi_{2,2} = \left[\psi_{1,1}\right] + \left[\psi_{3,1}\right] + \left[\psi_{1,2}\right] + \left[\psi_{1,3}\right],$$
$$\psi_{2,2}\psi_{2,1} = \left[\psi_{1,2}\right] + \left[\psi_{1,3}\right], \quad \psi_{2,1}\psi_{2,1} = \left[\psi_{1,1}\right] + \left[\psi_{3,1}\right]$$

Of course, the same multiplication tables hold for the $\xi_{p,q}$'s.

From these fusion rules, it is easy to determine whether an n-point function vanishes: one need only verify whether the identity operator $\psi_{1,1} \times \xi_{1,1}$, which is the only operator (primary or secondary) with a nonvanishing 1-point function, occurs in the operator product of the n fields. (For example, the n-point function of $\psi_{m-1,1}$, with m as in $(1a)$, vanishes if n is odd, since $\psi_{m-1,1}\psi_{m-1,1} = [\psi_{1,1}]$.)

In every case that I have checked, *the number of independent solutions to the BPZ differential equations for* $<\psi_{p_1,q_1}\cdots\psi_{p_n,q_n}>$ *simply equals the number of distinct ways that* $\psi_{1,1}$ *appears according to the fusion rules in the operator product of the* $\psi_{p,q}$*'s*. For example, in the Ising model, we have

$$\begin{aligned}&<\psi_{1,2}\,\psi_{1,2}\,\psi_{1,2}\,\psi_{1,2}>=<([\psi_{1,1}]+[\psi_{2,1}])([\psi_{1,1}]+[\psi_{2,1}])>\\&=<[\psi_{1,1}]+[\psi_{1,1}]+[\psi_{2,1}]+[\psi_{2,1}]>,\end{aligned}$$

consistent with our earlier finding that this correlator satisfies a second-order equation.

This rule allows us to make some general predictions for minimal models of arbitrary m, for instance: (i) inserting any even number of $\psi_{m-1,1}$'s into a correlator of $\psi_{p,q}$'s will not alter the number of distinct solutions; (ii) any n-point function of $\psi_{p,q}$'s of which at least $n-3$ of the fields are $\psi_{m-1,1}$ will be given as the (unique) solution to a system of $n-3$ first-order equations.

Appendix: 4-point functions in the tricritical Ising model

One can check from the fusion rules given in Section 4 that, of the 70 possible 4-point functions of the $\psi_{p,q}$'s in the tricritical Ising model that do not involve the identity $\psi_{1,1}$, only 27 are nonvanishing. Of these, $<\psi_{2,2}\,\psi_{2,2}\,\psi_{2,2}\,\psi_{2,2}>$

corresponds to a fourth-order equation, while nine correspond to second-order equations, namely:

$$< \psi_{1,2}\,\psi_{1,2}\,\psi_{1,2}\,\psi_{1,2} >,\ < \psi_{1,2}\,\psi_{1,2}\,\psi_{1,3}\,\psi_{1,3} >,\ < \psi_{1,2}\,\psi_{1,2}\,\psi_{2,2}\,\psi_{2,2} >,$$
$$< \psi_{1,2}\,\psi_{1,3}\,\psi_{2,2}\,\psi_{2,2} >,\ < \psi_{1,3}\,\psi_{1,3}\,\psi_{1,3}\,\psi_{1,3} >,\ < \psi_{1,3}\,\psi_{1,3}\,\psi_{2,2}\,\psi_{2,2} >,$$
$$< \psi_{2,1}\,\psi_{2,1}\,\psi_{2,1}\,\psi_{2,1} >,\ < \psi_{2,1}\,\psi_{2,1}\,\psi_{2,2}\,\psi_{2,2} >,\ < \psi_{2,1}\,\psi_{2,2}\,\psi_{2,2}\,\psi_{2,2} > .$$

Seventeen correlators correspond to first-order equations, easily obtained via the reduction algorithm; the solutions to these are given in Table I (here, g is defined as in Eq. (21)).

We should note the existence of an elegant alternative formulation of the tricritical Ising model in terms of superfields, which provides another means of calculating correlation functions in this theory.[2,10]

Acknowledgments

I am indebted to Tom Banks, Paul Griffin, Marek Karliner, Daniel Mattis, Michael Peskin, Steve Shenker and Chris Wendt for useful discussions.

Table I. "First-order" 4-point functions in the tricritical Ising model

$< \psi_{3,1}\, \psi_{3,1}\, \psi_{3,1}\, \psi_{3,1} >$:	$g = 7(\chi+\chi^{-1})^3 - 21(\chi+\chi^{-1})^2 + 45(\chi+\chi^{-1}) - 55$
$< \psi_{3,1}\, \psi_{3,1}\, \psi_{1,2}\, \psi_{1,2} >$:	$g = \chi^{-3}(\chi-1)^2(3\chi^2-7\chi+7)$
$< \psi_{3,1}\, \psi_{3,1}\, \psi_{1,3}\, \psi_{1,3} >$:	$g = \chi^{-3}(\chi-1)(3\chi^4-18\chi^3+25\chi^2-14\chi+7)$
$< \psi_{3,1}\, \psi_{3,1}\, \psi_{2,1}\, \psi_{2,1} >$:	$g = \chi^{-3}(\chi-2)(\chi-1)^{\frac{3}{2}}(7\chi^2-4\chi+4)$
$< \psi_{3,1}\, \psi_{3,1}\, \psi_{2,2}\, \psi_{2,2} >$:	$g = \chi^{-3}(\chi-2)(\chi-1)^{\frac{3}{2}}(\chi^2-28\chi+28)$
$< \psi_{3,1}\, \psi_{1,2}\, \psi_{1,2}\, \psi_{1,2} >$:	$g = \chi^{-1}(\chi-1)^2$
$< \psi_{3,1}\, \psi_{1,2}\, \psi_{1,3}\, \psi_{1,3} >$:	$g = \chi^{-1}(\chi-1)(\chi^2+\chi-1)$
$< \psi_{3,1}\, \psi_{1,2}\, \psi_{2,1}\, \psi_{2,2} >$:	$g = \chi^{-1}(\chi-1)^{\frac{3}{2}}(7\chi-6)$
$< \psi_{3,1}\, \psi_{1,2}\, \psi_{2,2}\, \psi_{2,2} >$:	$g = \chi^{-1}(\chi-2)(\chi-1)^{\frac{3}{2}}$
$< \psi_{3,1}\, \psi_{1,3}\, \psi_{2,1}\, \psi_{2,2} >$:	$g = \chi^{-2}(\chi-1)^{\frac{3}{2}}(7\chi^2-12\chi+4)$
$< \psi_{3,1}\, \psi_{1,3}\, \psi_{2,2}\, \psi_{2,2} >$:	$g = \chi^{-2}(\chi-1)^{\frac{3}{2}}(\chi^2-12\chi+12)$
$< \psi_{1,2}\, \psi_{1,2}\, \psi_{2,1}\, \psi_{2,2} >$:	$g = \chi^{\frac{2}{5}}(\chi-1)^{\frac{1}{10}}$
$< \psi_{2,1}\, \psi_{2,1}\, \psi_{1,2}\, \psi_{1,3} >$:	$g = \chi^{\frac{5}{8}}(\chi-1)^{-\frac{1}{8}}$
$< \psi_{1,2}\, \psi_{2,1}\, \psi_{1,2}\, \psi_{2,1} >$:	$g = \chi^{-\frac{1}{2}}(\chi-1)^{-\frac{3}{10}}(2\chi-1)$
$< \psi_{1,2}\, \psi_{2,1}\, \psi_{1,3}\, \psi_{2,2} >$:	$g = \chi^{-\frac{1}{2}}(\chi-1)^{\frac{1}{10}}(2\chi+1)$
$< \psi_{2,2}\, \psi_{2,1}\, \psi_{1,3}\, \psi_{1,3} >$:	$g = \chi^{\frac{1}{8}}(\chi-2)(\chi-1)^{-\frac{21}{40}}$
$< \psi_{1,3}\, \psi_{2,1}\, \psi_{1,3}\, \psi_{2,1} >$:	$g = \chi^{-1}(\chi-1)^{\frac{1}{5}}(4\chi^2-4\chi+3)$

REFERENCES

1. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.
2. D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **52** (1984) 1575.
3. Vl. S. Dotsenko, *J. Stat. Phys.* **34** (1984) 781; *Nucl. Phys.* **B235** (1984) 54.
4. G. E. Andrews, R. J. Baxter and P. J. Forrester, *J. Stat. Phys.* **35** (1984) 193; D. A. Huse, *Phys. Rev.* **B30** (1984) 3908.
5. B. L. Feigin and D. B. Fuchs, Moscow preprint, 1983.
6. Vl. S. Dotsenko and V. A. Fateev, *Nucl. Phys.* **B240** (1984) 312; *Nucl. Phys.* **B251** (1985) 691.
7. V. G. Kac, Lecture notes in physics **94** (1979) 441.
8. T. D. Schultz, D. C. Mattis and E. H. Lieb, *Rev. Mod. Phys.* **36** (1964) 856.
9. L. P. Kadanoff, *Phys. Rev.* **188** (1969) 859; A. Luther and I. Peschel, *Phys. Rev.* **B12** (1975) 3908.
10. D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **151B** (1985) 37; Z. Qiu, *Nucl. Phys.* **B270** (1986) 205; M. A. Bershadskii, V. G. Knizhnik and M. G. Teitelman, *Phys. Lett.* **151B** (1985) 31.